Algebraic connectivity: local and global maximizer graphs

Karim Shahbaz, Madhu N. Belur and Ajay Ganesh

***Abstract*—Algebraic connectivity is one way to quantify graph connectivity, which in turn gauges robustness as a network. In this paper, we consider the problem of maximizing algebraic connectivity both locally and globally overall simple, undirected, unweighted graphs with a given number of vertices and edges. We pursue this optimization by equivalently minimizing the largest eigenvalue of the Laplacian of the 'complement graph'. We establish that the union of complete subgraphs are largest eigenvalue *local* minimizer graphs. Further, under sufficient conditions satisfied by the edge/vertex counts, we prove that this union of complete components graphs are, in fact, Laplacian largest eigenvalue *global* maximizers; these results generalize the ones in the literature that are for just two components. These sufficient conditions can be viewed as quantifying situations where the component sizes are either 'quite homogeneous' or some of them are relatively 'negligibly small', and thus generalize known results of homogeneity of components. While a conjecture about global optimality of complete bipartite graphs' from the literature continues to remain open, assuming appropriate constraints we prove the conjecture and also formulate/prove a variant of this claim. We finally relate this central optimization problem in this paper with the Discrete Fourier Transform (DFT) and circulant graphs/matrices.**

***Index Terms*—Algebraic connectivity, Laplacian matrices, Circulant matrix, Discrete Fourier Transform (DFT).**
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1. INTRODUCTION

Graph connectivity finds application in networking, network security, transportation systems, multi-agent control and has been well studied in the literature. Connectivity of graph G is also a measure of robustness as a network. Algebraic connectivity [3] being one of the measures of graph connectivity is defined as the second smallest eigenvalue λ_F of the Laplacian matrix $L(G) \in \mathbb{R}^{n \times n}$ of the unweighted, undirected and simple graph G.

In this paper, we consider only simple undirected, unweighted graphs, i.e., with no self loops and no multiple edges between any pairs of the vertices. We study the problem of maximizing the algebraic connectivity of a graph for a given number of nodes and edges. We pursue this problem for: a *global* maximization across all graphs, and a *local* sense, in which we consider only one edge 'rearrangements' (See Definition 3.1 below). Since algebraic connectivity λ_F and the problem of maximizing it have received extensive attention and are well-understood, we quickly delve further into the problem formulation and then touch on other closely related work in the literature.

A. Notation

The notation we follow is standard and is included here for quick reference. The sets of real and complex numbers are denoted respectively by $\mathbb{R}$ and $\mathbb{C}$. The largest eigenvalue of a symmetric matrix is denoted by $\lambda_{\max}$. Given an undirected graph G, the number of vertices $|V(G)|$ is usually n, the number of edges $|E(G)|$ is usually m, and the number of components of the graph is usually p. Further, the maximum degree across all vertices is denoted by Δ, and d_{avg} is the average degree of vertices. The n eigenvalues of the Laplacian matrix $L(G)$ are denoted by $\lambda_1(L(G)) \geqslant \lambda_2(L(G)) \geqslant \cdots \geqslant \lambda_{n-1}(L(G)) \geqslant \lambda_n(L(G)) = 0$. Further note that for the largest eigenvalue of the Laplacian $\lambda_1(L(G)) =: \lambda_{\max}(L(G))$ and for the second smallest eigenvalue, $\lambda_{n-1}(L(G)) =: \lambda_F(L(G))$. When the matrix $L(G)$ and the graph G are clear from the context, we use just $\lambda_{\max}, \ldots, \lambda_F$ to denote the eigenvalues, and when comparing the maximum eigenvalues of Laplacian matrices of different graphs say G and G^c, we use $\lambda_{\max}(G)$ and $\lambda_{\max}(G^c)$. Note that, since L is symmetric, $\lambda_{\max}(L(G)) = \lambda_1(L(G)) = \max_{\|x\|_2=1} x^T L(G)x$.

We sometimes deal with integer-valued properties and their relation with other bounds, and in this context, we use the standard floor and the ceiling functions of x, denoted by $\lfloor x \rfloor$ and $\lceil x \rceil$, to mean the largest/smallest integer, not greater-than/not-smaller-than the real number x respectively.

The notion of complement graph G^c of a simple undirected graph G is straightforward: it is also a simple undirected graph with the same number (and indexing) of nodes and in which there is an edge in G^c between two nodes, by definition, if and only if there is no edge between those nodes in G.

The complete graph in n vertices is denoted by K_n, and the complete bipartite graph with vertex sets having cardinalities n_1 and n_2 is denoted by K_{n_1,n_2}. Of course, our paper deals with complete multi-partite graphs $K_{n_1,n_2,\ldots n_p}$, and in fact, with their complement graphs: which would then be a union of complete graphs, denoted by $\bigcup_{i=1}^{p} K_{n_i}$.

B. Problem formulation

The paper deals with the following two closely related problems. For easier referencing, they are listed below as Problems 1.1 and 1.2.

Problem 1.1. *The following sub-problems are interrelated for reasons clarified soon in the next section.*

(a) *For a given number of vertices $|V| = n$ and number of edges $|E| = m_1$, find an Algebraic Connectivity λ_F Maximizer graph $G_1 = (V, E)$.*
(b) *For a given number of vertices $|V| = n$ and number of edges $|E| = m_2$, find a Laplacian matrix L Largest Eigenvalue $\lambda_{\max}$ Minimizer graph $G_2 = (V, E)$.*

Further, each of the above optimizations can be pursued in one of two ways: globally and locally. For simplicity, we elaborate on the second one, i.e., the largest eigenvalue minimization: we study the 'global' case and the 'local' case. More precisely,

1) *finding a Largest Eigenvalue Global Minimizer (LEGM) graph that has the least largest eigenvalue possible for the given number of vertices and edges, and*

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2) *finding a Largest Eigenvalue Local Minimizer (LELM), with 'local minima' in the sense that all one-edge reconnect, one-edge addition and removal graphs (see Definition 3.1) have either the same largest eigenvalue $\lambda_{\max}$ or even higher $\lambda_{\max}$.*

Related work in the context of the above problem is pursued in the next section. The problem we consider in this paper is closely linked with circulant graphs (pursued further in Section 6) and DFT of time-symmetric vectors with entries from $\{0,1\}$. The remark after the problem formulation below makes this link precise.

Problem 1.2. *DFT magnitude minimization: given positive integers d and n with $1 \leqslant d \leqslant n-1$, consider the following steps/requirements.*
(a) *Define vector $x \in \{0,1\}^n$ with $x_1 = 0$ and $\|x\|_1 = d$.*
(b) *Let x satisfy 'time-symmetry', i.e. $x_i = x_{n+2-i}$, for $i = 2,..,n$.*
(c) *Use x to define $\bar{x} \in \mathbb{R}^n$ by $\bar{x}_1 := -d$, and $\bar{x}_i := x_i$, for all other i.*
(d) *Obtain the Discrete Fourier Transform (DFT)[1] of the vector $\bar{x}$ by $X = DFT(\bar{x})$. Notice that $X \in \mathbb{R}^n$ due to time-symmetry.*
(e) *Consider the minimization problem: find x satisfying the conditions above such that $\|X\|_\infty$ is minimized.*

Circulant matrices are pursued further in Section 6. The following remark motivates the assumptions within the problem formulations above.

Remark 1.3. *The following points relate to Problems 1.1 and 1.2 and Laplacian matrices of circulant graphs.*

1) *The condition $\bar{x}_1 = -d$ means that the 'DC part' of $\bar{x}$ is zero, hence $X_1 = 0$. Thus minimizing[2] $\|X\|_\infty$ means that the focus is on minimizing the maximum magnitude of all frequencies, except the DC.*
2) *Entries in X are nothing but the negative of the eigenvalues of the Laplacian of the graph G_C constructed from x, and G_C is regular (of degree d) and is circulant; i.e., the Laplacian matrix is a circulant matrix.*
3) *The operation of defining $\bar{x} \in \mathbb{R}^n$ from $x \in \{0,1\}^n$ is one of adding an appropriately scaled discrete time impulse δ; the impulse has an equal amount of all frequencies. The DFT operation is linear on the signal space, thus keeping the optimization focus on the non-DC part in the signal x.*
4) *The operation of defining $\bar{x}$ from x is like studying the eigenvalues of $A - D$ (i.e. $-L$) instead of the adjacency matrix A, and note that the diagonal matrix D (the degree matrix) is merely $d \cdot I$ for this regular and circulant graph.*
5) *A careful use of above points gives $\|X\|_\infty = \lambda_{\max}(L)$.*

[1]For convenience, we include the definition of DFT here: DFT(x)=$X \in \mathbb{C}^n$, with $X_k = \sum_{j=1}^n x_j e^{\{2\pi i(j-1)(k-1)/n\}}$. Note: for uniformity with the rest of this paper, we use indices of the vectors $x, \bar{x} \in \mathbb{R}^n$ and $X \in \mathbb{C}^n$ to vary from 1 to n, instead of the typical DFT convention of using indices from 0 to $n-1$.

