Imaginary axis eigenvalues of Hamiltonian matrix: controllability, defectiveness and the ϵ-characteristic

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ABSTRACT

The eigenstructure of imaginary axis eigenvalues of a Hamiltonian matrix is of importance in many fields of control systems, for example, in stability analysis of linear Hamiltonian systems, computation of the solutions of an algebraic Riccati equation (ARE), existence of a Lyapunov function for LTI systems, etc. The dynamical system consisting of all stationary trajectories for an optimal control problem – often called the Hamiltonian system – is known to admit, in its state space dynamical equation, a Hamiltonian matrix for the system matrix. In each of these cases, defectiveness of imaginary axis eigenvalues of a Hamiltonian matrix turns out to be of crucial importance. For example, defectiveness causes unboundedness of the oscillatory stationary trajectories in the Hamiltonian system. A characterization of the ARE solutions in terms of Lagrangian invariant subspaces of the Hamiltonian matrix when the imaginary axis eigenvalues are defective has been addressed in the literature for the *controllable* case. This paper focuses on the general case of uncontrollable systems; we formulate conditions under which the imaginary axis eigenvalues of the Hamiltonian matrix are non-defective: this is central for solutions corresponding to imaginary axis eigenvalues to not become unbounded. We provide conditions on the so-called ϵ-characteristic of the non-defective imaginary axis eigenvalue: this helps in the characterization of Lagrangian invariant subspaces. Also, as an extreme case of non-defectiveness, we formulate conditions under which a passivity based Hamiltonian matrix is *normal*, i.e. the matrix commutes with its transpose and we link normality of such Hamiltonian matrices with all-pass behavior.

In summary, this paper formulates results that link defectiveness of imaginary axis eigenvalues of Hamiltonian matrix to solvabilities of Lyapunov and Algebraic Riccati equations, controllability/observability, ϵ-characteristic and sign-controllability. We consider examples in the area of bounded-real transfer functions and RLC circuits, both controllable and uncontrollable, to study applicability of the results of this paper.

KEYWORDS

Controllability, Observability, Defective eigenvalues, Normal matrices, All-pass, Diagonalizability

1. Introduction

A Hamiltonian matrix $H \in \mathbb{R}^{2n\times 2n}$ is a matrix such that $J_{2n}H$ is symmetric where $J_{2n} \in \mathbb{R}^{2n\times 2n}$ is the skew-symmetric matrix:

$$J_{2n} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

and I_n is the identity matrix of size n (see [22]). Such matrices have widespread applications in the field of control and circuit analysis. In the context of RLC circuits and/or passive systems, Hamiltonian matrices are relevant, for example, in view of the fact that a state space system defined by the equations: $\dot{x} = Ax + Bu$, $y = Cx + Du$, $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times p}$, $D \in \mathbb{R}^{p\times p}$ is passive if and only if the corresponding Hamiltonian matrix (defined in equation (1), and Table 1) contains an n-dimensional invariant subspace of the form $\left[\begin{smallmatrix} I \\ K \end{smallmatrix}\right]$ with $K \in \mathbb{R}^{n\times n}$ symmetric and positive definite. Hamiltonian matrices also find applications in solving the algebraic Riccati equation (ARE) and depending on the problem at hand,

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Supply rate type	Supply rate	M	S	Q
Passivity	$u^T y$	$A - B(D + D^T)^{-1}C$	$-B(D + D^T)^{-1}B^T$	$C^T(D + D^T)^{-1}C$
Bounded-real	$u^T u - y^T y$	$A + B(I - D^T D)^{-1}D^T C$	$-B(I - D^T D)^{-1}B^T$	$C^T(I - DD^T)^{-1}C$
LQR	$x^T Qx + u^T Ru$	A	$-BB^T$	$-Q$

Table 1.: Different forms of Hamiltonian matrix: see equation (1)

different forms of the ARE are found in the literature. For example, the *continuous symmetric algebraic Riccati equation* (see [21, Chapter 7] for more details) arises in classical problems of system theory such as the linear quadratic regulator problem [23], the Kalman filter [14], optimal H_2/H_∞ control [32], differential games [25], passive network synthesis procedures, spectral factorization [1] and, more recently, [3] in the context of sign-indefinite quadratic/constant terms in the ARE. A standard method to solve the continuous symmetric ARE is to find suitable invariant subspaces of the Hamiltonian matrix.

The Hamiltonian matrix arising in many control areas has the form:

$$H := \begin{bmatrix} M & -S \\ -Q & -M^T \end{bmatrix}. \tag{1}$$

In most applications arising in dynamical systems, the matrices M, S, Q of the Hamiltonian matrix H in equation (1) depend on the system dynamics and an appropriate performance index that needs to be optimized. Table 1 shows the different forms of H that typically find applications in dynamical systems with a MIMO input-state-output (i/s/o) representation of the form $\dot{x} = Ax + Bu$ and $y = Cx + Du$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. Note that in Table 1, in case of passivity supply rate, it is assumed that the matrix $(D + D^T)$ is invertible and in case of the bounded real case, it is assumed that the matrix $(I - DD^T)$ is invertible.

1.1. Background, related work and motivation

Most of the classical results to find solutions of an ARE using the Hamiltonian matrix H assume that none of the eigenvalues of H are on the imaginary axis. Imaginary (i.e. imaginary axis) eigenvalues of Hamiltonian matrices pose numerical difficulties in the computation of Riccati equation solutions [11]. Such eigenvalues correspond to the presence of sustained oscillatory trajectories in the system and their defectiveness (see Definition 2.2) imply that the oscillations become unbounded. Only a few papers in the literature focus on studying the eigenstructure corresponding to the imaginary eigenvalues of the Hamiltonian matrix. In [10] and [11], control-theoretic conditions are formulated on the system under which the imaginary eigenvalues of the Hamiltonian matrices are defective, or more specifically, they formulate conditions when the size of all the Jordan blocks corresponding to the imaginary eigenvalues are of even size. For the case when imaginary eigenvalues of the Hamiltonian matrix have even sized Jordan blocks (i.e. are defective), the characterization of Lagrangian H-invariant subspaces (see Definition 2.7) has been pursued in [10]. The eigenstructure of imaginary eigenvalues of the Hamiltonian matrix has been discussed recently in [20, Theorem 2.1], where the dynamic behavior of the solutions of a Hermitian Riccati differential equation is characterized using the eigenstructure of the corresponding Hamiltonian matrix. Also in [19, Theorem 4.2,4.3 & 4.7], the asymptotic behavior of the structure preserving flows associated with the structure-preserving doubling algorithm (SDA) is characterized using the eigenstructure of the imaginary eigenvalues of a particular Hamiltonian matrix. In [4, Theorem 3] and [16, Theorem 5.3], the effect of perturbations on the eigenstructure of the imaginary eigenvalues of the Hamiltonian matrix has been formulated. As mentioned earlier, eigenspaces of Hamiltonian matrices are used for the computation of solutions to the algebraic Riccati equation. Usually, it is assumed that the quadratic terms in the Riccati equations are sign-semidefinite. Recently in [4, Theorem 2], under the assumption that imaginary eigenvalues in the corresponding Hamiltonian matrix are simple, sufficient conditions for existence of solutions of algebraic Riccati equations without the assumption of sign-semidefiniteness on the quadratic terms of the Riccati equation have been formulated. Central to many results in the context of Riccati equation solvability and Hamiltonian matrix methods is the question whether the Hamiltonian matrix has imaginary eigenvalues, and if so, whether these are simple, semi-simple or defective.

As mentioned above, imaginary eigenvalues of the Hamiltonian matrix indicate oscillatory phenomena of the optimal trajectories, and defectiveness of these eigenvalues, in some cases (see Section 4 for some counter examples), implies that the oscillations are not sustained and the trajectories become unbounded. This makes the system unstable. One application of this fact can be seen in the study of the stability of Hamiltonian systems. Non-defectiveness of imaginary eigenvalues of Hamiltonian matrices is important for determining the stability of linear autonomous Hamiltonian systems (see [24]). We expand on this more in Section 1.2. Moreover, the characterization of Lagrangian[1] H-invariant subsapces in Hamiltonian requires knowledge of whether the imaginary eigenvalue is defective or non-defective, see [10] for more details. Thus in this paper, we pursue the question: when do Hamiltonian matrices admit non-defective eigenvalues? We formulate conditions that result in non-defective eigenvalues. We also formulate a necessary condition on the ϵ-characteristic (see Definition 2.9) for the imaginary eigenvalues of the Hamiltonian matrix: this is useful in the characterization of the Lagrangian H-invariant subspaces, see [10] for more details.

The characterization of oscillatory trajectories, as mentioned previously, crucially depends on the algebraic and geometric multiplicity of the imaginary eigenvalues of H. Hence, later in Section 4, we consider four examples that reveal a common underlying structure due to which, in spite of the imaginary eigenvalues being repeated, the set of stationary trajectories do not become unbounded; this is thanks to either non-defectiveness of the eigenvalues, or the Lagrangian subspace involving just the eigenvector corresponding to these defective eigenvalues, and *not* involving the corresponding *generalized* eigenvector, and this is related to controllability properties of the underlying system.

The different areas that have a link with defectiveness of imaginary eigenvalues are shown in Figure 1: this paper focuses on formulating the link between defectiveness of the imaginary eigenvalues of H and various notions of system theory like periodic-lossless trajectories (see discussion within Examples 4.3 and 4.4), uncontrollability (see Theorem 3.1) and ϵ-characteristic (see Theorem 3.3). The following remark summarizes how despite eigenvalues of the submatrices comprising a Hamiltonian matrix H always being non-defective, the Hamiltonian structure can still admit defective eigenvalues.

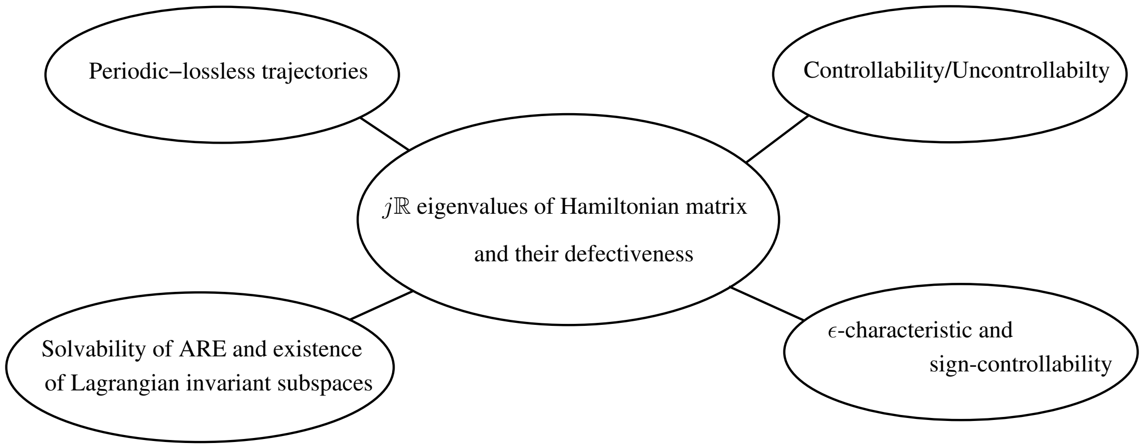

Figure 1.: Inter-relations of various notions with $j\mathbb{R}$ eigenvalues of Hamiltonian matrix and their defectiveness

Remark 1.1. Notice that eigenvalues of both symmetric and skew-symmetric matrices, when repeated, *cannot* be defective. Similarly, repeated eigenvalues of further examples of subsets of the set of normal[2] matrices, like circulant and unitary matrices, cannot be defective. On the other hand, loosely speaking,

[1]Corresponding to an appropriate Lagrangian subspace, one obtains stabilizing or semi-stabilizing ARE solution, which correspondingly satisfies positive (semi)-definiteness properties. This paper focuses on defectiveness of $j\mathbb{R}$ eigenvalues of the Hamiltonian matrix, and hence the possibility of the corresponding ARE solution not being semi-stabilizing.

[2]A matrix $A \in \mathbb{R}^{n\times n}$ is called normal if $AA^T = A^T A$.

'generically'[3] repeated eigenvalues of matrices are, in fact, defective. While a Hamiltonian matrix H is defined as the product of J and a symmetric matrix, and while H is also comprised of blocks of symmetric matrices, it is interesting as to how controllability, observability, ϵ-characteristic play a key role in causing defectiveness of imaginary eigenvalues: this is the focus of this paper.

1.2. Linear Hamiltonian systems and integrals of motion

In this section, we expand on the stability of linear autonomous Hamiltonian systems and connect it to the role of non-defectiveness of imaginary eigenvalues and we also provide a new proof of a well-known result in this context.