[2]Assume, for simplicity, the condition of time-symmetry, i.e. Condition (b) within Problem 1.2 is not imposed. Consider an example where $n = 10$ and $d = 4$. Then, it is easy to see that vectors $x = [0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 0]$, and $y = [0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 0]$, and $z = [0\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0]$, all have the same DC-value ($= d$). However, amongst the non-DC components in the corresponding DFTs, $DFT(x)$ has the highest ∞ norm, while $DFT(y)$ has the smallest ∞-norm and $DFT(z)$ has an intermediate value of ∞-norm. Thus, for multiple signals having same values of n and d, one studies the problem of minimizing the ∞ norm of the DFT of a vector $x \in \{0,1\}^n$ (excluding the DC-value within the DFT). This would thus amount to finding a discrete-time periodic signal having a lowest value of its maximum variation/frequency component.
Time-symmetry (i.e. Condition (b) of Problem 1.2) links this problem with symmetry of the circulant matrix, and hence to Adjacency/Laplacian matrices of a suitable simple, undirected and unweighted graph, and thus to the central problem in this paper.

C. Organization of the paper

The rest of this paper is organized as follows. The next section relates the problem we pursue with other work in the literature and how our work generalizes existing results. Section 3 contains the main results of this paper, about *locally* optimal graphs. Further, in the context of *globally* optimal graphs, our main results that improve upon results in the literature, formulate and prove for the case of many components are contained in Section 4. Section 6, relates our work to the Discrete Fourier Transform and circulant matrices/graphs. We consider some examples in Section 7. We conclude the paper in Section 8, where we also summarize the contribution of this paper.

2. Background and other work in this area

Recall that for a graph $G = (V, E)$, with V the vertex set and E the edge set, the Laplacian matrix is defined as $L(G) = D(G) - A(G)$ where $D(G)$ is the diagonal matrix with diagonal entries being the degree of vertices and $A(G)$ is the adjacency matrix of graph G. L is symmetric in this paper. The second smallest eigenvalue of L is defined as the algebraic connectivity of the graph G: see [3]. This eigenvalue is also called the Fiedler value and hence we denote it by λ_F.

Algebraic connectivity maximization of graphs has received much attention over the last few decades. Most recently in [12], a greedy heuristic is used to improve the Fiedler value from a random initial edge to have multiple solution and then a Markov chain Monte Carlo technique is applied to select fewer solutions. In another recent work [11], a heuristic algorithm based on the minimum degree and maximum distance is introduced. In the context of weighted graphs, [9] proposes an algorithm to find an edge to add to the graph to maximize algebraic connectivity; however, the edge weight here is a function of the distance between the vertices. The survey papers [5], [6], [8], [14], [17] and [18], contain a wealth of results about upper/lower bounds on the algebraic connectivity of a graph, many of which we use crucially in our paper too. In particular, given that we pursue maximum eigenvalue minimization on the complement graph instead of directly algebraic connectivity (second-smallest eigenvalue) maximization, it would help the reader to quickly review Proposition 2.1 below to see why this approach of focusing on the complement is equivalent.

Closely aligned with our paper, [16] pursues Algebraic Connectivity 'Local' Maximizers (ACLM) in the graph set of all one edge changes as in Definition 3.1 and global maximizers, where for a given number of vertices and edges, conditions guaranteeing global optimality are formulated. Propositions 2.4 and 2.5 contain the exact statements from [16] since this work is relevant to some of the main results in our paper. Both local and global optima obtained in [16] pursue the case when the complement has *two* components, while our paper generalizes to the case when the complement has multiple components and also slightly improves the bounds for the 2-components case. The rest of this section contains results we use and improve upon in this paper.

The following result crucially relates eigenvalues of the Laplacian matrices of graph G and its complement G^c.

Proposition 2.1. *[17, page 148] Let G be a simple undirected, unweighted graph and G^c be its complement. Then the largest eigenvalue of the graph $\lambda_{\max}(G)$, satisfies $\lambda_{\max}(G) \leqslant n$. Further,* *the eigenvalues of the Laplacian matrices of* G *and* G^c *are related by* $\lambda_i(G^c) = n - \lambda_{n-i}(G)$ *for* $i = 1, .., n-1$ *and* $\lambda_n(G) = \lambda_n(G^c) = 0$.

The following well-known result (from [19]) gives a lower bound for the maximum eigenvalue and also formulates the unique situation when the bound is tight.

Proposition 2.2. *[19, Theorem 3.19] Consider a connected graph* G *with at least one edge, vertex set* $V(G)$ *of cardinality* n*. Then the following hold.*

a) The maximum eigenvalue of the Laplacian matrix of the graph satisfies $\lambda_{\max}(L(G)) \geqslant \Delta + 1$.
b) $\lambda_{\max}(L(G)) = \Delta + 1$ *holds if and only if* $\Delta = n - 1$*, i.e., there exists a 'star node' in* G.

The following result gives a different lower bound for the maximum eigenvalue and the situation when this bound is tight.

Proposition 2.3. *([14, Theorem 3]) Let Graph* G *have* $n \geqslant 2$ *vertices and let* γ *denote its domination*[3] *number. Then,* $\lambda_{\max}(G) \geqslant \lfloor \frac{n}{\gamma} \rceil$ *and, further, equality holds if and only if* $G = G_a \bigcup G_b$ *such that:*

1) $|V(G_a)| = \lfloor \frac{n}{\gamma} \rfloor$ *and* $\gamma(G_a) = 1$*, and*
2) $\gamma(G_b) = \gamma(G) - 1$ *and* $\lambda_{\max}(G_b) \leqslant \lfloor \frac{n}{\gamma} \rfloor$.

Some of the main results in our paper generalize the following results from [16], and we generalize these results to the case of more than two components in the complement graph. For a specified number of vertices and edges, [16] studies the problem of Algebraic Connectivity Global Maximizer (ACGM) graphs and Algebraic Connectivity Local Maximizer (ACLM) graph. The precise statements are below.

Proposition 2.4. *[16, Thm 3] For integers* $a \in \mathbb{Z}^+$*, if* $a \leqslant \lceil \frac{n}{2} \rceil$ *and* $a - \frac{2a^2}{n} < 1$*, then for any* $n \geqslant 3$*, the complete bipartite graph* $K_{a,n-a}$ *is an ACGM in graphs with* n *vertices and* $a(n-a)$ *edges.*

Proposition 2.5. *[16, Thm 6] For integers* $a \in \mathbb{Z}^+$*, if* $a \leqslant \lceil \frac{n}{2} \rceil$*, then the complete bipartite graph* $K_{a,n-a}$ *is ACLM in graphs with* n *vertices and* $a(n-a)$ *edges.*

Proposition 2.6. *[18, Thm 3.1] Consider Graph* G *with at least one edge and independence*[4] *number* $\alpha(G)$*. Then,* $\lambda_{\max}(G) \geqslant \frac{n}{\alpha}$ *and, further, equality holds if and only if* α *is a factor of* n *and thus* G *then has* α *components, each being* $K_{\frac{n}{\alpha}}$.

We prove in this paper (Theorem 4.1) that the complement graph G^c made up of two complete components graph is LEGM under a very similar (and slightly relaxed) sufficient condition as compared to Proposition 2.4. We also extend the result (Theorem 4.5) of the complete two components to multi-components and prove that the graph is LEGM under an appropriately generalized sufficient condition. The notion of Algebraic Connectivity Local Maximizers (ACLM) graph was introduced in [16]. An ACLM graph G is one in which amongst all graphs obtained by changing one edge (i.e., one edge is added or removed or reconnected to a different set of vertices), then G's algebraic connectivity is not lower than among all such 'one edge changed' graphs. ACLM graphs thus need not be globally optimal, but are merely locally optimal. In [16], it has been shown that the complete bipartite graph $K_{a,n-a}$ is an Algebraic Connectivity Local Maximizer (ACLM) in G for n vertices and $a(n-a)$ edges graphs for $2 \leqslant a \leqslant \lfloor \frac{n}{2} \rfloor$; we generalize this result for the multipartite case i.e., the complement graph has not just two components but any number of components as Largest Eigenvalue Local Minimizers.

[3] The domination number of a graph $\gamma(G)$ is defined as the minimum size of the subset of vertices that are adjacent to every other vertex of the graph.

[4] The independence number of graph $\alpha(G)$ is defined as the cardinality of the largest set of vertices of the graph with no edge connection between any two amongst them.

3. Main results: locally optimal graphs

In this section, we present the main results of this paper, which concern 'locally' optimal graphs. The notion of local is made precise in the definition below. This notion coincides with that of [16]. Local optimality is important when only simple rearrangements of the topology of a set of multi-agents, for example, are allowed, and complicated rearrangements are disallowed. It helps to at least be locally optimal. Of course, globally optimal configurations would also need to satisfy this, and thus local optimality conditions are necessary conditions for global optimality too.

Definition 3.1. *(a)* ***One edge reconnect of*** G_0*: Let* $G_0(V, E_0)$ *be a simple graph with* $|V| = n$*, and* $|E_0| = m$*. We define* $G_1(V, E_1)$ *be a one-edge reconnect of* G_0 *if* G_1 *is also a simple graph and one or both of nodes of precisely one edge differ from that of* G_0*. Thus, we have one-edge reconnect if* G_1 *satisfies* $|E_1| = m$ *and* $\operatorname{rank}(L_1 - L_0) = 2$.
(b) ***One edge addition****: By one edge addition, we mean adding an edge to a graph while keeping the graph simple.*

Using the above notion of one edge reconnects and one edge additions, we define a local minimizer graph; this is w.r.t. the largest eigenvalue of the Laplacian.

Definition 3.2. ***Largest Eigenvalue Local Minimizer graph****: A graph* G_0 *is called a Largest Eigenvalue Local Minimizer (LELM) graph if* G_0 *has the least value of the Laplacian matrix's largest eigenvalue amongst all the simple graphs* G *obtained from* G_0 *by either one edge reconnect or one edge addition.*

In the context of various possibilities of an edge reconnection or addition, it helps to visualize the case using a figure. We include various figures, and the proof techniques vary depending on these cases. In summary: when we have a union of complete components, then an extra edge or an edge reconnection connects to complete components, and we make a distinction about whether the maximum degree of the full graph increases or remains same and whether the largest component (with vertex-size say n_1), or vertex-size slightly smaller than the largest (of size $n_1 - 1$), or further smaller was involved in the edge reconnection/addition. This distinction is needed to prove the local minimality of $\lambda_{\max}$ of the graph proposed in Theorem 3.6.

Lemma 3.3. *Suppose a connection is established between complete graph components* G_i *and* G_j *by adding an edge to give* G_{ij}^+ *and let* $L_{\text{new}} - L_{\text{old}} =: C_{\text{add}}$ *is the connection matrix. Then* $\operatorname{rank}(L_{\text{new}} - L_{\text{old}}) = 1$ *and the largest eigenvalue of* C_{add} *equals* 2 *(refer to Figure 1).*

Proof. Contribution to the Laplacian matrix of the graph due to an edge addition has the structure:

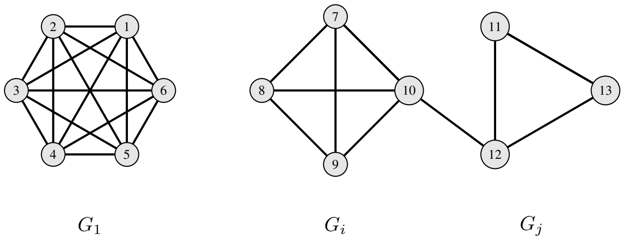


Figure 1: Connection established by one edge addition between G_i and G_j, where $|V(G_j)| \leqslant |V(G_i)| \leqslant |V(G_1)| - 2$

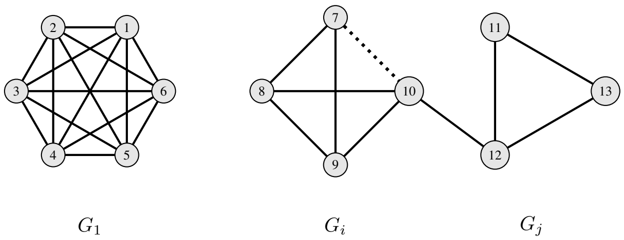


Figure 2: Reconnection without increasing the maximum degree of G_i, where $|V(G_j)| \leqslant |V(G_i)| \leqslant |V(G_1)| - 2$

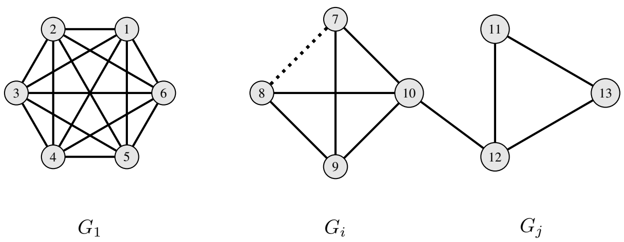


Figure 3: Reconnection with increasing the maximum degree of G_i, where $|V(G_j)| \leqslant |V(G_i)| \leqslant |V(G_1)| - 2$

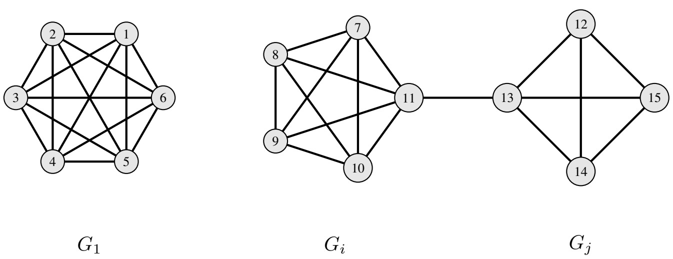


Figure 4: Connection established by one edge addition to G_i, where $|V(G_j)| \leqslant |V(G_i)| = |V(G_1)| - 1$

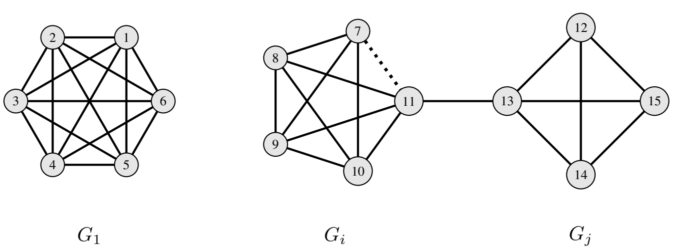


Figure 5: Reconnection without increasing the maximum degree of G_i, where $|V(G_j)| \leqslant |V(G_i)| = |V(G_1)| - 1$

$$C_{\text{add}} = \begin{array}{c} \begin{array}{cccc} \;\; G_i & & \;\; G_j & \end{array} \\ \left[\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ \hline 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

Clearly, the matrix C_{add} has rank one and the characteristic polynomial: $\chi_{C_{\text{add}}}(s) = s^3(s-2)$. So, $\lambda_{\max}(C_{\text{add}}) = 2$.

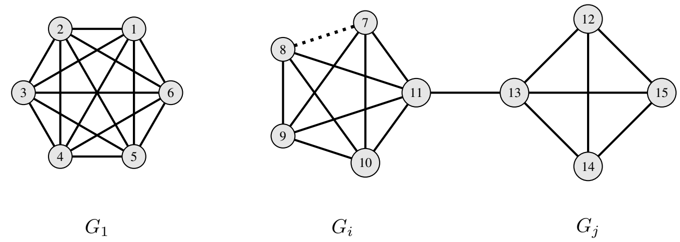


Figure 6: Reconnection with increasing maximum degree of G_i, where $|V(G_j)| \leqslant |V(G_i)| = |V(G_1)| - 1$

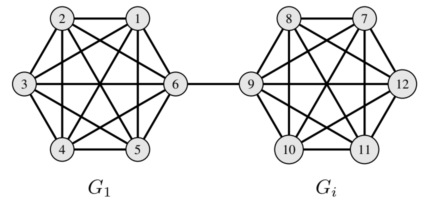


Figure 7: Connection established between largest size component G_1 with any size component by one edge addition

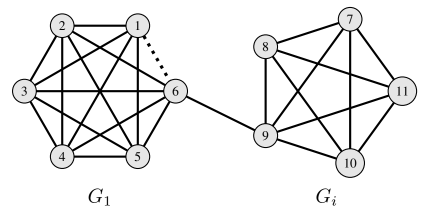


Figure 8: Reconnection of edge without increasing the maximum degree of largest size component G_1

For bigger or general size G_i and G_j with $|V(G_i)| + |V(G_j)| = a$, the structure of C_{add} remains the same but with zeros padded appropriately. Thus, C_{add} has rank one in general also and the lemma is proved. □

Lemma 3.4. *Suppose a connection is established between complete graph components G_i and G_j by reconnecting an edge by removing one edge e and adding elsewhere such that both nodes of e change, to give G_{ij}^+ and let $L_{\text{new}} - L_{\text{old}} =: C_{\text{re-incr}}$ is corresponding connection matrix. Then $\operatorname{rank}(L_{\text{new}} - L_{\text{old}}) = 2$ and the largest eigenvalue of $C_{\text{re-incr}}$ is 2 (refer to Figure 3).*