We first review essential concepts used in the analysis of linear Hamiltonian systems (see [2], [5], [24] for more on this topic). A linear constant Hamiltonian system is described by the following equation:

$$\frac{d}{dt}x = Hx, \text{ where } H \in \mathbb{R}^{2n\times 2n} \text{ is a Hamiltonian matrix.} \tag{2}$$

For a Hamiltonian system as given in equation (2), a Hamiltonian function is defined as $Q(x) := x^T Qx$, where $Q := JH$ and $J := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Define the linear map $\mathcal{L}_A : \text{Sym}(2n) \to \text{Sym}(2n)$ by:

$$\mathcal{L}_A(P) := H^T P + PH.$$

Note that the Lie derivative of the quadratic function $x^T Px$ along the trajectories of (2) is equal to: $\frac{d}{dt}x^T Px = x^T \mathcal{L}_A(P)x$. Now if $P \in \texttt{ker}(\mathcal{L}_A)$, then $x^T Px$ becomes an integral of motion (which are conserved quantities and related to lossless trajectories) for the system (2). For more details on such integrals of motion refer to [5]. This relation between integral-of-motion and Lyapunov equation $H^T P + PH = 0$ results in a natural link between semi-definiteness condition of an integral-of-motion and the eigenstructure of the imaginary eigenvalues of the Hamiltonian matrix. Non-defectiveness of imaginary eigenvalues of the Hamiltonian matrix and semi-definiteness of integral-of-motions is important in the stability analysis of linear Hamiltonian systems. The following theorem is a reformulation of the famous Krein-Gelfand theorem [24] for stability of Hamiltonian systems. This theorem formulates a condition for stability in terms of Hamiltonian functions. Since Hamiltonian functions are also candidates for the integrals of motion, the following theorem is relevant to the focus of this paper: non-defectiveness of $j\mathbb{R}$ eigenvalues of a Hamiltonian matrix.

Proposition 1.2. *(Krein-Gelfand theorem, [24]) Suppose all the eigenvalues of the Hamiltonian matrix $H \in \mathbb{R}^{2n\times 2n}$ are imaginary. Then, for the linear Hamiltonian system given in equation* (2)*, there exists a positive definite matrix $P \in \mathbb{R}^{2n\times 2n}$ satisfying the Lyapunov equation $H^T P + PH = 0$ if and only if no eigenvalue of H is defective.*

We provide a new proof for the above classical result.

Proof. **If:** Suppose a sign-definite matrix P satisfies the Lyapunov equation $H^T P + PH = 0$. Since P is sign-definite (say positive definite), there exists a nonsingular, lower triangular matrix F such that $P = FF^T$. Hence, the Lyapunov equation $H^T P + PH = 0$ can be rewritten as:

$$H^T FF^T + FF^T H = 0. \tag{3}$$

[3] This can be made precise as follows. Amongst matrices with repeated eigenvalues, the set of diagonalizable matrices forms a 'thin set', i.e. repeated eigenvalues of almost all matrices are defective. Alternatively, almost any arbitrarily small perturbation of a diagonalizable matrix within the set of matrices with repeated eigenvalues results in defectiveness of repeated eigenvalues.

Now, pre- and post-multiplying equation (3) with F^{-1} and F^{-T}, respectively, we obtain:

$$F^{-1}H^TF + F^THF^{-T} = 0. \tag{4}$$

Define $\widetilde{H} := F^THF^{-T}$. From equation (4) we therefore have $\widetilde{H} + \widetilde{H}^T = 0$, i.e., $\widetilde{H}$ is skew-symmetric, hence a normal matrix. Since F^T is nonsingular, the Hamiltonian matrix H is similar to a normal matrix $\widetilde{H}$. Hence, the Hamiltonian matrix H must be diagonalizable and thus no eigenvalue of H is defective.
Only if: Since the matrix H is diagonalizable, there exists a nonsingular matrix L such that $L^{-1}HL = \widehat{H}$, where $\widehat{H} = \mathtt{diag}(H_1, H_2, \ldots, H_k)$ and $H_j = \begin{bmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{bmatrix}$, $j = 1, 2, \ldots, k$, and $\alpha_j \in \mathbb{R}$. Defining $\widehat{P} := L^TPL$ we rewrite $H^TP + PH$ as follows

$$\begin{aligned} H^TP + PH &= (L\widehat{H}L^{-1})^TP + P(L\widehat{H}L^{-1}) \\ &= L^{-T}\widehat{H}^TL^TP + PL\widehat{H}L^{-1} \\ &= L^{-T}\left(\widehat{H}^T\widehat{P} + \widehat{P}\widehat{H}\right)L^{-1}. \end{aligned} \tag{5}$$

Since $\widehat{H}$ is a block diagonal matrix in real-Jordan form, it is evident that $\widehat{H}^T\widehat{P} + \widehat{P}\widehat{H} = 0$ if $\widehat{P} = I_n$. Therefore, for $P = L^{-T}\widehat{P}L^{-1} = L^{-T}L^{-1}$, we infer from equation (5) that $H^TP + PH = 0$. □

1.3. Summary of contribution and organization of the paper

We summarize below the contribution of the paper. We use a Kalman-based decomposition of the spaces of state and co-state variables to obtain necessary and sufficient conditions for non-defectiveness of imaginary eigenvalues of the Hamiltonian matrix in Section 3: this result extends controllability based results in the literature to the general uncontrollable case. This section also formulates sufficient conditions for defectiveness based on open-loop poles/zeros of the MIMO transfer matrix. The notion of ϵ-characteristic is also used to obtain conditions on defectiveness of imaginary eigenvalues.

As elaborated in Remark 1.1, normality of a matrix is, loosely speaking, an extreme opposite of defectiveness: a natural question about when are Hamiltonian matrices normal is answered in Theorem 3.5, where it is shown that under Hurwitz assumptions, only the trivially zero transfer function admits a normal Hamiltonian matrix, while for unstable systems, all-pass behavior is linked to normality of the Hamiltonian matrix.

Some of these seemingly unrelated themes arising in systems theory get linked through the common theme: defectiveness of the Hamiltonian matrix; this is summed up in Figure 1. We also apply these results to circuit examples and transfer functions where the corresponding Hamiltonian matrix features imaginary eigenvalues: this is addressed in Section 4.

The organization of the paper is as follows: Section 2 contains notation and preliminaries needed in this paper. Section 3 contains main results regarding defectiveness of imaginary eigenvalues of a Hamiltonian matrix, namely necessary/sufficient conditions and also sufficient conditions regarding defectiveness, and link with classical results about Lyapunov functions and integrals of motion. This section also formulates conditions for a Hamiltonian matrix to be a normal matrix, which is a kind of opposite of defectiveness. Section 4 consists of a few examples: RLC circuits and bounded-real transfer functions, and studies the applicability of the results. Section 5 has a few concluding remarks about the results in the paper.

2. Notation and Preliminaries

In this section we discuss the notation used in the paper and we also review preliminaries needed to prove certain results in this paper.

2.1. Notation

We use $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{N}$ to respectively denote the sets of real, complex and natural numbers. The symbols $\overline{\mathbb{C}}_+$ and $\overline{\mathbb{C}}_-$ are used to respectively denote the sets of complex numbers with non-negative and non-positive real parts. Further, the symbols $\mathbb{C}_+$ and $\mathbb{C}_-$ are used to denote the sets of complex numbers with strictly positive and strictly negative real parts, respectively. The symbol $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ matrices with elements from $\mathbb{R}$. Symbol I_n is used for $n \times n$ identity matrix. The symbol 0_k is used to denote a $k \times 1$ zero column vector. We use the symbol $\sigma(A)$ to denote the set of eigenvalues of $A \in \mathbb{R}^{n\times n}$, where an eigenvalue is included in the set as many times as it appears as a root of $\texttt{det}(sI_n - A)$. The symbols $\texttt{ker}(A)$ and $\texttt{img}(A)$ denote the kernel and image of the matrix A respectively. Further, the symbol $\texttt{dim}\,\texttt{ker}(A)$ represents the dimension of the kernel of the matrix A. The symbol $\mathcal{R}_1 \oplus \mathcal{R}_2$ represents the direct sum of the subspaces $\mathcal{R}_1$ and $\mathcal{R}_2$. The symbol $\mathrm{Sym}(n)$ denotes the set of symmetric matrices in $\mathbb{R}^{n\times n}$. The symbol F^{-T} is used to denote the transpose of the inverse of a matrix F, i.e., $(F^{-1})^T$. A block diagonal matrix G is represented as $\texttt{diag}(G_1,\ldots,G_{\texttt{m}})$, where each of $G_1,\ldots,G_{\texttt{m}}$ are square matrices of possibly different sizes. The symbol $\texttt{col}(B_1,B_2)$ denotes $\left[\begin{smallmatrix} B_1 \\ B_2 \end{smallmatrix}\right]$.

2.2. Dissipativity and the Hamiltonian matrix

The notion of dissipativity plays a crucial role in this paper and we review it next [4].

Definition 2.1. *Consider a system $\mathfrak{B}$ with an i/s/o representation $\dot{x} = Ax + Bu$ and $y = Cx + Du$, where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$. Let $\Sigma = \Sigma^T \in \mathbb{R}^{(m+p)\times(m+p)}$ and (A,B) is controllable and (C,A) is observable. The system $\mathfrak{B}$ is called* dissipative *with respect to Σ if there exists $K = K^T \in \mathbb{R}^{n\times n}$ such that*

$$\frac{d}{dt}\left(x^T K x\right) \leqslant \begin{bmatrix} u \\ y \end{bmatrix}^T \Sigma \begin{bmatrix} u \\ y \end{bmatrix} \text{ for all } (u,y) \in \mathfrak{B}. \tag{6}$$

and $x^T K x$ is called a storage function for the Σ-dissipative system.

In inequality (6), Σ is called the *supply rate* of the system. In particular, the supply rate $\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$ is called the *passivity supply rate* and the supply rate $\begin{bmatrix} I_m & 0 \\ 0 & -I_p \end{bmatrix}$ is called the *bounded-real supply rate*. Most of the results in this paper are for systems that are dissipative with respect to the passivity supply rate. Throughout this paper, we assume a certain strictness of dissipativity at 'frequency equal to ∞', namely at very high frequencies (see [6]); this assumption results in invertibility of $(D + D^T)$ for the passivity supply rate case. In fact, the submatrices in Hamiltonian matrix for the various cases considered in Table 1 are under this strictness at ∞ assumption, and we retain this throughout this paper. In particular, the Hamiltonian matrix corresponding to the passivity supply rate is

$$H = \begin{bmatrix} A - B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -(A - B(D+D^T)^{-1}C)^T \end{bmatrix}. \tag{7}$$

In addition to $x^T K x$, with a slight abuse of nomenclature, we call K to be a storage function of $\mathfrak{B}$, as well. It is known in the literature that for a controllable and dissipative system $\mathfrak{B}$, the set of storage functions admits a maximal and a minimal element, call them $K_{\max}$ and $K_{\min}$ respectively. For a vector $a \in \mathbb{R}^n$, the quantity $a^T(K_{\max} - K_{\min})a \geqslant 0$ and signifies the maximum energy lost when charging a system initially at rest to the state $x(0) = a$ and again discharging the system to rest. The set of vectors such that $a^T(K_{\max} - K_{\min})a = 0$ correspond to trajectories of the system for which the energy required to be supplied to the system equals to that which can be extracted from the system, and are

[4] In this definition, existence of a storage function requires that the system is minimal, i.e. (A,B) is controllable and (C,A) is observable. But as remarked in [13] and [15], the observability condition can be dropped. Also, recently conditions for existence of a storage function even when (A,B) is uncontrollable were formulated for passive systems in [13] and for dissipative systems in [15].

in a sense, *lossless trajectories* and are of utmost importance for this paper. Lossless trajectories are trajectories along which no energy is dissipated. The system of minimal dissipation is described by the linear Hamiltonian system [29]

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} A - B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -(A-B(D+D^T)^{-1}C)^T \end{bmatrix}\begin{bmatrix} x \\ z \end{bmatrix}.$$

2.3. Sign-controllability and c-sets

In order to prove the main results of this paper we crucially use a result in [10] (stated later below as Proposition 2.5). We need the notion of defective eigenvalues, sign-controllability and 'complementary'-set (a 'c-set', for short): see [10], [21] for more on sign-controllability and c-sets. For ease of reference, we review the definitions of these notions next.

Definition 2.2. *An eigenvalue $\lambda \in \mathbb{C}$ of a matrix $A \in \mathbb{R}^{q\times q}$ is called defective if the geometric multiplicity of λ is less than the algebraic multiplicity of λ. The algebraic multiplicity n_a of an eigenvalue $\lambda \in \mathbb{C}$ is defined as the multiplicity of λ as a root of $\det(\lambda I - A)$, and the geometric multiplicity n_g is defined as $q -$ rank $(A - \lambda I)$; thus λ is called defective if $n_g < n_a$.*

Definition 2.3. *A pair (A, B) with $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ is called sign-controllable if for any $\lambda \in \mathbb{C}$ at least one of the two matrices $\begin{bmatrix}\lambda I - A & B\end{bmatrix}$ and $\begin{bmatrix}\lambda I + A & B\end{bmatrix}$ has full rank.*

From Definition 2.3 it is evident that if (A, B) is sign-controllable and if $\pm j\omega \in \sigma(A)$, then eigenvalues $\pm j\omega$ are both controllable.

Definition 2.4. *Consider a matrix $P \in \mathbb{R}^{n\times n}$ such that its set of eigenvalues $\sigma(P)$ satisfies $\sigma(P) = \sigma(-P)$. Consider a subset $\Lambda \subset \sigma(P)$ and define $-\bar{\Lambda} := \{\lambda \in \mathbb{C} \mid -\bar{\lambda} \in \Lambda\}$. The set $\Lambda \subset \sigma(P)$ is called <u>c-set</u> if $\Lambda \cap -\bar{\Lambda} = \emptyset$ and $\Lambda \cup -\bar{\Lambda} = \sigma(P)\backslash j\mathbb{R}$.*

From Definition 2.4, it is evident that if Λ is a c-set then $-\bar{\Lambda}$ is also a c-set. We call $-\bar{\Lambda}$ to be the *complementary c-set* of Λ.

In the next proposition, we assume that given a Hamiltonian matrix $H \in \mathbb{R}^{2n\times 2n}$, the imaginary axis eigenvalues of H are denoted by

$$\underbrace{\mu_1,\, \mu_1,\, \ldots,\, \mu_1}_{2\alpha_1 \text{ times}}, \underbrace{\mu_2,\, \mu_2,\, \ldots,\, \mu_2}_{2\alpha_2 \text{ times}}, \ldots, \underbrace{\mu_k,\, \mu_k,\, \ldots,\, \mu_k}_{2\alpha_k \text{ times}} \qquad \text{with} \quad \sum_{i=1}^{k} \alpha_i =: w.$$

On the other hand, the eigenvalues of H with non-zero real parts are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_p$. Note that we are considering the general case, where the eigenvalues λ_i might not be distinct. Further, since H has $2n$ eigenvalues, we must have $p + 2w = 2n$. Without loss of generality, we assume that the first $\frac{p}{2} = n - w$ eigenvalues, i.e., $\{\lambda_1, \lambda_2, \ldots, \lambda_{n-w}\} \subset \mathbb{C}_-$ and $\{\lambda_{n+1-w}, \lambda_{n+2-w}, \ldots, \lambda_{2n-2w}\} \subset \mathbb{C}_+$.

Proposition 2.5. *[10, Theorem 3.3] Consider the ARE:*

$$A^TK + KA + Q - KSK = 0. \tag{8}$$

with the corresponding Hamiltonian matrix defined as $H := \begin{bmatrix} A & -S \\ -Q & -A^T \end{bmatrix}$, where $A, Q \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $S = \pm BB^T$ and (A, B) is sign-controllable. Let $\sigma(H) \cap j\mathbb{R} = \{\mu_1, \mu_2, \ldots, \mu_k\}$ where $\mu_i \in \mathbb{C}$ has multiplicity $2\alpha_i$ and $\sum_{i=1}^{k} \alpha_i =: w$. Define $\mathcal{R}_\mu := \texttt{ker}\,(\mu I_{2n} - H)^{2n}$. Then the following are equivalent:

(i) The ARE in equation (8) *has an unmixed Hermitian*[5] *solution.*

(ii) The ARE in equation (8) *has a Hermitian solution.*

(iii) The imaginary eigenvalues of H have even partial multiplicities.

(iv) There exists a c-set $\Lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_{n-w}\} \subset \sigma(H)$ and a w-dimensional H-invariant subspace[6] *$\mathcal{I} \subset \bigoplus_{i=1}^{k} \mathcal{R}_{\mu_i}$ such that with*

$$\mathtt{img}\begin{bmatrix} I_n \\ K \end{bmatrix} = \mathcal{I} \oplus \mathcal{R}_{\lambda_1} \oplus \mathcal{R}_{\lambda_2} \oplus \cdots \oplus \mathcal{R}_{\lambda_{n-w}} \tag{9}$$

K is the unique unmixed solution of the ARE: $A^T K + KA + Q - KSK = 0$ with $\sigma(A - SK) = \Lambda$.

In Proposition 2.5, the set Λ is a c-set and the solution of the ARE obtained using the root-subspace $\mathcal{R}_\lambda$ of Λ is K. Corresponding to the complementary c-set $-\Lambda$, a graph-subspace of the form in equation (9) exists. Let $\mathtt{img}\begin{bmatrix} I_n \\ \widehat{K} \end{bmatrix}$ be the graph-subspace corresponding to the complementary c-set $-\Lambda$. Then, $\widehat{K}$ is also the unique solution of the ARE with $\sigma(A - S\widehat{K}) = -\bar{\Lambda}$. In this paper, we call $\widehat{K}$ to be a *complementary solution* of the ARE with respect to K. Let $-\bar{\Lambda} = \{\lambda_{n+1-w}, \lambda_{n+2-w}, \ldots, \lambda_{2n-2w}\}$. Then it is clear that

$$\begin{aligned}
&\mathtt{img}\left(\begin{bmatrix} I_n \\ K \end{bmatrix}\right) \cap \mathtt{img}\left(\begin{bmatrix} I_n \\ \widehat{K} \end{bmatrix}\right) \\
&= \left(\mathcal{I} \oplus \mathcal{R}_{\lambda_1} \oplus \cdots \oplus \mathcal{R}_{\lambda_{n-w}}\right) \cap \left(\mathcal{I} \oplus \mathcal{R}_{\lambda_{n+1-w}} \oplus \cdots \oplus \mathcal{R}_{\lambda_{2n-2w}}\right) \\
&= \mathcal{I}.
\end{aligned} \tag{10}$$

In equation (10), we have used the fact that, by definition of a c-set, $\left(\oplus_{i=1}^{n-w} \mathcal{R}_{\lambda_i}\right) \cap \left(\oplus_{i=n+1-w}^{2n-2w} \mathcal{R}_{\lambda_i}\right) = 0$.

Since in this paper we explore the relation between eigenvalues of a Hamiltonian matrix on the imaginary axis[7] and their defectiveness, we present a proposition next that would be required to prove one of the main results of this paper (Theorem 3.4).

Proposition 2.6. *[21, Lemma 7.3.3] Let $U, V \in \mathbb{R}^{n\times n}$ be such that V is positive-semidefinite. Define $\mathcal{R}_\lambda(U) := \mathtt{ker}(\lambda I_n - U)^n$ and $\mathcal{C}_{U,V} := \mathtt{img}\left(\begin{bmatrix} V & UV & \cdots & U^{n-1}V \end{bmatrix}\right)$. Let $W := \begin{bmatrix} U & V \\ 0 & -U^T \end{bmatrix}$. Suppose $\mathcal{R}_\lambda(U) \subseteq \mathcal{C}_{U,V}$ for every eigenvalue λ of U on the imaginary axis. Then, the partial multiplicities of each eigenvalue λ of W on the imaginary axis are all even.*

Another notion that is used throughout this paper is the notion of Lagrangian subspaces. We define this next.

Definition 2.7. *A subspace $\mathcal{L} \subset \mathbb{R}^{2n}$ is called a* Lagrangian subspace *if it has dimension n and*

$$x^T J_{2n}\, y = 0 \text{ for all } x, y \in \mathcal{L} \text{ with } J_{2n} := \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

One can verify that for any symmetric matrix $K \in \mathbb{R}^{n\times n}$, the image of $\begin{bmatrix} I_n \\ K \end{bmatrix}$ is a Lagrangian subspace.

[5] A Hermitian solution $K \in \mathbb{R}^{n\times n}$ of an ARE is called *unmixed* if $V := \mathtt{col}(I_n, K) \in \mathbb{R}^{2n\times n}$ is such that $HV = V\Gamma$, where $H \in \mathbb{R}^{2n\times 2n}$ is the corresponding Hamiltonian matrix and, importantly, $\sigma(\Gamma)$ is a c-set of H.

[6] A basis for the subspace $\mathcal{I}$ is given by a suitable selection of the eigenvectors and generalized eigenvectors corresponding to the eigenvalues of H on the imaginary axis. The procedure to select such vectors is given in [10]. Note that the dimension of $\mathcal{I}$ is w and $w = \sum_{i=1}^{k} \alpha_i$.

[7] Note that we no longer make the distinction between the symbols μ and λ being eigenvalues on $j\mathbb{R}$ and $\mathbb{C}\backslash j\mathbb{R}$, respectively. We made the distinction only for Proposition 2.5 to improve readability.

2.4. Canonical form of a Hamiltonian matrix and the ϵ-characteristic

In this section we review the notion of the ϵ-*characteristic* of a Hamiltonian matrix. This is essential for one of the main results of this paper. We first review the definition of the canonical form of Hamiltonian matrices, for which we define the following notation needed for the Jordan Canonical Form (Γ) for a positive integer g and complex number λ:

$$\Gamma_{\mathtt{g}}(\lambda) := \begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \lambda & 1 \\ 0 & 0 & \dots & 0 & \lambda \end{bmatrix} \in \mathbb{C}^{\mathtt{g}\times\mathtt{g}} \text{ and } Z_{\mathtt{g}} := \begin{bmatrix} 0 & 0 & \cdots & 0 & (-1)^{\mathtt{g}-1} \\ 0 & 0 & \cdots & (-1)^{\mathtt{g}-2} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathbb{R}^{\mathtt{g}\times\mathtt{g}},$$

and, correspondingly, $\widetilde{\Gamma}_{2\mathtt{g}}(\lambda) := \mathtt{diag}\left(\Gamma_{\mathtt{g}}(\lambda), \Gamma_{\mathtt{g}}(-\bar{\lambda})\right)$ and $\widetilde{Z}_{2\mathtt{g}} := \begin{bmatrix} 0 & Z_{\mathtt{g}}^T \\ Z_{\mathtt{g}} & 0 \end{bmatrix}$. Next we review a result in [10] that provides a method to construct the canonical form of a Hamiltonian matrix that leads to the notion of ϵ-characteristic.

Proposition 2.8. *[10, Theorem 2.1] Let $H \in \mathbb{R}^{2n\times 2n}$ be a Hamiltonian matrix. Then, there exists a Jordan basis $v_1, v_2, \dots, v_{2n} \in \mathbb{C}^{2n}$ of generalized eigenvectors of H, $\alpha, \beta \in \mathbb{N}\cup\{0\}$ with $0 \leqslant \alpha \leqslant \beta$ and numbers $\epsilon_1, \dots, \epsilon_\alpha \in \{1, -1, j, -j\}$ such that with $V := \begin{bmatrix} v_1 & v_2 & \cdots & v_{2n} \end{bmatrix}$*

$$\begin{aligned} V^{-1}HV &= \mathtt{diag}\left(\Gamma_{\mathtt{g}_1}(\lambda_1),\cdots,\Gamma_{\mathtt{g}_\alpha}(\lambda_\alpha),\widetilde{\Gamma}_{2\mathtt{g}_{\alpha+1}}(\lambda_{\alpha+1}),\cdots,\widetilde{\Gamma}_{2\mathtt{g}_\beta}(\lambda_\beta)\right), \\ V^*(jE)V &= \mathtt{diag}\left(\epsilon_1 Z_{\mathtt{g}_1},\cdots,\epsilon_\alpha Z_{\mathtt{g}_\alpha},\widetilde{Z}_{2\mathtt{g}_{\alpha+1}},\cdots,\widetilde{Z}_{2\mathtt{g}_\beta}\right), \end{aligned} \tag{11}$$

where $E := -\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$, $\lambda_1, \dots, \lambda_\alpha \in j\mathbb{R}$ and $\lambda_{\alpha+1}, \cdots, \lambda_\beta \notin j\mathbb{R}$.

The pair $\left(V^{-1}HV, V^*(jE)V\right)$ in Proposition 2.8 is called a canonical form of the pair (H, jE). This canonical form is unique up to permutation of blocks. The numbers α and β are unique here.