Proof. Contribution to the Laplacian matrix of the graph due to an edge reconnection as specified in the lemma has the following structure:

$$C_{\text{re-incr}} = \begin{array}{c} \begin{array}{cccc} \;\; G_i & & \;\; G_j & \end{array} \\ \left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \hline 0 & 0 & -1 & 1 \end{array}\right] \end{array}.$$

The matrix $C_{\text{re-incr}}$ has rank two and the characteristic polynomial: $\chi_{C_{\text{re-incr}}}(s) = s^2(s^2 - 2^2)$. Thus, $\lambda_{\max}(C_{\text{re-incr}}) = 2$. Again, for the general case, zeros get padded appropriately, and the lemma is thus proved. □

Lemma 3.5. *Suppose a connection is established between complete graph components G_i and G_j by reconnecting an edge by removing one edge e and adding an edge such that only one node of e is changed, to give G_{ij}^+, and let $L_{\text{new}} - L_{\text{old}} =: C_{\text{re-same}}$ be the corresponding connection matrix. Then $\operatorname{rank}(L_{\text{new}} - L_{\text{old}}) = 2$ and the largest eigenvalue of $C_{\text{re-same}}$ is $\sqrt{3}$ (refer to Figure 2).*

Proof. Contribution to the Laplacian matrix of the graph due to an edge reconnection as specified in the lemma has the following structure:

$$C_{\text{re-same}} = \begin{array}{c} \quad G_i \qquad\qquad G_j \\ \left[\begin{array}{ccc|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \hline 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}.$$

The matrix $C_{\text{re-same}}$ has rank two and the characteristic polynomial: $\chi_{C_{\text{re-same}}}(s) = s^3(s^2-3)$. Thus, $\lambda_{\max}(C_{\text{re-same}}) = \sqrt{3}$. Again, for the general case, zeros get padded appropriately and the lemma is thus proved. □

With the help of the above lemmas, we now state and prove the first main result of this paper.

Theorem 3.6. *Define $\mathcal{G}_{KC}$ as the family of graphs of union of complete components. A graph $G \in \mathcal{G}_{KC}$, (i.e., $G = \cup_i G_i$, where each G_i is a complete graph) is a Largest Eigenvalue Local Minimizer (LELM). In other words, for graph G of n number of vertices, m number of edges and p number of complete components of $|V(G_i)|$ size such that $\sum_{i=1}^{p}|V(G_i)| = n$ and $m = \sum_{i=1}^{p} {}^{|V(G_i)|}C_2$, then the value $\lambda_{\max}(G)$ is locally minimized, i.e. minimized w.r.t. all one edge reconnects, one edge removals, and one edge additions as defined in Definition 3.1. Further, $\lambda_{\max}(G) = \max_{i\in\{1,2,\ldots,p\}} |V(G_i)|$.*

Proof. Let $G_1, G_2, ..., G_p$ be the components and the number of nodes involved in those components be $|V(G_i)| = n_i$ then total edges, $|E(G)| = \sum_{i\in\{1,2,..,p\}} {}^{|V(G_i)|}C_2$. Without loss of generality, we assume that the size of the components have the following relation between them: $|V(G_1)| \geqslant |V(G_2)| \geqslant |V(G_3)| \geqslant ...|V(G_p)| > 0$. Thus the largest eigenvalue $\lambda_{\max}(G)$ of the graph G is: $\lambda_{\max}(G) = |V(G_1)|$, since $\lambda_{\max}(L(K_{n_1})) = n_1$. In this setup, if one edge is reconnected or one edge is added, it can be connected in the following three ways:

Case 1: Between components of smaller sizes G_i and G_j such that $|V(G_j)| \leqslant |V(G_i)| \leqslant |V(G_1)| - 2$, i.e. both components G_i and G_j are at least two or more nodes smaller than the largest component's size (G_1).

Case 2: Between component G_i and G_j with $|V(G_j)| \leqslant |V(G_i)| = |V(G_1)| - 1$.

Case 3: Between G_1 and any other component: same size as G_1 or smaller.

We now prove the theorem for each of the 3 cases. Note that for each case, we have three subcases: (a) Addition of an edge, (r_s) Removal and addition of an edge e such that only one vertex of e is changed, and (r_i) Removal and addition of an edge e such that both vertices of e are changed. We are not analyzing the one edge removal for local minimizer because (except the trivial case of removal from K_2) removing only one edge from a complete component graph, does not change its Laplacian largest eigenvalue. Thus, the proposed graph G is LELM w.r.t. removal.

Case 1: When a connection is established between two components of smaller sizes G_i and G_j (without loss of generality assuming $|V(G_i)| \geqslant |V(G_j)|$) such that $|V(G_i)| \leqslant |V(G_1)| - 2$, i.e. both G_i & G_j are at least two nodes smaller than the largest component G_1:

$1a$) By one edge addition (refer to Figure 1): If a connection is established between G_i and G_j to give G_{ij}^+ by adding an edge, then the connection matrix C_{add} of Lemma 3.3, gets added to $L(G_i \oplus G_j)$.
Thus, due to the edge addition in between components we get, $L(G_{ij}^+) = L(G_i \oplus G_j) + C_{\text{add}}$.
Also, $\lambda_{\max}(L(G_{ij}^+)) = \max_{\|x\|_2=1} x^T L(G_{ij}^+)x = \max_{\|x\|_2=1} [x^T L(G_i \oplus G_j)x + x^T C_{\text{add}}\, x]$.
Using Lemma 3.3, we have $\lambda_{\max}(C_{\text{add}}) = 2$, which implies that $\lambda_{\max}(L(G_{ij}^+)) \leqslant \lambda_{\max}(L(G_i \oplus G_j)) + 2 = \lambda_{\max}(L(G_i)) + 2 \leqslant \lambda_{\max}(G_1)$.
Therefore, $\lambda_{\max}(G) = \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2), \ldots, \lambda_{\max}(G_{ij}^+)\} = \lambda_{\max}(G_1)$. This proves that $\lambda_{\max}(G)$ remains same and the proposed graph G is a $\lambda_{\max}(G)$ local minimizer.

$1r$) One edge reconnect: If the connection established between G_i and G_j to give G_{ij}^+ by reconnecting one edge, then the following two different types of C connection matrix get added to $L(G_i \oplus G_j)$ depending upon how the reconnection of an edge is done.

$1r_s$) Reconnection without increasing the maximum degree of G_i (refer to Figure 2):
Due to the reconnection, the connection matrix $C_{\text{re-same}}$ of Lemma 3.5 gets added and we get $L(G_{ij}^+) = L(G_i \oplus G_j) + C_{\text{re-same}}$.
Also, $\lambda_{\max}(L(G_{ij}^+)) = \max_{\|x\|_2=1} x^T L(G_{ij}^+)x = \max_{\|x\|_2=1} [x^T L(G_i \oplus G_j)x + x^T C_{\text{re-same}}\, x]$
Using Lemma 3.5, we have $\lambda_{\max}(C_{\text{re-same}}) = \sqrt{3}$.
$\implies \lambda_{\max}(L(G_{ij}^+)) \leqslant \lambda_{\max}(L(G_i \oplus G_j)) + \sqrt{3} = \lambda_{\max}(L(G_i)) + \sqrt{3} < \lambda_{\max}(L(G_i)) + 2 \leqslant \lambda_{\max}(G_1)$.
Therefore, $\lambda_{\max}(G^+) = \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2), ..., \lambda_{\max}(G_{ij}^+)\} = \lambda_{\max}(G_1)$. $\lambda_{\max}(G)$ remains the same, and our graph is local minimizer.

$1r_i$) Reconnection with increasing the maximum degree of G_i (refer to Figure 3):
Due to reconnection, the connection matrix $C_{\text{re-incr}}$ of Lemma 3.4 gets added and we get $L(G_{ij}^+) = L(G_i \oplus G_j) + C_{\text{re-incr}}$.
Also, $\lambda_{\max}(L(G_{ij}^+)) = \max_{\|x\|_2=1} x^T L(G_{ij}^+)x = \max_{\|x\|_2=1} [x^T L(G_i \oplus G_j)x + x^T C_{\text{re-incr}}\, x]$
Using Lemma 3.4, $\lambda_{\max}(C_{\text{re-incr}}) = 2$.
$\implies \lambda_{\max}(L(G_{ij}^+)) \leqslant \lambda_{\max}(L(G_i \oplus G_j)) + 2 = \lambda_{\max}(L(G_i)) + 2 \leqslant \lambda_{\max}(G_1)$.
Therefore, $\lambda_{\max}(G^+) = \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2), ..., \lambda_{\max}(G_{ij}^+)\} = \lambda_{\max}(G_1)$. $\lambda_{\max}(G)$ remains the same and the proposed graph G graph is a local minimizer. This completes the proof of Case 1.