Definition 2.9. *Consider H to be a Hamiltonian matrix that admits purely imaginary eigenvalues. Then the vector $\epsilon(H) := \mathtt{col}(\epsilon_1, \epsilon_2, \dots, \epsilon_\alpha)$, with $\epsilon_1, \epsilon_2, \dots, \epsilon_\alpha$ defined in equation* (11)*, is unique up to permutation and is called the ϵ-characteristic of H.*
Moreover, for $1 \leqslant k \leqslant \alpha$ with α and $\mathtt{g}_k$ as defined in Proposition 2.8

$$\epsilon_k \in \begin{cases} \{1, -1\} & \text{if } \mathtt{g}_k \text{ is odd,} \\ \{j, -j\} & \text{if } \mathtt{g}_k \text{ is even.} \end{cases} \tag{12}$$

In particular for $i \in \{1, 2, \dots, \alpha\}$ and $\lambda_i \in \sigma(H)$, we call $\epsilon_i \in \epsilon(H)$ the ϵ-characteristic of λ_i.

3. Main results

The primary question in the study of non-defective imaginary eigenvalues of a Hamiltonian matrix is: when does a Hamiltonian matrix corresponding to a linear time-invariant system admit non-defective imaginary eigenvalues? It is known in the literature that for controllable systems, the imaginary eigenvalues of the corresponding Hamiltonian matrix are defective [10]. Hence, in the sequel, we deal with the more interesting case of uncontrollable systems. Section 3.1 therefore pertains to the relation between the imaginary eigenvalues of an uncontrollable system and that of its corresponding Hamiltonian matrix. Once we establish a condition on the existence of non-defective imaginary eigenvalues of Hamiltonian matrix, we present a result on the ϵ-characteristic of non-defective imaginary eigenvalues of a Hamiltonian matrix. Followed by this result, in Section 3.3 we present results on the defectiveness of imaginary eigenvalues of passivity based Hamiltonian matrices. In particular, we present a condition when the passivity based Hamiltonian matrix becomes, in fact, a normal matrix. This is relevant because normal matrices are diagonalizable, and hence an extreme case of non-defectiveness of a Hamiltonian matrix is when such a matrix is normal.

3.1. Existence of non-defective imaginary eigenvalues in a Hamiltonian matrix

In order to formulate the conditions for non-defectiveness of $j\mathbb{R}$ eigenvalues, we write the matrices M, S and Q in equation (1) in a suitable basis. Let $\mathcal{C}(M,S) := \mathtt{img}\begin{bmatrix} S & MS & \cdots & M^{n-1}S \end{bmatrix}$ be the controllable subspace of (M,S) and $\mathcal{C}(M,S)^{\perp}$ represents a complement of $\mathcal{C}(M,S)$, i.e, $\mathbb{R}^n = \mathcal{C}(M,S) \oplus \mathcal{C}(M,S)^{\perp}$. Assume $\mathtt{dim}\,(\mathcal{C}(M,S)) =: \mathtt{k}_1$ and define $\mathtt{k}_2 := n - \mathtt{k}_1$. Then, the matrices M, S and Q rewritten with respect to a basis of $\mathcal{C}(M,S)$ and $\mathcal{C}(M,S)^{\perp}$ are of the following form (see [12, Section 3] for more on this form of a Hamiltonian matrix):

$$M = \begin{bmatrix} M_1 & M_{12} \\ 0 & M_2 \end{bmatrix}, S = \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix}, \text{ and } Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix}, \tag{13}$$

where $M_1 \in \mathbb{R}^{\mathtt{k}_1 \times \mathtt{k}_1}$, $M_2 \in \mathbb{R}^{\mathtt{k}_2 \times \mathtt{k}_2}$, $S_1 = S_1^T \in \mathbb{R}^{\mathtt{k}_1 \times \mathtt{k}_1}$, $Q_1 = Q_1^T \in \mathbb{R}^{\mathtt{k}_1 \times \mathtt{k}_1}$, and $Q_3 = Q_3^T \in \mathbb{R}^{\mathtt{k}_2 \times \mathtt{k}_2}$. With respect to this decomposition, (M_1, S_1) is controllable. The Hamiltonian matrix H correspondingly decomposes to the block-form:

$$H = \begin{bmatrix} M_1 & M_{12} & -S_1 & 0 \\ 0 & M_2 & 0 & 0 \\ -Q_1 & -Q_2 & -M_1^T & 0 \\ -Q_2^T & -Q_3 & -M_{12}^T & -M_2^T \end{bmatrix}. \tag{14}$$

Consider the permutation matrix

$$P := \begin{bmatrix} I_{\mathtt{k}_1} & 0 & 0 & 0 \\ 0 & 0 & I_{\mathtt{k}_2} & 0 \\ 0 & I_{\mathtt{k}_1} & 0 & 0 \\ 0 & 0 & 0 & I_{\mathtt{k}_2} \end{bmatrix}. \tag{15}$$

On pre- and post-multiplying H in equation (14) with P^{-1} and P, we obtain:

$$P^{-1}HP = \left[\begin{array}{cc|c|c} M_1 & -S_1 & M_{12} & 0 \\ -Q_1 & -M_1^T & -Q_2 & 0 \\ \hline 0 & 0 & M_2 & 0 \\ \hline -Q_2^T & -M_{12}^T & -Q_3 & -M_2^T \end{array}\right]. \tag{16}$$

Define the matrix

$$\widetilde{M} := \begin{bmatrix} M_1 & -S_1 \\ -Q_1 & -M_1^T \end{bmatrix}. \tag{17}$$

From the structure of $P^{-1}HP$ in equation (16), it is evident that

$$\sigma(H) = \sigma(\widetilde{M}) \cup \sigma(M_2) \cup \sigma(-M_2), \tag{18}$$

counted appropriately with multiplicities. Next we present the main result of this paper, a necessary and sufficient condition for the existence of non-defective imaginary eigenvalues in a Hamiltonian matrix: this result pinpoints the conditions needed on the uncontrollable part for the non-defectiveness. We use the Kalman-type controllable and uncontrollable decomposition of the Hamiltonian matrix as in equation (14) above.

Theorem 3.1. *Consider the Hamiltonian matrix H as given in equation* (14)*. Suppose the ARE: $M^TK + KM - KSK + Q = 0$ admits a symmetric solution. Consider the Kalman-based controllable/uncontrollable subspace decomposition of M and H defined in equations* (13)-(17) *and the matrices $\widetilde{M}$ and M_2 defined there. Suppose $j\omega$ is an eigenvalue of H. Then, $j\omega$ is non-defective if and only if the following two conditions hold:*

(1) $j\omega \notin \sigma(\widetilde{M})$.
(2) $j\omega \in \sigma(M_2)$ and $j\omega$ is non-defective as an eigenvalue of M_2.

Proof. **If:** Let K be a symmetric solution of the ARE: $M^TK + KM - KSK + Q = 0$. Then, we have

$$\begin{bmatrix} I & 0 \\ -K & I \end{bmatrix}\begin{bmatrix} M & -S \\ -Q & -M^T \end{bmatrix}\begin{bmatrix} I & 0 \\ K & I \end{bmatrix} = \begin{bmatrix} M - SK & -S \\ 0 & -(M^T - SK) \end{bmatrix} =: \widehat{H} \tag{19}$$

Conforming to the partition of M in equation (13), partition K as $K =: \left[\begin{smallmatrix} K_1 & K_2 \\ K_2^T & K_3 \end{smallmatrix}\right] \in \mathbb{R}^{n\times n}$, $K_1 = K_1^T \in \mathbb{R}^{\mathtt{k}_1\times \mathtt{k}_1}$ and $K_3 = K_3^T \in \mathbb{R}^{\mathtt{k}_2\times\mathtt{k}_2}$. Then, $\widehat{H}$ in equation (19) can be rewritten as

$$\widehat{H} = \begin{bmatrix} M_1 - S_1K_1 & M_{12} - S_1K_2 & -S_1 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & -(M_1 - S_1K_1)^T & 0 \\ 0 & 0 & -(M_{12} - S_1K_2)^T & -M_2^T \end{bmatrix}. \tag{20}$$

Since H and $\widehat{H}$ are related by a similarity transform, we have $\sigma(H) = \sigma(\widehat{H})$. Hence, from equation (20), it is evident that $\sigma(H) = \sigma(M_1 - S_1K_1) \cup \sigma\left(-(M_1 - S_1K_1)\right) \cup \sigma(M_2) \cup \sigma(-M_2)$. Also, from equation (18), we have $\sigma(H) = \sigma(\widetilde{M}) \cup \sigma(M_2) \cup \sigma(-M_2)$. Thus,

$$\sigma(M_1 - S_1K_1) \cup \sigma\left(-(M_1 - S_1K_1)\right) = \sigma(\widetilde{M}). \tag{21}$$

Since $j\omega \notin \sigma(\widetilde{M})$, $j\omega$ must be an eigenvalue of M_2 and $-M_2$.

Next, since $j\omega$ is non-defective in M_2, there exists linearly independent vectors $v_1, v_2, \ldots, v_{\mathtt{m}_{j\omega}} \in \mathbb{R}^{\mathtt{k}_2}$ such that $M_2v_i = j\omega v_i$ for all $i \in \{1, 2, \ldots, \mathtt{m}_{j\omega}\}$, with $\mathtt{m}_{j\omega}$ being the algebraic multiplicity of $j\omega$. Using the same line of reasoning, we infer that $-M_2^T$ admits linearly independent eigenvectors $w_1, w_2, \ldots, w_{\mathtt{m}_{j\omega}} \in \mathbb{R}^{\mathtt{k}_2}$ corresponding to eigenvalue $j\omega$. Using these eigenvectors we show next that H admits $2 \times \mathtt{m}_{j\omega}$ linearly independent eigenvectors corresponding to eigenvalue $j\omega$.

First we show that there exists $x_i \in \mathbb{R}^{\mathtt{k}_1}$ such that $\left[\begin{smallmatrix} x_i \\ v_i \\ 0_{\mathtt{k}_1} \\ w_i \end{smallmatrix}\right]$ is an eigenvector of $\widehat{H}$ corresponding to $j\omega$. For that to happen the vector $\left[\begin{smallmatrix} x_i \\ v_i \\ 0_{\mathtt{k}_1} \\ w_i \end{smallmatrix}\right]$ must satisfy:

$$\begin{bmatrix} M_1 - S_1K_1 & M_{12} - S_1K_2 & -S_1 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & -(M_1 - S_1K_1)^T & 0 \\ 0 & 0 & -(M_{12} - S_1K_2)^T & -M_2^T \end{bmatrix}\begin{bmatrix} x_i \\ v_i \\ 0_{\mathtt{k}_1} \\ w_i \end{bmatrix} = j\omega\begin{bmatrix} x_i \\ v_i \\ 0_{\mathtt{k}_1} \\ w_i \end{bmatrix}$$

From the above equation, we have $M_2v_i = j\omega v_i$ and $-M_2^Tw_i = j\omega w_i$. This is trivially true. Further,

$$\begin{aligned} &(M_1 - S_1K_1)x_i + (M_{12} - S_1K_1)v_i = j\omega x_i \\ \Rightarrow &(M_1 - S_1K_1 - j\omega I_{\mathtt{k}_1})x_i + (M_{12} - S_1K_1)v_i = 0 \end{aligned} \tag{22}$$

Since $j\omega \notin \sigma(\widetilde{M}) \Rightarrow j\omega \notin \sigma(M_1 - S_1K_1)$, we must have

$$x_i = (M_1 - S_1K_1 - j\omega I_{\mathtt{k}_1})^{-1}(M_{12} - S_1K_1)v_i \tag{23}$$

For each of the eigenvectors of M_2 corresponding to $j\omega$ we compute x_i using equation (23). Since $v_1, v_2, \ldots, v_{\mathtt{m}_{j\omega}}$ are linearly independent, the following matrix is full-column rank

$$T := \begin{bmatrix} x_1 & x_2 & \cdots & x_{\mathtt{m}_{j\omega}} \\ v_1 & v_2 & \cdots & v_{\mathtt{m}_{j\omega}} \\ 0 & 0 & \cdots & 0 \\ w_1 & w_2 & \cdots & w_{\mathtt{m}_{j\omega}} \end{bmatrix}$$

Thus, the columns of T are $\mathtt{m}_{j\omega}$ eigenvectors of $\widehat{H}$ corresponding to eigenvalue $j\omega$. Since H and $\widehat{H}$ are related by a similarity transform (see equation (19)), the columns of the following matrix gives $\mathtt{m}_{j\omega}$ linearly independent eigenvectors of H corresponding to eigenvalue $j\omega$.

$$P := \begin{bmatrix} I_{\mathtt{k}_1} & 0 & 0 & 0 \\ 0 & I_{\mathtt{k}_2} & 0 & 0 \\ -K_1 & -K_2 & I_{\mathtt{k}_1} & 0 \\ -K_2 & -K_3 & 0 & I_{\mathtt{k}_2} \end{bmatrix} T \tag{24}$$

Clearly, P is a full column rank matrix of rank $\mathtt{m}_{j\omega}$. Now note that

$$\begin{bmatrix} M_1 - S_1K_1 & M_{12} - S_1K_2 & -S_1 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & -(M_1 - S_1K_1)^T & 0 \\ 0 & 0 & -(M_{12} - S_1K_2)^T & -M_2^T \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ w_i \end{bmatrix} = j\omega \begin{bmatrix} 0 \\ 0 \\ 0 \\ w_i \end{bmatrix},$$

for all $i \in \{1, 2, \cdots, \mathtt{m}_{j\omega}\}$. Thus, the columns of the following matrix are the eigenvectors of H corresponding to $j\omega$.

$$\begin{aligned} Q &:= \begin{bmatrix} I_{\mathtt{k}_1} & 0 & 0 & 0 \\ 0 & I_{\mathtt{k}_2} & 0 & 0 \\ -K_1 & -K_2 & I_{\mathtt{k}_1} & 0 \\ -K_2 & -K_3 & 0 & I_{\mathtt{k}_2} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ w_1 & w_2 & \cdots & w_{\mathtt{m}_{j\omega}} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ w_1 & w_2 & \cdots & w_{\mathtt{m}_{j\omega}} \end{bmatrix} \end{aligned} \tag{25}$$

From equations (24) and (25) it is clear that the matrix $\begin{bmatrix} P & Q \end{bmatrix}$ has the following structure $\left[\begin{smallmatrix} T_1 & 0 \\ T_2 & T_3 \end{smallmatrix}\right]$ with

$$T_1 = \begin{bmatrix} x_1 & x_2 & \cdots & x_{\mathtt{m}_{j\omega}} \\ v_1 & v_2 & \cdots & v_{\mathtt{m}_{j\omega}} \end{bmatrix} \text{ and } T_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ w_1 & w_2 & \cdots & w_{\mathtt{m}_{j\omega}} \end{bmatrix}.$$

Since T_1 and T_3 are both full column rank, $\begin{bmatrix} P & Q \end{bmatrix}$ must have rank $2 \times \mathtt{m}_{j\omega}$. Thus, we have $2 \times \mathtt{m}_{j\omega}$ linearly independent eigenvectors of H corresponding to $j\omega$. Therefore, $j\omega$ is non-defective in H.

Only if: Consider the transformed Hamiltonian matrix $\widehat{H}$:

$$\widehat{H} = \begin{bmatrix} M_1 - S_1K_1 & M_{12} - S_1K_2 & -S_1 & 0 \\ 0 & M_2 & 0 & 0 \\ 0 & 0 & -(M_1 - S_1K_1)^T & 0 \\ 0 & 0 & -(M_{12} - S_1K_2)^T & -M_2^T \end{bmatrix}$$

where $K = \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_3 \end{bmatrix}$, $K_1 = K_1^T \in \mathbb{R}^{\mathtt{k}_1 \times \mathtt{k}_1}$, $K_2 \in \mathbb{R}^{\mathtt{k}_1 \times \mathtt{k}_2}$ and $K_3 = K_3^T \in \mathbb{R}^{\mathtt{k}_2 \times \mathtt{k}_2}$ is a solution of the ARE: $M^T K + KM - KSK + Q = 0$. Using the permutation matrix P in equation (15), we define $\widetilde{H} := P^{-1}\widehat{H}P$. Then we have

$$\widetilde{H} = \begin{bmatrix} M_1 - S_1K_1 & -S_1 & M_{12} - S_1K_2 & 0 \\ 0 & -(M_1 - S_1K_1)^T & 0 & 0 \\ 0 & 0 & M_2 & 0 \\ 0 & -(M_{12} - S_1K_2)^T & 0 & -M_2^T \end{bmatrix}.$$

Further note that $\sigma(\widehat{H}) = \sigma(\widetilde{H})$. From the structure of $\widetilde{H}$ we obtain

$$\begin{aligned} &\mathtt{dim\,ker}(j\omega I_n - H) = \mathtt{dim\,ker}(j\omega I_n - \widetilde{H}) = \\ &2\left(\mathtt{dim\,ker}(j\omega I_{\mathtt{k}_1} - M_1 + S_1K_1) + \mathtt{dim\,ker}(j\omega I_{\mathtt{k}_2} - M_2)\right). \end{aligned} \tag{26}$$

We next consider 3 separate cases. Since $j\omega \in \sigma(H)$, exactly one of the following cases hold.

Case 1: $j\omega \in \sigma(M_1 - S_1K_1)$ and $j\omega \notin \sigma(M_2)$.
Case 2: $j\omega \in \sigma(M_1 - S_1K_1)$ and $j\omega \in \sigma(M_2)$.
Case 3: $j\omega \notin \sigma(M_1 - S_1K_1)$ and $j\omega \in \sigma(M_2)$.

(Case 1): Define

$$U := \begin{bmatrix} M_1 - S_1K_1 & -S_1 \\ 0 & -(M_1 - S_1K_1)^T \end{bmatrix}.$$

Let $\mathtt{m}_{j\omega}$ be the algebraic multiplicity $j\omega$ in U. Therefore, if $j\omega$ is non-defective in H, we must have $\mathtt{dim\,ker}(j\omega I_n - H) = 2 \times \mathtt{m}_{j\omega}$. Note that since $(M_1 - S_1K_1, S_1)$ is controllable, by Proposition 2.6 we know that $j\omega$ is defective in U. This means that $\mathtt{dim\,ker}(j\omega I_{\mathtt{k}_1} - (M_1 - S_1K_1)) < \mathtt{m}_{j\omega}$. Therefore, from equation (26) we have

$$\mathtt{dim\,ker}(j\omega I_n - H) = 2 \times \mathtt{dim\,ker}(j\omega I_{\mathtt{k}_1} - (M_1 - S_1K_1)) < 2\mathtt{m}_{j\omega}.$$

This is a contradiction to the fact that $j\omega$ is non-defective in H. Therefore, Case 1 is not possible.
(Case 2): Similar to Case 1, using Proposition 2.6 we know that $j\omega$ is defective in U. Hence,

$$\mathtt{dim\,ker}(j\omega I_n - H) < 2\left(\mathtt{m}_{j\omega} + \mathtt{dim\,ker}(j\omega I_{\mathtt{k}_2} - M_2)\right).$$

This means that $j\omega$ is defective in H which is a contradiction to the fact that $j\omega$ is non-defective in H. Case 2 is not possible.
(Case 3): Since $j\omega \notin \sigma(M_1 - S_1K_1)$ and $j\omega \in \sigma(M_2)$, we must have

$$\mathtt{dim\,ker}(j\omega I_n - H) = 2 \times \mathtt{dim\,ker}(j\omega I_{\mathtt{k}_2} - M_2). \tag{27}$$

Further, since $j\omega$ is non-defective in H, therefore, we infer from equation (27) that $j\omega$ is non-defective in M_2. Further $j\omega \notin \sigma(M_1 - S_1K_1) \Leftrightarrow j\omega \notin \sigma\left(-(M_1 - S_1K_1)^T\right)$. Hence, from equation (21) it is evident that $j\omega \notin \sigma(M_1 - S_1K_1) \Rightarrow j\omega \notin \sigma(\widetilde{M})$. □

Theorem 3.1 can broadly be classified into two different cases. In order to highlight the difference between these two cases, we define these cases next (see [12]).

Definition 3.2. *Consider the Kalman-based controllable/uncontrollable subspace decomposition of M and H defined in equations* (13)-(17) *with M_2 and $\widetilde{M}$ as defined in equation* (13) *and equation* (17)*,* *respectively. Then, the ARE:* $M^TK + KM - KSK + Q = 0$ *is called* regular *if*

$$\sigma(\widetilde{M}) \cap \overline{\sigma(M_2)} = \emptyset, \tag{28}$$

and non-regular *otherwise.*

For the regular case, Conditions 1 and 2 of Theorem 3.1 imply that all the (M, S) controllable imaginary eigenvalues of H are defective. However, for the non-regular case, there can be scenarios where an eigenvalue of H is both (M, S) controllable and (M, S) uncontrollable, as well. Such eigenvalues of H are defective by Theorem 3.1. Only those eigenvalues of H that are (M, S) uncontrollable are the ones that may turn out to be non-defective.

3.2. The ϵ-characteristic and defectiveness

Next we present a result on the ϵ-characteristic of a Hamiltonian matrix that admit non-defective imaginary eigenvalues. Recall the notion of the ϵ-characteristic from Section 2.4. The following result formulates conditions on the imaginary eigenvalue $j\omega$ under which its ϵ-characteristic $\epsilon_{j\omega}$ does not have elements j and $-j$, but has just 1 and -1.

Theorem 3.3. *Consider the Hamiltonian matrix*

$$H = \begin{bmatrix} M & -S \\ -Q & -M^T \end{bmatrix}$$

with $M, S = S^T, Q = Q^T \in \mathbb{R}^{n\times n}$ *having the structure as in equation* (13). *Assume the Riccati equation* $M^TK + KM - KSK + Q = 0$ *has a symmetric solution. Let* $j\omega$ *be a non-defective imaginary eigenvalue of* H *with algebraic multiplicity* $\mathtt{m}_{j\omega}$ *and let* $\epsilon_{j\omega}$ *be the* ϵ*-characteristic of* $j\omega$*. Then the following are true:*

(1) $j\omega$ *has even algebraic multiplicity.*
(2) $\epsilon_{j\omega} \in \mathbb{R}^{\mathtt{m}_{j\omega}}$ *and elements of* $\epsilon_{j\omega}$ *are from the set* $\{1, -1\}$*.*

In particular, if $\mathtt{m}_{j\omega} = 2$*, then the elements of* $\epsilon_{j\omega}$ *have opposite signs.*

Proof: (1) Note that since $j\omega$ is a non-defective eigenvalue of H, we must have $j\omega \in \sigma(M_2)$ (see Condition 2 of Theorem 3.1). Further, since $j\omega$ is imaginary, we must have $j\omega \in \sigma(-M_2)$. Thus, the algebraic multiplicity of $j\omega$ must be a multiple of 2 and hence even.
(2) Since $j\omega$ is non-defective, the Jordan-blocks corresponding to $j\omega$ in the canonical form of H (see Proposition 2.8) are of dimension 1×1. Therefore, the total number of Jordan-blocks corresponding to $j\omega$ in the canonical form of H is equal to the algebraic multiplicity of $j\omega$, i.e., $\mathtt{m}_{j\omega}$. Hence, the epsilon-characteristic corresponding to $j\omega$ must be a vector of size $\mathtt{m}_{j\omega} \times 1$, i.e., $\epsilon_{j\omega} \in \mathbb{R}^{\mathtt{m}_{j\omega}}$. Further, since the size of the Jordan-blocks corresponding to $j\omega$ are odd, in particular 1, from equation (12) we must have the elements of $\epsilon_{j\omega}$ from the set $\{1, -1\}$.

For the case when $\mathtt{m}_{j\omega} = 2$, we assume that v_1 and v_2 are the eigenvectors of $j\omega$. Using Proposition 2.8, we must have $V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ and from equation (11) we have

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix}^* (jE) \begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 \\ 0 & \epsilon_2 \end{bmatrix}. \tag{29}$$

Since the Riccati equation has a solution, H must admit an invariant Lagrangian subspace. Assume the space is spanned by $\{v_1, v_2\}$. By the property of Lagrangian subspace, for γ_1 and $\gamma_2 \in \mathbb{C}$, we must have

$$(\gamma_1 v_1 + \gamma_2 v_2)^* E(\gamma_1 v_1 + \gamma_2 v_2) = 0,$$

$$\text{which also means } (\gamma_1 v_1 + \gamma_2 v_2)^*(jE)(\gamma_1 v_1 + \gamma_2 v_2) = 0,$$

$$\text{which can be rewritten as } \begin{bmatrix}\gamma_1 & \gamma_2\end{bmatrix} \begin{bmatrix}v_1 & v_2\end{bmatrix}^* (jE) \begin{bmatrix}v_1 & v_2\end{bmatrix} \begin{bmatrix}\gamma_1 \\ \gamma_2\end{bmatrix} = 0. \tag{30}$$

Using equation (29) in equation (30), we have

$$\begin{aligned}&(\gamma_1 v_1 + \gamma_2 v_2)^* E(\gamma_1 v_1 + \gamma_2 v_2) = 0 \\ &\quad \Rightarrow \begin{bmatrix}\gamma_1 & \gamma_2\end{bmatrix} \begin{bmatrix}\epsilon_1 & 0 \\ 0 & \epsilon_2\end{bmatrix} \begin{bmatrix}\gamma_1 \\ \gamma_2\end{bmatrix} = 0 \Rightarrow \gamma_1^2 \epsilon_1 + \gamma_2^2 \epsilon_2 = 0.\end{aligned} \tag{31}$$

Note that if the nonzero values: $\epsilon_1 = \epsilon_2$, then for equation (31) to be true we must have $\gamma_1^2 + \gamma_2^2 = 0 \Rightarrow \gamma_1 = \gamma_2 = 0$. This means we have a trivial Lagrangian subspace, this is not true since the Riccati equation has a solution. Therefore, we must have $\epsilon_1 = -\epsilon_2 = 1$ or $\epsilon_1 = -\epsilon_2 = -1$. Thus, the elements of $\epsilon_{j\omega}$ must have opposite signs. □

3.3. Passivity based Hamiltonian matrices, their normality and all-pass behavior

The conditions for the existence of non-defective imaginary axis eigenvalues of a Hamiltonian matrix obtained in the previous section will be utilized to establish a relation between the poles/zeros of a passive system and its corresponding Hamiltonian matrix in this section.