Case 2: When a connection is established between two smaller size components with at least one component has precisely one vertex less than the largest one. In other words, the connection between component G_i of size $|V(G_i)| = |V(G_1)| - 1$ and any other component G_j of equal or smaller size than G_i i.e. $|V(G_j)| \leqslant |V(G_i)| = |V(G_1)| - 1$:

$2a$) By one edge addition (refer to Figure 4): Suppose a connection is established between G_i and G_j to give G_{ij}^+ by adding an edge (using Proposition 2.2 b),
$\lambda_{\max}(G_{ij}^+) > |V(G_i)| + 1 = |V(G_1)| = \lambda_{\max}(G_1)$.

$\lambda_{\max}(G^+) = \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2), ..., \lambda_{\max}(G_{ij}^+)\} = \lambda_{\max}(G_{ij}^+) > \lambda_{\max}(G_1)$. Thus, proposed graph G is a local minimizer.

$2r$) One edge reconnect: Suppose a connection is established between G_i and G_j to give G_{ij}^+ by relocating an edge, then following two different types of C connection matrix gets added to $L(G_i \oplus G_j)$ depending upon how reconnection of an edge is done.

$2r_s$) Reconnection without increasing the maximum degree of G_i (refer to Figure 5):
Due to the reconnection, the connection matrix $C_{\text{re-same}}$ of Lemma 3.5 gets added and we get $L(G_{ij}^+) = L(G_i \oplus G_j) + C_{\text{re-same}}$. Also, $\lambda_{\max}(L(G_{ij}^+)) =$

$$\max_{\|x\|_2=1} x^T L(G_{ij}^+)x = \max_{\|x\|_2=1} [x^T L(G_i \oplus G_j)x + x^T C_{\text{re-same}}\, x]$$

Using Lemma 3.5, $\lambda_{\max}(C_{\text{re-same}}) = \sqrt{3}$.
$\lambda_{\max}(L(G_{ij}^+)) \leqslant \lambda_{\max}(L(G_i \oplus G_j)) + \sqrt{3} = \lambda_{\max}(L(G_i)) + \sqrt{3} = \lambda_{\max}(L(G_1)) + \sqrt{3} - 1$.
So, in case of reconnecting without increasing maximum degree, we use the following relation:
$\lambda_{\max}(G_i) = \lambda_{\max}(G_1) - 1 < \lambda_{\max}(G_{ij}^+) \leqslant \lambda_{\max}(G_1) + \sqrt{3} - 1$.
Thus, $\lambda_{\max}(G_1) \leqslant \lambda_{\max}(G^+) \leqslant \lambda_{\max}(G_1) + \sqrt{3} - 1$ implies $\lambda_{\max}(G^+)$ either increases or remains the same. Therefore again the proposed graph G is an LELM.

$2r_i$) Reconnection with increasing the maximum degree of G_i (refer to Figure 6): Suppose the connection is established between G_i and G_j to give G_{ij}^+ by reconnecting an edge with increasing maximum degree of G_i, we get: (using Proposition 2.2),
$\lambda_{\max}(G_{ij}^+) > |V(G_i)| + 1 = |V(G_1)| = \lambda_{\max}(G_1)$.
$\lambda_{\max}(G^+) = \max\{\lambda_{\max}(G_1), \lambda_{\max}(G_2), .., \lambda_{\max}(G_{ij}^+)\}$ which equals $\lambda_{\max}(G_{ij}^+)$. Hence, $\lambda_{\max}(G)$ increases. Thus, the proposed graph G is an LELM.

This completes the proof of Case 2.

Case 3: When a connection is established between the largest size component G_1, and any other component:

$3a$) By one edge addition (refer to Figure 7): Before the addition of edge, we have $\lambda_1(G) = |V(G_1)|$. Then the connection is established in two ways: between the two largest size components and between the largest and any other size components. Thus, the addition of edge between components $K_{|V(G_1)|}$ and $K_{|V(G_i)|}$ leads to $\lambda_1(G) > |V(G_1)|$ (using Proposition 2.2) [19], [5]). Therefore, the proposed graph is a $\lambda_1(G)$ local minimizer (LELM).

$3r$) Reconnecting of edge with or without increasing the maximum degree of G_1 (refer to Figure 8): Here, the connection is established in cases with the largest size component G_1 by re-connecting $K_{|V(G_1)|}$ and $K_{|V(G_i)|}$ either by increasing maximum degree of G_1 or not; similarly like the addition of edge, the reconnection leads to $\lambda_1(G) > |V(G_1)|$ (using Proposition 2.2) [19], [5]). Hence the proposed graph G is again an LELM for this case also.

This completes the proof of Case 3 and also the proof of the theorem. □

The following corollary is an easy consequence of Theorem 3.6.

Corollary 3.7. *Consider graph G of n number of vertices and m number of edges forming a complete p-partite graph, $K_{n_1,n_2,...n_p}$ such that $\sum_{i=1}^p n_i = n$ and $m = |E(G)| = \prod_{i=1}^p n_i$. Then the p-partite graph $G = K_{n_1,n_2,...n_p}$ is an Algebraic Connectivity Local Maximizer (ACLM).*

4. Main results: globally optimal graphs

In this section, we obtain sufficient conditions for the union of complete graphs to be a global minimizer of the largest eigenvalue. The first main result of this section (Theorem 4.1) is a slight improvement (though claimed and proved on the *complement* graph using different proof techniques) to Proposition 2.4. The second main result of this section (Theorem 4.5) is a generalization to the case of more than two components and also gives the first one as a corollary, except in the case of equality within the sufficient condition: equation (1).

Theorem 4.1. *Consider graph $G = (V, E)$ of n number of vertices and m number of edges consisting of two complete components K_ℓ and $K_{n-\ell}$, i.e. $m = |E(G)| = {}^\ell C_2 + {}^{n-\ell}C_2$. Let without loss of generality $\ell \leqslant \frac{n}{2}$. Assume*

$$\ell - \frac{2\ell^2}{n} \leqslant 1. \tag{1}$$

Then the graph, $G = K_\ell \bigcup K_{n-\ell}$ is a Largest Eigenvalue Global Minimizer (LEGM).

Proof. This proof involves two cases depending on whether the inequality $\ell - \dfrac{2\ell^2}{n} \leqslant 1$ is strict or met with equality. First, notice that when $\ell = \dfrac{n}{2}$, we get $\ell - \dfrac{2\ell^2}{n} = 0$ and $\ell < \dfrac{n}{2}$ is the same as $0 < \ell - \dfrac{2\ell^2}{n}$. Hence, the assumption in the theorem $\ell \leqslant \frac{n}{2}$ gives $0 \leqslant \ell - \dfrac{2\ell^2}{n}$ and on adding condition (1), we get $0 \leqslant \ell - \dfrac{2\ell^2}{n} < 1$ (Case 1) or holds with equality (Case 2).

Case 1: Condition $\ell - \dfrac{2\ell^2}{n} < 1$.
In order to prove the theorem, we obtain the average degree of the graph.
Average degree (d_{avg}) of the graph $G = K_\ell \bigcup K_{n-\ell}$:

$$\begin{aligned} d_{avg} = \frac{2m}{n} &= \frac{2}{n}\left\{\frac{\ell^2 - \ell}{2} + \frac{(n-\ell)^2 - (n-\ell)}{2}\right\}, \\ &= \frac{1}{n}\left\{2\ell^2 + n^2 - 2n\ell - n\right\}, \\ &= n - \ell - 1 - \left(\ell - \frac{2\ell^2}{n}\right). \end{aligned}$$

We use the maximum degree of the graph, $\Delta \geqslant d_{avg}$. We also use that Δ should be an integer which implies $\Delta \geqslant \lceil d_{avg} \rceil$. If $0 \leqslant \ell - \frac{2\ell^2}{n} < 1$, then the maximum degree, $\Delta \geqslant n - \ell - 1$.
Using Proposition 2.2a), for any graph that has as many edges as m, we get $\lambda_{\max}(G) \geqslant \Delta + 1$ and thus $\lambda_{\max}(G) \geqslant n - \ell$ for any graph having as many edges as $G = K_\ell \bigcup K_{n-\ell}$.
For the proposed graph G, the largest eigenvalue of the graph, $\lambda_{\max}(G) = \max\{\ell, n - \ell\} = n - \ell$.
Hence, the proposed graph G of theorem $K_\ell \bigcup K_{n-\ell}$ is an LEGM.

Case 2: Condition $\ell - \dfrac{2\ell^2}{n} = 1$.

Consider $\ell - \dfrac{2\ell^2}{n} = 1 \implies 2\ell^2 - n\ell + n = 0$

whose roots are: $\ell = \dfrac{n \pm \sqrt{n^2 - 8n}}{4}$.