Theorem 3.4. *Consider a system $\frac{d}{dt}x = Ax + Bu$, $y = Cx + Du$, where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$, $D = D^T \in \mathbb{R}^{m\times m}$ with (A, B) possibly uncontrollable and $(D + D^T) > 0$. Assume the system has no uncontrollable imaginary poles or zeros. Let the system be dissipative with respect to the passivity supply rate. The Hamiltonian matrix H corresponding to the passivity supply rate is as defined in equation* (7)*. Then the following statements are true:*

(1) All the imaginary poles and zeros of $G(s)$ are defective in H.
(2) If $j\omega$ is neither a pole nor zero of $G(s)$ and $j\omega \in \sigma(H)$, then $j\omega$ is defective in H.

Proof: Without loss of generality, we assume that the matrices (A, B, C, D) are in the following form:

$$A = \begin{bmatrix}A_{11} & A_{12} \\ 0 & A_{22}\end{bmatrix}, B = \begin{bmatrix}B_1 \\ 0\end{bmatrix}, C = \begin{bmatrix}C_1 & C_2\end{bmatrix},$$

where $A_{11} \in \mathbb{R}^{\mathbf{k}_1\times \mathbf{k}_1}$, $A_{22} \in \mathbb{R}^{\mathbf{k}_2\times \mathbf{k}_2}$, $B_1, C_1^T \in \mathbb{R}^{\mathbf{k}_1\times m}$. Hence, the Hamiltonian matrix has the form given in equation (14).

$$\begin{aligned}&M_1 = A_{11} - B_1(D + D^T)^{-1}C_1,\ M_{12} = A_{12} - B_1(D + D^T)^{-1}C_2, \\ &M_2 = A_{22}, S_1 = -B_1(D + D^T)^{-1}B_1^T,\ Q_1 = C_1^T(D + D^T)^{-1}C_1, \\ &Q_2 = C_1^T(D + D^T)^{-1}C_2,\ Q_3 = C_2^T(D + D^T)^{-1}C_2.\end{aligned}$$

(1) Let $j\mu$ be an arbitrary pole or zero of $G(s)$. Since the system does not admit any uncontrollable imaginary poles or zeros, we have $j\mu \notin \sigma(A_{22}) = \sigma(M_2)$. Hence, using equation (18) we conclude that $j\mu \in \sigma(\widetilde{M})$, where $\widetilde{M}$ is as defined in (17). Thus, from Theorem 3.1 it is evident that $j\mu$ is a defective eigenvalue of H.
(2) Since $j\omega \in \sigma(H)$, we have three possibilities:

(a) $j\omega \in \sigma(\widetilde{M})$ and $j\omega \in \sigma(M_2)$.

(b) $j\omega \notin \sigma(\widetilde{M})$ and $j\omega \in \sigma(M_2)$
(c) $j\omega \in \sigma(\widetilde{M})$ and $j\omega \notin \sigma(M_2)$.

Since the system does not admit uncontrollable imaginary poles or zeros, $\sigma(A_{22}) \cap j\mathbb{R} = \emptyset \Rightarrow \sigma(M_2) \cap j\mathbb{R} = \emptyset$. Hence, case (a) and (b) are not possible and therefore, $j\omega \in \sigma(\widetilde{M})$. Thus, from Theorem 3.1 it is evident that $j\omega$ is a defective eigenvalue of H. □

Next we provide a necessary and sufficient condition for the Hamiltonian matrix to be normal. In the following theorem we assume that matrix A is normal, i.e. A satisfies $AA^T = A^TA$. Further we also assume $B = C^T$. Such an assumption is valid for *internally symmetric systems,* for example. Such systems are part of a broader class of systems known as relaxation systems (see [31]) in the literature. These systems correspond to physical systems which have only one "type" of energy storage element, e.g. an RC network or an RL network. Also, systems with zeros interlacing poles (ZIP) also admit such realizations [28].

Theorem 3.5. *Consider a system $\dot{x} = Ax + Bu$, $y = Cx + Du$, where $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$ and $D + D^T > 0$. Suppose the Hamiltonian matrix is defined as*

$$H := \begin{bmatrix} A - B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -(A - B(D+D^T)^{-1}C)^T \end{bmatrix}.$$

Assume A is semi-Hurwitz and normal and let $B = C^T$. Then, the passivity based Hamiltonian matrix H is normal if and only if $B = C^T = 0$.

In order to prove Theorem 3.5, we first formulate and prove the following lemma.

Lemma 3.6. *Consider the Algebraic Riccati Equation: $XA + A^TX - 4X^2 = 0$ with the matrix $A \in \mathbb{R}^{n\times n}$. Then, the following statements hold.*

(1) If A is semi-Hurwitz, then $X = 0$ is the maximal symmetric solution.
(2) If A is semi-anti-Hurwitz, then $X = 0$ is the minimal symmetric solution.

Proof: The Hamiltonian matrix corresponding to the ARE: $XA + A^TX - 4X^2 = 0$ is $H := \begin{bmatrix} -A & 4I \\ 0 & A \end{bmatrix}$. Note that $X = 0$ is a symmetric solution of the given ARE. From the Hamiltonian matrix H, it is evident that the graph subspace corresponding to the solution $X = 0$ is given by $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$. Note that $\begin{bmatrix} I_n \\ 0 \end{bmatrix}$ is the eigenspace corresponding to the eigenvalues of $-A$. If A is semi-Hurwitz then $X = 0$ is the maximal symmetric solution of the given ARE (see [21, Theorem 7.5.1]). This proves Statement 1. Similarly, if A is semi-anti-Hurwitz, then $X = 0$ is the minimal symmetric solution (again see [21, Theorem 7.5.1]). □

We prove Theorem 3.5 next using the above auxiliary result.
Proof of Theorem 3.5: **If:** When $B = C^T = 0$, then the Hamiltonian matrix becomes: $H = \begin{bmatrix} A & 0 \\ 0 & -A^T \end{bmatrix}$. Since A is normal, the matrix H is also normal.
Only if: We assume H is normal. Hence $HH^T = H^TH$. Since $D + D^T > 0$, the matrix $(D + D^T)^{-1}$ can be factorized as $(D + D^T)^{-1} = RR^T$, $R \in \mathbb{R}^{m\times m}$ using Cholesky factorization and the Hamiltonian matrix is re-written as:

$$H = \begin{bmatrix} A - BRR^TC & BRR^TB^T \\ -C^TRR^TC & -(A - BRR^TC)^T \end{bmatrix} = \begin{bmatrix} A - \widetilde{B}\widetilde{C} & \widetilde{B}\widetilde{B}^T \\ -\widetilde{C}^T\widetilde{C} & -(A - \widetilde{B}\widetilde{C})^T \end{bmatrix},$$

where $\widetilde{B} = BR$ and $\widetilde{C} = R^TC$. Thus, $B = C^T \Rightarrow \widetilde{B} = \widetilde{C}^T$. From the expansion of HH^T and H^TH in terms of $(A, \widetilde{B}, \widetilde{C})$, it follows that H is normal if and only if the following two equations are satisfied by $(A, \widetilde{B}, \widetilde{C})$:

$$A^T A - AA^T + A\widetilde{C}^T\widetilde{B}^T + \widetilde{B}\widetilde{C}A^T - A^T\widetilde{B}\widetilde{C} - \widetilde{C}^T\widetilde{B}^T A \\ + \widetilde{C}^T(\widetilde{C}\widetilde{C}^T + \widetilde{B}^T\widetilde{B})C - \widetilde{B}^T(\widetilde{C}\widetilde{C}^T + \widetilde{B}^T\widetilde{B})\widetilde{B}^T = 0, \tag{32}$$

$$\widetilde{B}\widetilde{B}^T A + A^T\widetilde{B}\widetilde{B}^T + A\widetilde{C}^T\widetilde{C} + \widetilde{C}^T\widetilde{C}A^T - \\ \widetilde{B}(\widetilde{C}\widetilde{C}^T + \widetilde{B}^T\widetilde{B})\widetilde{C} - \widetilde{C}^T(\widetilde{C}\widetilde{C}^T + \widetilde{B}^T\widetilde{B})\widetilde{B}^T = 0. \tag{33}$$

Substituting $\widetilde{B} = \widetilde{C}^T$ in equation (33), we get:

$$\widetilde{B}\widetilde{B}^T(A + A^T) + (A + A^T)^T\widetilde{B}\widetilde{B}^T - 4\widetilde{B}\widetilde{B}^T\widetilde{B}\widetilde{B}^T = 0. \tag{34}$$

Note that since A is semi-Hurwitz, $(A + A^T)$ is also semi-Hurwitz. On using Lemma 3.6, it is evident from equation (34) that $\widetilde{B}\widetilde{B}^T = 0$ is the maximal symmetric solution of the ARE (34). Hence, any other solution of the ARE, if another exists, must be negative-semidefinite. Since $\widetilde{B}\widetilde{B}^T \geqslant 0$, any other nonzero solution of the ARE (34) cannot be decomposed into the form $\widetilde{B}\widetilde{B}^T$. Therefore, the only solution of the ARE (34) that can be decomposed into the form $\widetilde{B}\widetilde{B}^T$ is 0. Therefore, for H to be normal, we need $\widetilde{B}\widetilde{B}^T = 0$ which implies $B = 0 = C^T$. □

Theorem 3.5 formulates conditions under which the Hamiltonian matrix H is normal. Of course, the necessary and sufficient condition: $B = C^T = 0$ means the transfer function is merely a constant; the assumptions, $B = C^T$, $D + D^T > 0$ and semi-Hurwitzness of A together play an important role in this conclusion. Below is an example of a non-constant $G(s)$ in which H is normal.

Example 3.7. *Consider the system:*

$$\dot{x} = 2x + \sqrt{2}\, u,\ y = \sqrt{2}x + \frac{1}{2}\, u.$$

The Hamiltonian matrix H corresponding to the passivity supply rate is as defined in equation (7)*. Hence, $H = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Clearly, H is normal. Note that the system is not dissipative with respect to the passivity supply rate, and in fact, unstable.*

Using Lemma 3.6, we show in the next theorem that for a SISO system with a single state, the Hamiltonian matrix H considered in Theorem 3.5 is normal if and only if the corresponding system is all-pass (after appropriately scaling the transfer function to have $d = 0.5$).

Theorem 3.8. *Consider a single state, SISO system $\dot{x} = ax + bu$, $y = cx + du$, where $a > 0$, $b = c \neq 0$ and $d = \frac{1}{2}$. Consider the passivity-based Hamiltonian matrix:*

$$H := \begin{bmatrix} a - bc & b^2 \\ -c^2 & -a + bc \end{bmatrix} \tag{35}$$

Then, H is normal if and only if the system is all-pass.

Proof: **If**: The transfer function of the given system is $G(s) := \frac{1}{2} + \frac{cb}{s-a} = \frac{s+2cb-a}{2(s-a)}$. Since $G(s)$ is all-pass and $\lim_{s\to\infty} G(s) = \frac{1}{2}$, $|G(j\omega)| = \frac{1}{2}$ for all $\omega \in \mathbb{R}$. Thus, we have

$$\frac{\sqrt{\omega^2 + (2cb - a)^2}}{2\sqrt{\omega^2 + a^2}} = \frac{1}{2} \text{ which simplifies to } 2cb - a = \pm a \Rightarrow 2b^2 = a \pm a.$$

Since $b = c \neq 0$, $b = \pm\sqrt{a}$. Using these values of b, c and $d = \frac{1}{2}$ in equation (35), we have $H = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$. Thus, $HH^T = H^TH$ and hence H is normal.