Notice that for ℓ to be an integer, the discriminant $n^2 - 8n$ needs to be a perfect square, i.e. $n^2 - 8n = s^2$, where $s, n \in \mathbb{Z}^+$.
With the above constraints, non-negative integer solution n exists only for $n = 8$ and $n = 9$, which makes ℓ as 2 and 3, respectively.
For case (a): This is the case when $n = 8$, $\ell = 2$, i.e. $K_2 \bigcup K_6$, we have ${}^2C_2 + {}^6C_2 = 1 + 15 = 16$ edges, and $\lambda_{\max}(G) = 6$. The complement of this graph is a complete bipartite graph $G^c = K_{2,6}$ with minimum degree $\delta(G^c) = 2$. Also, we know from [3, Thm 4.1] $\lambda_F(G^c) \leqslant \delta(G^c) = 2$. But for complete bipartite graph, algebraic connectivity $\lambda_F(G^c) = \min(2, 6) = 2$ [3, Thm 3.11] which means in the complement graph space, algebraic connectivity is maximized. Hence in the original graph space, due to Proposition 2.1, the largest eigenvalue is minimized.
For case (b): This is the case when $n = 9$, $\ell = 3$, i.e. $K_3 \bigcup K_6$, we have ${}^3C_2 + {}^6C_2 = 3 + 15 = 18$ edges, and $\lambda_{\max}(G) = 6$. The complement of this graph is a complete bipartite graph $G^c = K_{3,6}$ with a minimum degree $\delta(G^c) = 3$. Also, we know from [3] that $\lambda_F(G^c) \leqslant \delta(G^c) = 3$. However, for a complete bipartite graph, the algebraic connectivity $\lambda_F(G^c) = \min(3, 6) = 3$ [3, Theorem 3.11], which means in complement graph space, algebraic connectivity is maximized. Hence in the original graph space, $\lambda_{\max}$ is minimized due to Proposition 2.1.
Thus, for both cases (a) and (b), we conclude that $K_2 \bigcup K_6$ and $K_3 \bigcup K_6$ respectively, are LEGM: this completes the proof. □

Theorem 4.1 yields the following corollary about λ_F.

Corollary 4.2. *Consider graph $G = (V, E)$ of n number of vertices and m number of edges forming a complete bipartite graph $K_{\ell,n-\ell}$, i.e. $m = |E(G)| = \ell(n - \ell)$. Let without loss of generality $\ell \leqslant \frac{n}{2}$. Assume*

$$\ell - \frac{2\ell^2}{n} \leqslant 1. \tag{2}$$

Then the complete bipartite graph $K_{\ell,n-\ell}$ is an Algebraic Connectivity Global Maximizer with $\lambda_F = \ell$.

The next theorem uses the above results to formulate conditions under which a complete bipartite graph is an ACGM; see also Footnote 5.

Theorem 4.3. *Consider integers n, ℓ such that $2 \leqslant \ell \leqslant \frac{n}{2}$ and*

$$n \leqslant \frac{2\ell^2}{\ell - 1}. \tag{3}$$

Then, among all the graphs of n vertices and $m = \ell(n - \ell)$ edges, a graph which maximizes the algebraic connectivity is the complete bipartite graph $K_{\ell,n-\ell}$ with $\lambda_F(K_{\ell,n-\ell}) = \ell$.

Proof. The proof follows from Corollary 4.2: consider a graph G with n vertices and $m = \ell(n-\ell)$ edges, the condition $\ell \leqslant \frac{n}{2}$ and (2) gives $n\ell - 2\ell^2 \leqslant n$. Upon solving for n, we get $2\ell^2 \geqslant n\ell - n$ which yields $n \leqslant \frac{2\ell^2}{\ell-1}$. Hence, any complete bipartite graph satisfying the given conditions on ℓ and n is an ACGM. □

While equation (3) within Theorem 4.3 is a sufficient condition, Example 7.2 later in this paper shows how this sufficient condition's violation (for $n = 16$, $\ell = 4$) results in a circulant graph having a higher λ_F. Next, as a special case of Theorem 4.3 above, for bipartite graphs, we formulate Corollary 4.4. The assumption of constraints on the number of vertices makes this corollary a special case of [10, Conjecture 1.5], leaving the conjecture[5] open for the general case.

Corollary 4.4. *Among all graphs of n vertices and $m = 2(n-2)$ edges with $4 \leqslant n \leqslant 8$, a graph which maximizes the algebraic connectivity is the complete bipartite graph $K_{2,n-2}$ with $\lambda_F(K_{2,n-2}) = 2$.*

We now generalize Theorem 4.1 and Proposition 2.6 to the number of components $p > 2$.

Theorem 4.5. *Consider graph G of n number of vertices and m number of edges consisting of p complete components $K_{n_1}, K_{n_2}, \ldots K_{n_p}$ such that $\sum_{i=1}^{p} n_i = n$ and $m = |E(G)| = \sum_{i=1}^{p} {}^{n_i}C_2$. Let without loss of generality $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_p$. Assume*

$$n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} < 1. \tag{4}$$

Then the graph, $G = K_{n_1} \bigcup K_{n_2} \ldots \bigcup K_{n_p}$ is a Largest Eigenvalue Global Minimizer (LEGM) with $\lambda_{\max} = n_1$.

Proof. For the graph $G = K_{n_1} \bigcup K_{n_2} \ldots \bigcup K_{n_p}$, first notice that $n_1 - \dfrac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} \geqslant 0$.
This is because $\dfrac{n \times n_1 - (n_1^2 + n_2^2 + \ldots + n_p^2)}{n} = \dfrac{n_2(n_1 - n_2) + n_3(n_1 - n_3) + \ldots + n_p(n_1 - n_p)}{n}$ and thus only when $\frac{n}{p} \in \mathbb{Z}$ (and hence $\frac{n}{p} = n_1 = n_2 = \ldots = n_p$), we have $n_1 - \dfrac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} = 0$. For any other value of n and of n_i, we have $0 < n_1 - \dfrac{n_1^2 + n_2^2 + \ldots + n_p^2}{n}$ and thus $0 \leqslant n_1 - \dfrac{n_1^2 + n_2^2 + \ldots + n_p^2}{n}$ in general.
The average degree (d_{avg}) of the graph:

$$\begin{aligned} d_{avg} &= \frac{2m}{n} = 2\sum_{i=1}^{p} \frac{{}^{n_i}C_2}{n}, \\ &= \sum_{i=1}^{p} \frac{n_i^2 - n_i}{n} = \frac{n_1^2 + n_2^2 + \ldots + n_p^2 - n}{n}, \\ &= n_1 - 1 - (n_1 - \frac{n_1^2 + n_2^2 \ldots + n_p^2}{n}). \end{aligned}$$

We next use the maximum degree of the graph, $\Delta \geqslant d_{avg}$. We also know that Δ should be an integer which implies $\Delta \geqslant \lceil d_{avg} \rceil$. If $0 \leqslant n_1 - \dfrac{n_1^2 + n_2^2 \ldots + n_p^2}{n} < 1$, then the maximum degree $\Delta \geqslant n_1 - 1$.
Using Proposition 2.2a), for any graph that has as many edges as m, we get $\lambda_{\max}(G) \geqslant \Delta + 1 \implies \lambda_{\max}(G) \geqslant n_1$.
Finally, it remains to show that the proposed graph G satisfies $\lambda_{\max}(G) = n_1$. Since $n_1 \geqslant n_i$ and $\lambda_{\max}(K_{n_i}) = n_i$, we conclude that the graph proposed is LEGM. This completes the proof of Theorem 4.5. □

The following corollary is a restatement of the above theorem in terms of global maximization of λ_F.

[5] Conjecture 1.5 of [10] claims that among all graphs on n vertices and $m = 2(n-2)$ edges (with $n \geqslant 4$), a graph which maximizes the algebraic connectivity is the complete bipartite graph $K_{2,n-2}$ with $\lambda_F(K_{2,n-2}) = 2$.

Corollary 4.6. *Consider graph G of n number of vertices and m number of edges forming a complete p-partite graph, $K_{n_1,n_2,\ldots n_p}$ such that $\sum_{i=1}^{p} n_i = n$ and $m = |E(G)| = \prod_{i=1}^{p} n_i$. Let without loss of generality $n_1 \geqslant n_2 \geqslant \ldots \geqslant n_p$. Assume*

$$n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} < 1. \tag{5}$$

Then the p-partite graph $G = K_{n_1,n_2,\ldots n_p}$ is an Algebraic Connectivity Global Maximizer with $\lambda_F = n_1$.

The following remark interprets the sufficient conditions in the main results above and explains why our results are a generalization of existing results of the literature.