Only if: For H to be normal, i.e, $H^TH = HH^T$ the variables a, b and c must satisfy the equation $ab^2 = b^4$. Using Theorem 3.5, it follows that if $a \leqslant 0$, then H is normal if $b = c = 0$ but $b = c \neq 0$. Hence, a must be a positive real number, i.e., $a > 0$. Thus $ab^2 = b^4 \Rightarrow b = \pm\sqrt{a} = c$. Hence, the transfer function of the system becomes $G(s) = c(s-a)^{-1}b + d = \frac{a}{s-a} + \frac{1}{2} = \frac{s+a}{2(s-a)}$. Since $\frac{1}{4} - G(s)G(-s) = 0$, the system is all-pass. □

4. Examples

In this section, we consider four examples and analyze the defectiveness aspects of the imaginary eigenvalues of the corresponding Hamiltonian matrices.

Example 4.1. *Consider the bounded real transfer function* $G_1(s) = \dfrac{0.1}{s^2 + 0.1s + 1}$ *and a corresponding state-space representation with* $A = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0.1 & 0 \end{bmatrix}$ *and* $D = 0$. *The Bode magnitude plot of* $G_1(s)$ *is given in Figure 2. It is clear from the plot that the* H_∞ *norm of* $G_1(s)$ *is equal to* 1, *and this is attained at the frequency* $\omega = 1$. *Also, the eigenvalues of the corresponding Hamiltonian matrix*

$$H_1 = \begin{bmatrix} A + B(\gamma^2 I - D^T D)^{-1} D^T C & B(\gamma^2 I - D^T D)^{-1} B^T \\ C^T(DD^T - \gamma^2 I)^{-1} C & -(A + B(\gamma^2 I - D^T D)^{-1} D^T C)^T \end{bmatrix}$$

at $\gamma = 1$ *are* $\pm j$ *and both* j *and* $-j$ *are defective in* H_1. *We see that for this example, all imaginary*

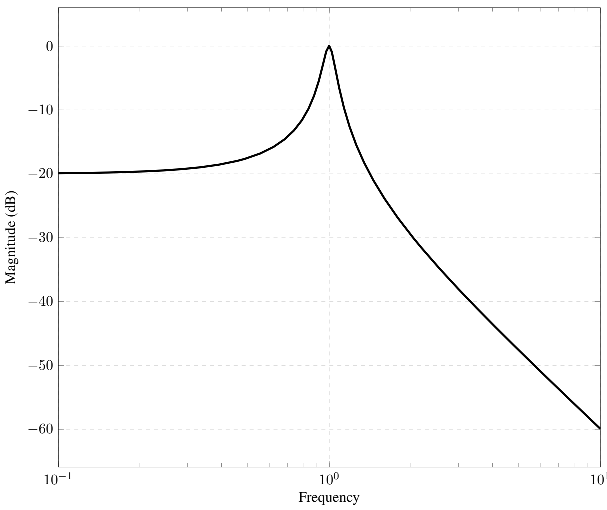

Figure 2.: Bode magnitude plot of $G_1(s)$ in Example 4.1

eigenvalues of the Hamiltonian matrix are defective. This can also be concluded Theorem 3.1 as there are no imaginary eigenvalues in this example that are uncontrollable.

In the context of RLC circuits, and positive real systems in more generality, imaginary eigenvalues of the Hamiltonian matrix correspond to no loss of energy in the system at that frequency. Such trajectories along which there is no loss of energy in the system are referred to as lossless/stationary trajectories of the system.

Example 4.2. *Consider the positive real transfer function* $G_2(s) = \dfrac{1.0931s^2 + 1.0931s + 0.187442}{s^2 + s + 2}$ *and consider its state representation with* $A = \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} -1.998758 & 0 \end{bmatrix}$ *and* $D = 1.0931$. *The corresponding Hamiltonian matrix*

$$H_2 = \begin{bmatrix} A - B(D + D^T)^{-1}C & B(D + D^T)^{-1}B^T \\ -C^T(D + D^T)^{-1}C & -(A - B(D + D^T)^{-1}C)^T \end{bmatrix}$$

contains the eigenvalues $\pm j$ *and both* j *and* $-j$ *are defective.*

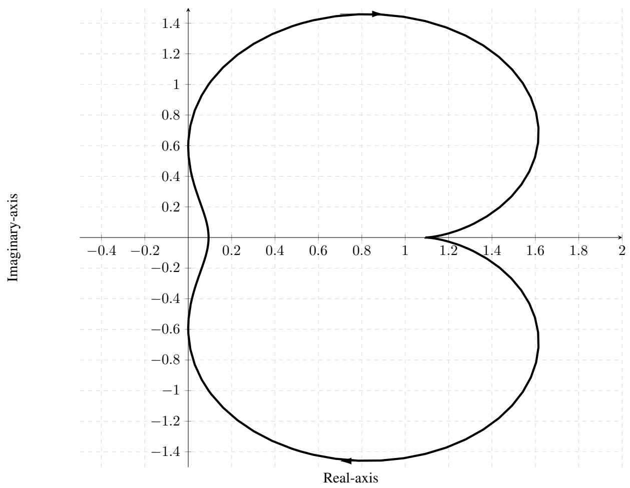

Figure 3.: Nyquist plot of $G_2(s)$ in Example 4.2

When studying the Nyquist plot of $G_2(s)$ *(see Figure 3), the points at which the graph touches the imaginary axis are exactly the frequencies corresponding to the lossless trajectories of the system. In Figure 3, the graph touches the imaginary axis at* $\pm j$ *which are also the eigenvalues of the matrix* H_2. *At the frequency* 1 *rad/s, the positive real system corresponding to the transfer function* $G_2(s)$ *absorbs only reactive power, and active/real power absorbed is zero. We see that for this example as well that all imaginary eigenvalues of the Hamiltonian matrix are defective. This can also be concluded Theorem 3.1 as the are no imaginary eigenvalue in this example that are uncontrollable.*

In the next two examples we consider one controllable and one uncontrollable system each of which admit Hamiltonian matrices with imaginary eigenvalues. Further, we also analyze the lossless trajectories of such systems with modes corresponding to the imaginary eigenvalues of the corresponding Hamiltonian systems.

Example 4.3. *Consider the RLC circuit (as in Figure 4) with a minimal i/s/o representation given by*

$$\frac{d}{dt}x = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 10 \end{bmatrix} u, \qquad y = \begin{bmatrix} 0 & 1 \end{bmatrix} x + u. \tag{36}$$

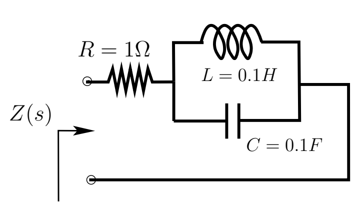

Figure 4.: A controllable RLC circuit (Example 4.3)

The Hamiltonian matrix corresponding to the system described by equation (36) *is*

$$H_3 = \begin{bmatrix} 0 & 10 & 0 & 0 \\ -10 & -5 & 0 & 50 \\ 0 & 0 & 0 & 10 \\ 0 & -0.5 & -10 & 5 \end{bmatrix},$$

and this gives rise to

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = H_3 \begin{bmatrix} x \\ z \end{bmatrix}, \tag{37}$$

with x being the state vector of the system and z the corresponding 'co-state' vector. The lossless trajectories of the circuit in Figure 4 can be characterized by confining the trajectories of the Hamiltonian system in equation (37) *to suitable trajectories. We show this procedure next.*

It is easy to verify that the eigenvalues of H_3 are $\pm 10j$. Further, both the eigenvalues of H_3 are defective. Using the eigenvectors and generalized eigenvectors of H_3 define

$$V := \begin{bmatrix} 1 & 1 & 1 & 1 \\ j & 0.1 + j & -j & 0.1 - j \\ 0.1 & 0.14 & 0.1 & 0.14 \\ 0.1j & 0.01 + 0.14j & -0.1j & 0.01 - 0.14j \end{bmatrix}.$$

The first and the third columns of V above are the eigenvectors of H_3 corresponding to eigenvalues $10j$ and $-10j$, respectively, while the second and the fourth columns of V are the generalized eigenvectors corresponding to $10j$ and $-10j$, respectively. The trajectories of the Hamiltonian system in equation (37) *are:*

$$\begin{bmatrix} x \\ z \end{bmatrix} = V e^{\widetilde{H}_3 t} V^{-1} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, \text{ where } \widetilde{H}_3 := \mathtt{diag}(D_1, D_2) \tag{38}$$

with $D_1 := \begin{bmatrix} 10j & 1 \\ 0 & 10j \end{bmatrix}$ *and* $D_2 := \begin{bmatrix} -10j & 1 \\ 0 & -10j \end{bmatrix}$.

The primary step in finding lossless trajectories of the system in equation (36) *is to choose $z_0 = Kx_0$ in equation* (38)*, where K is a solution of the ARE: $A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0$. In order to construct a solution*[8] *of the ARE we choose a suitable H_3-invariant subspace, e.g., the space spanned by the eigenvectors of H_3 corresponding to eigenvalue $\pm 10j$. Then, we have*

$$\mathtt{img} \begin{bmatrix} 1 & 1 \\ j & -j \\ 0.1 & 0.1 \\ 0.1j & -0.1j \end{bmatrix} = \mathtt{img} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0 \\ 0 & 0.1 \end{bmatrix} =: \mathtt{img} \begin{bmatrix} I_2 \\ K \end{bmatrix}$$

[8]Since an ARE can have multiple solutions, we choose any one solution among the different solutions of the ARE to compute the lossless trajectories. On choosing a different ARE solution we get a different set of lossless trajectories with the modes of such trajectories depending on the choice of the invariant subspace used to compute the solution of the ARE.

Hence, on choosing $z_0 = Kx_0$, we have

$$\begin{bmatrix} x \\ z \end{bmatrix} = Ve^{\widetilde{H}_3 t}V^{-1}\begin{bmatrix} I_2 \\ K \end{bmatrix} x_0 = Ve^{\widetilde{H}_3 t}\begin{bmatrix} 1 & -j \\ 0 & 0 \\ 1 & j \\ 0 & 0 \end{bmatrix} 0.5x_0$$

$$= 0.5V \begin{bmatrix} e^{10jt} & -je^{10jt} \\ 0 & 0 \\ e^{-10jt} & je^{-10jt} \\ 0 & 0 \end{bmatrix} x_0 = \begin{bmatrix} \cos 10t & \sin 10t \\ -\sin 10t & \cos 10t \\ 0.1\cos 10t & 0.1\sin 10t \\ -0.1\sin 10t & 0.1\cos 10t \end{bmatrix} x_0$$

Thus, corresponding to initial condition x_0, the lossless trajectories of the system in equation (36) *are given by:*

$$x(t) = \begin{bmatrix} \cos 10t & \sin 10t \\ -\sin 10t & \cos 10t \end{bmatrix} x_0 \tag{39}$$

From equation (39) *it is clear that although the Hamiltonian system admits defective imaginary eigenvalues, yet the lossless trajectories do not admit any terms involving $t\cos 10t$ or $t\sin 10t$ and thus the lossless trajectories are* bounded. *Again for this example, we see that all imaginary eigenvalues of the Hamiltonian matrix are defective. This can also be concluded from Theorem 3.1 as there are no imaginary eigenvalues that are uncontrollable.*

In the next example we find the lossless trajectories of an uncontrollable system.