Remark 4.7. *Theorem 4.1 and Theorem 4.5 establish that when the components are of 'almost similar sizes' or the largest component is 'much larger than the smallest', we get a Largest Eigenvalue Global Minimizer graph. Both sufficient conditions, equations* (1) *and* (4)*, are to be viewed as relaxation on the condition 'α is a factor of n' in Proposition 2.6. This is elaborated as follows. From Proposition 2.6, it is clear that for any integer n_1, when we have $\bigcup_{i=1}^{p} K_{n_1}$, then this graph is an LEGM. Intuitively, when n_p is 'slightly smaller' than n_1, then too LEGM would continue to hold: for example, the graph in the first row in Table I. On the other hand, by addition of a 'sufficiently small' component $K_{n_{p+1}}$, i.e. when $0 < n_{p+1} \ll n_1$, then LEGM would continue to hold: for example, the last row in Table I. In other words, not just when all components are of the same size, but also when the components are 'quite homogeneous' or some of them are relatively 'negligibly small', then too LEGM property continues to hold: in that sense, the sufficient condition $n_1 - \frac{n_1^2 + n_2^2 + \ldots + n_p^2}{n} < 1$ is relaxation of the condition 'α is a factor of n' of Proposition 2.6.*

5. Algorithmic construction of LELM graphs

In this section we consider the problem of constructing LELM graphs. Given positive integers n and m, construct a graph with n vertices and m edges which is an LELM graph. We propose the algorithm below and prove its optimality later in Theorem 5.2.

Remark 5.1. *Note the following points due to which the algorithm achieves the objective of a 'best' LELM (see Theorem 5.2). In order to obtain a small $\lambda_{\max}$, it is essential to have a low value of q_i, the vertex cardinality of each complete component. For each component's size q_i, from amongst the remaining vertices n_i^{rem}, we construct ℓ_i number components of K_{q_i} with $\ell_i \leqslant \ell_{\max} := \lfloor \frac{n_i^{rem}}{q} \rfloor$. Each K_{q_i} accommodates ${}^{q_i}C_2$ edges and we try to have as many q_i-size components as possible. Loosely speaking, a slight increase in q_i helps accommodate significantly more edges (due to the fact that the edge-count is* quadratic *in the node-count for complete components). However, this is at the cost of an increase in $\lambda_{\max}$ (proportional to increase in q_i) and a lesser number of components ℓ_i. On the other hand, a smaller q_i aids in decreasing $\lambda_{\max}$ quite proportionately but would perhaps be unable to accommodate enough edges.*

The optimality property of the graph constructed by Algorithm 1 is formulated below.

Theorem 5.2. *Consider Algorithm 1 which takes input n, number of vertices and $m_{desired}$, desired number of edges. Suppose the algorithm terminates with $m_{actual} = m_{desired}$, then the constructed graph G is LELM. Consider $\mathcal{G}_{KC}$, the family of graphs defined in Theorem 3.6. Amongst all graphs in $\mathcal{G}_{KC}$, the graph G has the least $\lambda_{\max}$. Further, if the sufficient condition of Theorem 4.5 is met, the proposed graph is also an LEGM.*

Algorithm 1: Edges inclusion in the graph

- **Input**: Vertices count: n, number of edges desired: m^{des}.
- **Output**: Graph G of n nodes, number of edges $m^{act} \leqslant m^{des}$ and union of complete components with least $\lambda_{\max}(L)$: see Theorem 5.2.
- **Initialize** Set $i, j \leftarrow 0$, $q \leftarrow 2$, $n_0^{rem} \leftarrow n$ and $m_0^{rem} \leftarrow m^{des}$.
- **while** $m_i^{rem} > 0$ & $n_i^{rem} \geqslant 0$ & $q \leqslant n^{rem}$ **do**
  - $\ell_{max} := \lfloor \frac{n_i^{rem}}{q} \rfloor$
  - **if** $m_i^{rem} > {}^{n_i^{rem}}C_2$ **then** // m^{rem} too high for any complete graph
    - $\ell \leftarrow 1$ and $q \leftarrow n_i^{rem}$
    - Set $i \leftarrow i+1$, $n_i^{rem} \leftarrow n_{i-1}^{rem} - q\ell$ and $m_i^{rem} \leftarrow m_{i-1}^{rem} - \ell \cdot {}^{q}C_2$
    - **for** $k = 1, 2, \ldots, \ell$ **do** // storing the components sizes
      - Set $j \leftarrow j+1$ and $q_j \leftarrow q$
    - **end**
  - **else if** $m^{rem} \leqslant \ell_{max} \cdot {}^{q}C_2$ **then** // q & ℓ_{max} can take in m_{rem}
    - isTypical ← False // look for a corner case: addressed later below
    - **for** $\ell = \ell_{max}, \ell_{max}-1, \ldots, 1$ **do** // to find largest ℓ
      - **if** ${}^{q}C_2 \cdot \ell \leqslant m^{rem}$ **then** // not the corner case
        - isTypical ←True & break // found ℓ, break out of 'for loop'
      - **end**
    - **end**
    - **if** isTypical = False **then** // Need to ↓ q since $\ell \cdot K_{q-1} \leqslant m^{rem} < K_q$
      - $q \leftarrow q-1$ // decrease q slightly for the corner case
      - $\ell_{max} := \lfloor \frac{n_i^{rem}}{q} \rfloor$
      - **for** $\ell = \ell_{max}, \ell_{max}-1, .., 1$ **do** // to find largest ℓ
        - **if** ${}^{q}C_2 \cdot \ell \leqslant m^{rem}$ **then**
          - break // found ℓ, break out of the 'for loop'
        - **end**
      - **end**
    - **end**
    - Set $i \leftarrow i+1$, $n_i^{rem} \leftarrow n_{i-1}^{rem} - q\ell$ & $m_i^{rem} \leftarrow m_{i-1}^{rem} - \ell \cdot {}^{q}C_2$
    - **for** $k = 1, 2, \ldots, \ell$ **do** // storing the components sizes
      - Set $j \leftarrow j+1$ and $q_j \leftarrow q$
    - **end**
    - Set $q \leftarrow 2$ // re-enter while loop with smallest possible q
  - **else**
    - $q \leftarrow q+1$ // need to increase q to accommodate m_i^{rem}
  - **end**
- **end**
- Return the graph $\bigcup_{k=1}^{j} K_{q_k}$. // construct the graph of sizes defined

Proof. The claims in the theorem are straightforward; hence we summarize and dwell on only the key arguments in this proof. Remark 5.1 also contains various relevant arguments. The algorithm constructs components: largest first and then smaller ones until all vertices are used up, and the maximum number of edges (up to $m_{desired}$) are accommodated through the following steps/features.

- By construction, the obtained graph is clearly an LELM.
- Within the while loop, the condition $q_i \leqslant n_i^{rem}$ ensures that the new components do not exceed the remaining number of vertices.
- Setting $\ell_{max} := \lfloor \frac{n_i^{rem}}{q} \rfloor$ ensures that the ℓ_i components, each of vertex cardinality q_i, do not over-exhaust the remaining vertices.
- The condition $m_i^{rem} \leqslant \ell_{max} \cdot {}^{q}C_2$ verifies that it is possible to accommodate the desired number of edges with the current q value.
- The check ${}^{q}C_2 \cdot \ell_i \leqslant m_i^{rem}$ with decreasing ℓ_i ensures that ℓ_i is as large as possible for a given component size q_i to accommodate the desired number of edges.

Thus, the construction procedure accommodates the desired number of edges with as small size components of complete graphs K_{q_i} as possible and hence is an LELM with $\lambda_{\max} = q_1$. Further, from the procedure, $\lambda_{\max}$ is the least amongst the family of LELM graphs: $\mathcal{G}_{KC}$. □

6. Circulant Graphs

Propositions 2.1 and 2.2 are about relations between the Laplacian eigenvalues for a graph and its complement and about the maximum degree Δ providing a lower bound for the $\lambda_{\max}$. In particular, the lower bound $\Delta + 1$ is tight for the case when the graph contains a star node, i.e., the domination number γ (see Footnote 3) equals 1. This naturally suggests that a relatively equitable distribution of edges keeps the max-degree Δ low and thus also helps keep the maximum eigenvalue $\lambda_{\max}$ low. Circulant matrices are such matrices: they are regular and contain a symmetry that indeed makes them LEGM for certain cases; we pursue this link in this section.

A matrix $C \in \mathbb{R}^{n\times n}$ is called *circulant* if each entry $c_{i,j}$, the entry in the i-th row and the j-th column satisfies: $c_{i,j} = c_{i+k,j+k}$, where the indices are considered to be modulo-n and - for this reason, and just for this sentence - indices i, j vary from 0 to $n-1$. It is well-known (see [1]) that the set of circulant matrices form an n-dimensional subspace of $\mathbb{R}^{n\times n}$, and the entries of only the first row of C need to be specified for specifying the $n \times n$ matrix C. A *circulant graph* is one whose Laplacian is a circulant matrix after a permutation/re-ordering of the nodes, if needed. Define the matrix $J \in \mathbb{R}^{n\times n}$ such that $J_{ij} = 1$ for all $i, j \in \{1, 2, \ldots, n\}$. Notice that $nI - J$ is a circulant matrix with generating row as $[n-1, -1, -1, \ldots, -1]$. The Laplacian of this circulant graph is same as the Laplacian of K_n, i.e. $nI - J$. This means that if G is a circulant graph, so is its complement G^c.