Example 4.4. *Consider the system given in Figure 5. The matrices (A, B, C, D) for the given system are:*

$$A = \begin{bmatrix} 0 & 10 & 0 & 0 \\ -10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10 \\ 0 & 0 & -10 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 10 \\ 0 \\ 10 \end{bmatrix}, \ C^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \ D = 1.$$

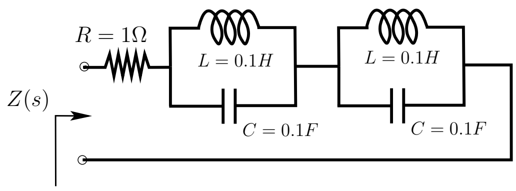

Figure 5.: An uncontrollable RLC circuit (Example 4.4)

The Hamiltonian matrix corresponding to this system is:

$$\frac{d}{dt}\begin{bmatrix} x \\ z \end{bmatrix} = H_4 \begin{bmatrix} x \\ z \end{bmatrix}, \textit{ where} \tag{40}$$

$$H_4 := \begin{bmatrix} 0 & 10 & 0 & 0 & 0 & 0 & 0 & 0 \\ -10 & -5 & 0 & -5 & 0 & 50 & 0 & 50 \\ 0 & 0 & 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & -5 & -10 & -5 & 0 & 50 & 0 & 50 \\ 0 & 0 & 0 & 0 & 0 & 10 & 0 & 0 \\ 0 & -0.5 & 0 & -0.5 & -10 & 5 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\ 0 & -0.5 & 0 & -0.5 & 0 & 5 & -10 & 5 \end{bmatrix}$$

The eigenvalues of matrix H_4 are $\pm 10j$; each of $+10j$ and $-10j$ have an algebraic multiplicity of 4 each, and a geometric multiplicity of 3. Define

$$V_1 := \begin{bmatrix} 1 & 1 & 1 & 1 \\ j & 0.1+j & j & j \\ 1 & 1 & -1+10j & -1+10j \\ j & 0.1+j & -10-j & -10-j \\ 0.1 & 1 & 1+j & 1 \\ 0.1j & 0.01+j & -1+j & j \\ 0.1 & -0.76 & -1 & -1+j \\ 0.1j & 0.01-0.76j & -j & -1-j \end{bmatrix}.$$

and define $V := \begin{bmatrix} V_1 & V_1^* \end{bmatrix}$. The first, third, and fourth columns of V are the eigenvectors of H corresponding to eigenvalue $10j$ and the second column of V is a corresponding generalized eigenvector of H. Similarly, the fifth, seventh, and eighth columns of V are the eigenvectors corresponding to $-10j$ and the sixth column is a corresponding generalized eigenvector. The trajectories of the Hamiltonian system are given by:

$$\begin{bmatrix} x \\ z \end{bmatrix} = V e^{\widetilde{H}_4 t} V^{-1} \begin{bmatrix} x_0 \\ z_0 \end{bmatrix}, \text{ where } \widetilde{H}_4 = \mathtt{diag}(D_1, D_2)$$

with $D_1 = \begin{bmatrix} 10j & 1 & 0 & 0 \\ 0 & 10j & 0 & 0 \\ 0 & 0 & 10j & 0 \\ 0 & 0 & 0 & 10j \end{bmatrix}$ *and* $D_2 = \begin{bmatrix} -10j & 1 & 0 & 0 \\ 0 & -10j & 0 & 0 \\ 0 & 0 & -10j & 0 \\ 0 & 0 & 0 & -10j \end{bmatrix}$*. Similar to the procedure in Example 4.3, in order to compute the lossless trajectories of the system we need to confine the initial conditions of the Hamiltonian system to* $\mathtt{img} \left[\begin{smallmatrix} I \\ K \end{smallmatrix} \right]$*, where K is a solution of the ARE: $A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) = 0$. One may verify that $K = 0.1 \cdot I_4$ is a solution of the ARE. Further, it is easy to verify that* $VS = \begin{bmatrix} I_4 \\ K \end{bmatrix}$*, where*

$$S := \frac{1}{100} \begin{bmatrix} 49-5j & -5-49j & 1+5j & 5-1j \\ 0 & 0 & 0 & 0 \\ -4+1j & 1+4j & 4-1j & -1-4j \\ 5+4j & 4-5j & -5-4j & -4+5j \\ 49+5 & -5+49j & 1-5j & 5+1j \\ 0 & 0 & 0 & 0 \\ -4-1j & 1-4j & 4+1j & -1+4j \\ 5-4j & 4+5j & -5+4j & -4-5j \end{bmatrix}$$

Hence, upon choosing $z_0 = Kx_0$, the trajectories of the system become:

$$\begin{bmatrix} x \\ z \end{bmatrix} = V e^{\widetilde{H} t} V^{-1} \begin{bmatrix} I_4 \\ K \end{bmatrix} x_0 = V e^{\widetilde{H} t} S x_0. \tag{41}$$

Since the second and fifth rows of S are zero, the elements $te^{\pm 10jt}$ in $e^{\widetilde{H}t}$ do not appear in the trajectories computed using equation (41). Hence, the lossless trajectories $x(t)$ obtained from equation (41) are periodic and bounded. For this example, we see that all imaginary eigenvalues of the Hamiltonian matrix are defective even though there exist uncontrollable imaginary eigenvalues. This can also be concluded using Theorem 3.1. Since there are common elements in the set of uncontrollable and controllable imaginary eigenvalues which violates condition (i) in Theorem 3.1, the imaginary eigenvalues of the Hamiltonian matrix in this example are defective.

5. Concluding remarks

We studied imaginary axis eigenvalues of Hamiltonian matrices and their relevance in various control theory problems, in particular focussing on imaginary axis eigenvalues and their defectiveness. Many results in the literature have addressed this issue only for controllable systems: this paper focussed on extending these results for the more general case of uncontrollable systems. Figure 1 shows the various links that are formulated in this paper in the context of defectiveness of the imaginary eigenvalues of a Hamiltonian matrix for the uncontrollable case. We summarize the contribution of this paper.

(i) In Theorem 3.1, using a Kalman-based controllable/uncontrollable subspace decomposition of the concerned matrices, we proved a necessary and sufficient condition for the non-defectiveness of an imaginary eigenvalue of a Hamiltonian matrix: this thus extends the controllability-based results from the literature to the general uncontrollable case.

(ii) In Theorem 3.4, we formulated the following sufficient conditions for defectiveness of $j\mathbb{R}$ eigenvalues of the Hamiltonian matrix: for a system that is dissipative with respect to the passivity supply rate, defectiveness arises either if there were no open-loop poles or zeros of the system on the imaginary axis, or alternatively, if the concerned imaginary eigenvalues of the Hamiltonian matrix are *controllable and observable* open-loop poles/zeros.

(iii) Further, we also showed in Theorem 3.3 that non-defective imaginary eigenvalues of Hamiltonian matrix have their corresponding ϵ-characteristic with elements from the set $\{1, -1\}$ only.

(iv) Utilizing our approach to the extreme case of uncontrollable systems, namely autonomous systems, we have provided a new proof (Proposition 1.2) of the classical Krein-Gelfand theorem [24].

(v) We formulated conditions under which a Hamiltonian matrix is normal (Theorem 3.5). Normality of the Hamiltonian matrix ensures that all the eigenvalues of the matrix are non-defective. Assuming matrix A to be semi-Hurwitz and normal, and $B = C^T$ (i.e. 'collocated sensors/actuators'), we proved that the passivity-based Hamiltonian matrix is normal if and only if $B = C^T = 0$.

(vi) Further, we showed that for a single state SISO system the passivity-based Hamiltonian matrix corresponding to the passivity supply rate is normal if and only if the system is all-pass (Theorem 3.8).

In Section 4, we considered controllable and uncontrollable RLC circuits and also examples of transfer functions from bounded-realness and passivity (Examples 4.1-4.4): these systems exhibited Hamiltonian matrices with imaginary eigenvalues. In the context of passivity-based Hamiltonian matrices that are also normal, Example 3.7 revealed all-pass behavior. For each of these cases, we related example-specific system theoretic properties to the results in this paper.

References

[1] B.D.O. Anderson, S. Vongpanitlerd, *Network Analysis and Synthesis: A Modern Systems Theory Approach*, Dover Publications, New York, 2006.

[2] V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, New York, 1989.

[3] N.E. Barabanov, Kalman-Yakubovich lemma in general finite dimensional case, *International Journal of Robust and Nonlinear Control*, vol. 17, pages 369-386, 2007.

[4] N.E. Barabanov, Necessary and sufficient conditions for solvability of the Riccati inequalities in the general case, *Linear Algebra and its Applications*, vol. 513, pages 33-54, 2017.

[5] S.P. Bhat, D.S. Bernstien, Average-preserving symmetries and energy equipartition in linear Hamiltonian systems, *Mathematics of Control, Signals, and Systems*, vol. 21, no. 2, pages 127–146, 2009.

[6] C. Bhawal, D. Pal and M.N. Belur, Closed form solutions of singular KYP lemma: strongly passive systems, and fast lossless trajectories", *International Journal of Control*, vol. 93, no. 5, pages 1204-1217, 2020.

[7] R.U. Chavan, V.P. Samuel, K. Mallick, M.N. Belur, Optimal charging/discharging and commutativity properties of ARE solutions for RLC circuits, *Proceedings of the 21st International Sympo-*

sium on Mathematical Theory of Networks and Systems (MTNS), Groningen, Netherlands, 2014.
[8] W.A. Coppel, Matrix quadratic equations, *Bulletin of the Australian Mathematical Society*, vol. 10, no. 3, pages 377-401, 1974.
[9] A.S.A. Dilip and H.K. Pillai, Yet another characterization of solutions of the Algebraic Riccati Equation, *Linear Algebra and its Applications*, vol. 481, pages 1-35, 2015.
[10] G. Freiling, On the existence of Hermitian solutions of general algebraic Riccati equations, *Proceedings of the 14th International Symposium on Mathematical Theory of Networks and Systems (MTNS)*, Perpignan, France, 2000.
[11] G. Freiling, V. Mehrmann, H. Xu, Existence, uniqueness, and parametrization of Lagrangian invariant subspaces, *SIAM Journal on Matrix Analysis and Applications*, vol. 23, no. 4, pages 1045-1069, 2002.
[12] I. Gohberg, P. Lancaster, L. Rodman, On hermitian solutions of the symmetric algebraic Riccati equation, *SIAM Journal on Control and Optimization*, vol. 24, no. 6, pages 1323-1334, 1986.
[13] T.H. Hughes, A theory of passive linear systems with no assumptions, *Automatica*, vol. 86, pages 87-97, 2017.
[14] R.E. Kalman, A new approach to linear filtering and prediction problems, *Journal of Basic Engineering*, vol. 82, no. 1, pages 35-45, 1960.
[15] S. Karikalan, M. N. Belur, C. D. Athalye and R. A.. Razak, Uncontrollable dissipative systems: observability and embeddability, *International Journal of Control* , vol. 87, no. 1, pages 101-119, 2014.
[16] K. Kissi, Solvability of Matrix Riccati Inequalities, *arXiv preprint arXiv:1505.04861*, 2015.
[17] A. Kothyari, C. Bhawal, M.N. Belur, D. Pal, Defective Hamiltonian matrix imaginary eigenvalues and losslessness, *Proceedings of the 5th IEEE Indian Control Conference (ICC)*, Delhi, 9-11 Jan 2019.
[18] V. Kučera, A contribution to matrix quadratic equations, *IEEE Transactions on Automatic Control*, vol. 17, no. 3, pages 344-347, 1972.
[19] Y.C. Kuo, H.E. Lin, and S.F. Shieh, Structure-preserving flows of symplectic matrix pairs, *SIAM Journal on Matrix Analysis and Applications*, vol. 37, no. 3, pages 976-1001, 2016.
[20] Y.C. Kuo, H.E. Lin, and S.F. Shieh, Time-asymptotic Dynamics of Hermitian Riccati Differential Equations, *Taiwanese Journal of Mathematics*, 2019.
[21] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*, Oxford University Press, 1995.
[22] W.W. Lin, V. Mehrmann and H. Xu, Canonical forms for Hamiltonian and symplectic matrices and pencils, *Linear Algebra and its Applications*, vol. 302, pages 469-533, 1999.
[23] A.G.J. MacFarlane, An eigenvector solution of the optimal linear regulator problem, *International Journal of Electronics*, vol. 14, no. 6, pages 643-654, 1963.
[24] K. Meyer and G.R. Hall, *Introduction to Hamiltonian Systems and the N-Body Problem*, Springer-Verlag, Berlin, 1991.
[25] I.R. Petersen, Linear quadratic differential games with cheap control, *Systems & Control Letters*, vol. 8, no. 2, pp. 181–188, 1986.
[26] T.J. Richardson and R.H. Kwong, On positive definite solutions to the algebraic Riccati equation, *Systems & Control Letters*, vol. 7, no. 2, pages 99-104, 1986.
[27] M.A. Shayman, Geometry of the Algebraic Riccati Equation, Part I, *SIAM Journal of Control and Optimization*, vol. 21, no. 3, pages 375-394, 1983.
[28] B. Srinivasan and P. Myszkorowski, Model reduction of systems with zeros interlacing the poles, *Systems & Control Letters*, vol. 30, no. 1, pages 19-24, 1997.
[29] H.L. Trentelman, H.B. Minh and P. Rapisarda, Dissipativity preserving model reduction by retention of trajectories of minimal dissipation, *Mathematics of Control, Signals, and Systems*, vol. 21, no. 3, pages 171-201, 2009.
[30] J.C. Willems, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Transactions on Automatic Control*, vol. 16,no. 6,pages 621-634, 1971.
[31] J.C. Willems, Realization of systems with internal passivity and symmetry constraints, *Journal of the Franklin Institute*, vol. 301, no. 6, pages. 605-621, 1976.
[32] K. Zhou, J.C. Doyle, and K. Glover, *Robust and Optimal Control*, vol. 40, Prentice Hall, New Jersey, 1996.