We further pursue Problem 1.2 and note that the DFT of the first row of a circulant matrix C are precisely the eigenvalues of C. Given integers n and m, the number of vertices and edges, due to the implicit regularity of a circulant graph, $2 \times m$ has to be divisible by n for a circulant graph $G(V, E)$ to exist such that $|V| = n$ and $|E| = m$.

Below is our first result in this context. We also consider some related examples in this and the following section.

Theorem 6.1. *Consider positive integers n and m satisfying Conditions (A) and (B) below:*

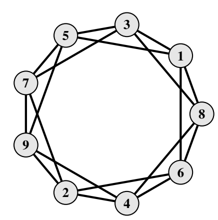
(A) n is a factor of $2m$, and *(B) $\dfrac{2m}{n} + 1$ is a factor of n.*

Then, the following hold.

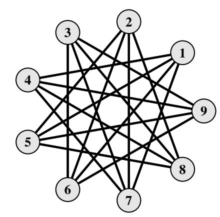
1) *There exists a circulant graph G_c having n vertices and m edges.*
2) *G_c is an LEGM.*
3) *The first row of the adjacency matrix of G_c solves Problem 1.2.*
4) *G_c^c, the complement of G_c, is also a circulant graph with the highest algebraic connectivity, i.e., G_c^c is an ACGM.*

Proof. Notice that the condition (A) allows regularity and the condition (B) on m allows one to construct $G := \underbrace{\bigcup K_q}_{\ell \text{ times}}$, with $q := \dfrac{2m+n}{n}$ and $\ell := \frac{n}{q}$. This union of complete components graph G so obtained can be made into circulant graph G_c by carefully renumbering the vertices in G, thanks to the two divisibility assumptions in the theorem. Note that renumbering vertices is merely premultiplying and postmultiplying the Laplacian L by permutation matrices P and P^T, a unitary similarity transformation; hence no change in the eigenvalues of L. Further observe that this obtained circulant graph G_c also satisfies the condition of Theorem 4.5 (as $n_1 = n_2 = \ldots = n_p$), hence it is an LEGM. It is easy to see that the complement of circulant graph G_c^C is also a circulant graph and G_c^C is an Algebraic Connectivity Global Maximizer.
Also, it is clear from Remark 1.3 that the DFT Problem 1.2 is linked with the circulant graph ACGM Problem. This completes the proof. □

Of course, the condition specified in the theorem is only a sufficient condition for a circulant matrix to be an LEGM. Example 7.1 is included in the next section: it is about when the sufficient condition of Theorem 4.1 is met with equality and the resulting union of complete components is an LEGM. Further, this case also admits a circulant matrix, which also is an LEGM, though it is not a union of complete components. Example 7.2 shows that for $a = 4$ and $n = 16$, the complete bipartite graph $K_{a,n-a}$ is not be an ACGM graph and a circulant matrix with same number vertices and edges provides a higher $\lambda_F(G_c) = 4.15$ than the $K_{4,12}$ with $\lambda_F(K_{4,12}) = 4$ (for $n = 16$ and $m = 72$).



(a) *Circulant graph with its Laplacian's largest eigenvalue minimized globally: $\lambda_{\max}(G_c) = 6$. This graph also solves the DFT Problem 1.2; i.e. the adjacency matrix A's first row minimizes its largest frequency component amongst all adjacency matrices with first row having the same number of ones as that of A (refer to Example 7.1).*



(b) *Circulant graph with Laplacian's largest eigenvalue $\lambda_{\max}(G_c) = 6.88$: maximized amongst all circulant graphs with same node/edge cardinalities, including graph of Figure 9a. This graph does not solve the DFT Problem 1.2.*

Figure 9: Circulant graphs related to Problem 1.2

7. Examples

In this section, we consider some examples. Table I contains many typical values of n and m (the total number of vertices and edges) and also lists which are LEGM (in addition to being LELM). Some more examples are elaborated in this section.

Table I: $\lambda_{\max}$ for complete components K_i having n vertices and m edges

n	m	$\bigcup K_i$	$\lambda_{\max}$	LEGM/LELM
10	16	5, 4	5	LEGM
9	10	4, 3, 2	4	LEGM
9	12	4, 4	4	LEGM
10	20	5, 5	5	LEGM
15	34	7, 5, 3	7	LELM
20	22	4, 4, 4, 2, 2, 2, 2	4	LEGM
20	50	8, 6, 4, 2	8	LELM
25	66	8, 8, 4, 3, 2	8	LELM
25	132	12, 12	12	LEGM
30	235	22, 2, 2, 2, 2	22	LELM
32	136	10, 10, 10, 2	10	LEGM

The following example shows LEGM graph satisfying the sufficient condition inequality (1) with equality. It also show non-uniqueness of LEGM with a circulant graph.

Example 7.1. *Suppose the number of vertices, $n = 9$, and the number of edges, $m = 18$. The graph $K_6 \bigcup K_3$ with $\lambda_{\max}(K_6 \bigcup K_3) = 6$, is an LEGM with the nonstrict inequality (1) satisfied with equality.*
Further, the circulant graph G_c with degree 4, represented by the circulant adjacency matrix having its first row as $[0\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0]$ also has $\lambda_{\max}(G_c) = 6$. Thus $K_6 \bigcup K_3$ is not the unique LEGM, but the circulant graph G_c has the same $\lambda_{\max}$ value and is an LEGM too.

The following example shows a circulant graph minimizing $\lambda_{\max}$ better than the complement of bipartite graph or union of two complete components (LELM).

Example 7.2. *Consider the case when the vertex/edge counts are $n = 16$, $m = 72$. In this case, the sufficient condition of Theorem 4.1 is violated quite severely and $G = K_{12} \bigcup K_4$ is indeed not an LEGM, but just an LELM, with $\lambda_{\max}(G) = 12$.*
Further, it may be verified that the circulant graph G_c obtained by the circulant adjacency matrix having its first row as $[0\ 0\ 1\ 0\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0\ 1\ 0]$ has $\lambda_{\max}(G_c) = 11.85$. This is a case where a circulant graph provides a lower $\lambda_{\max}$ than the LELM graph $K_{12} \bigcup K_4$. So, on complementing G_c, we get the circulant graph G_c^C which provides higher algebraic connectivity, $\lambda_F(G_c^C) = 4.15$ than the complete bipartite graph $K_{4,12}$ which has $\lambda_F(K_{4,12}) = 4$.

8. Concluding remarks

In this paper we considered the problem of designing/placing edges to maximize the algebraic connectivity of a graph: this has applications in various areas in control, sensor network design, input/output network design: especially in the study of structured systems: see [13], for example. More precisely, in the context of undirected, unweighted graphs with a given number of vertices and edges, and when studying their Laplacian matrices' eigenvalues, we showed how the graphs comprised of two or more complete components locally minimize the Laplacian's largest eigenvalue (LELM graphs): Theorem 3.6. The proof involved a meticulous case-by-case analysis of various possibilities: see Figures 1 to 8, and Lemmas 3.3-3.5. Further, if the components sizes are either 'quite homogeneous' or some of them are relatively 'negligibly small' (as elaborated in Remark 4.7, which interpreted Equations (1) and (4) of Theorems 4.1 and 4.5), then this graph is not just local, but also a global minimizer of the largest eigenvalue for that many vertices and edges. This thus extends existing results (Propositions 2.1, 2.2 and 2.3) in different and appropriate ways. We also proposed an algorithm to construct such a locally/globally optimum graph (Algorithm 1). Assuming suitable restrictions on size, and also generalizing along a digression, we proved Theorem 4.3 & Corollary 4.4: special cases of [10, Conjecture 1.5], with the conjecture remaining open for the general case.

We also related our results to the well-studied class of graphs called circulant graphs: the significance being that due to a symmetric and uniform distribution of edges across nodes within such graphs, circulant graphs appear like the opposite of graphs that have a 'star node' (see Proposition 2.2, Statement b)), and hence are potential candidates when minimizing the largest eigenvalue of the Laplacian. The link between circulant graphs/matrices and the Discrete Fourier Transform is well-known, and the central problem considered in this paper thus translates to minimization of the maximum magnitude across all nonzero frequencies in a periodic discrete time signal (see Problem 1.2, Remark 1.3, and Theorem 6.1) comprising of only 0 and 1's.

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