Moment matching using Arnoldi/Lanczos methods

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Outline

- Moments of a transfer function (about $s_0 \in \mathbb{C}$)
- Moments about $s = \infty$
- Relative degree of a transfer function
- Hessenberg form
- Tridiagonal form (for symmetric A in state space realization)
- Arnoldi method and Lanczos method for Hessenberg reduction
- Matching of moments
- Conclusion

Moments

- Recall Taylor series expansion (about $x = a$)

$$f(x) = f(a) + f'(a)(x-a) + f''(a)\frac{(x-a)^2}{2!} + \cdots$$

provided $f(a)$ is bounded (i.e. no 'pole' at $x = a$)

- If $G(s)$ has no pole at $s = 0$, then

$$G(s) = g(0) + g'(0)s + g''(0)\frac{s^2}{2!} + \cdots$$

- At $x = a$, all terms (except first term) are zero.
- $g(0)$ is zeroth moment of $G(s)$ about $s = 0$.
- $g'(0)$ is first moment of $G(s)$ about $s = 0$, etc.
- Consider series in negative powers of s

$$G(s) = G(\infty) + G_1 s^{-1} + G_2 \frac{s^{-2}}{2!} + \cdots$$

- For $s \to \infty$, we get $s^{-1} \to 0$.
- $G(\infty)$ is zeroth moment of $G(s)$ about $s = \infty$.
- G_1 is first moment of $G(s)$ about $s = \infty$, etc.

Moments

- Of course, for such an expansion about $s = a$, $G(s)$ ought not have pole at $s = a$.
- For expansion in series in negative powers in s, $G(s)$ should have no pole at $s = \infty$.
- When $s \to \infty$, we want $G(s)$ should not go unbounded
- $G(s) = \frac{n(s)}{d(s)}$ (with polynomials $n(s)$ and $d(s)$) is proper: numerator degree $\leqslant$ denominator degree
- For MIMO systems: proper $\equiv$ each entry in transfer matrix $G(s)$ is proper
- State space realization (A, B, C, D) exists, and then $G(\infty) = D$: the feedthrough term
- In fact, for proper MIMO $G(s)$,

$$G(s) = D + \frac{CB}{s} + \frac{CAB}{s^2} + \frac{CA^2B}{s^3} + \frac{CA^3B}{s^4} \cdots$$

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  the first k moments about $s = s_0$ of $\hat{G}$ and G are equal.
- Match moments of G.

Meaning

- Steady state analysis $\equiv s = 0$
- $G(0)$ (zeroth moment about $s = 0$) is steady state value for step input
- Higher moments $\equiv$ 'rate of approaching' steady state value
- Immediate transients ($t \in (0, \epsilon)$ for small positive ϵ)
  moments of $G(s)$ about $s = \infty$.
- In fact, $G(\infty)$: the value step response jumps to at $t = 0^+$.
- Match moments about $s = \infty \equiv$ transient response approximation
- Relevant for 'piece-wise' approximation (transients' analysis)

More meaning

- Markov parameters $\equiv$ moments of $G(s)$ about $s = \infty$
- In fact, impulse response $h(t)$ of $G(s)$

$$h(t) = D\delta + CBt + CABt^2 + \cdots$$

- More generally, Padé approximation:
  moment matching (about different points s_0)
- Put Markov parameters in special structure: Hankel matrix

Hankel matrix

For $G(s) = D + \frac{CB}{s} + \frac{CAB}{s^2} + \frac{CA^2B}{s^3} + \frac{CA^3B}{s^4} \cdots$, define

$$H = \begin{bmatrix} D & CB & CAB & \ldots \\ CB & CAB & CA^2B & \ldots \\ CAB & CA^2B & CA^3B & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- Though H is defined to have infinitely many rows and columns, rank is finite.
- Suppose D is zero and leave first column (for simplicity).
- Define H_{nn}: the first N rows and first N columns of above H.
- H_{NN} (from this new matrix) is

$$H_{nn} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{N-1} \end{bmatrix} \begin{bmatrix} B & AB & \ldots A^{N-1}B \end{bmatrix}$$

Realization theory link

- By Cayley Hamilton theorem, we know observability and controllability matrix ranks cannot keep increasing: atmost N each (if $G(s)$ has order N).
- Hankel matrix: very well-studied for state space realization from Markov paramters
- Given Markov parameters, find state space realization (A, B, C, D) that has precisely these Markov parameters
- For rational $G(s)$, Markov parameters are 'dependent' after N moments
- Like $\frac{10}{27} = 0.370\ 370\ 370\ 370\ 370\ \ldots$ (for any rational number)
- 'Dependency': Hankel matrix rank is bounded for rational $G(s)$
- Well-studied 50 years ago!

Model order reduction??

- Often state space realization of $G(s)$ given: but N very large
- Mechanical systems, FEM or FDM: we get A, B, C and D.
- D plays no role. 'Feedthrough' term needs no states: assume $D = 0$.
- Too much (computational) effort to calculate moments and then build lower order $\hat{G}$ from the computed moments
- Large memory involved in storing/computing for large matrices
- Floating point error accumulation due to ill-conditioned $[B \ AB \ A^2B \ ]$ etc.
- Power method tells for almost any nonzero v (and A), the vectors v, Av, A^2v, become parallel (after normalizing) $\Rightarrow$ ill-conditioning

Ill-conditioned matrices

- Recall for square matrix A: ill-conditioned means very large $\|A\| \times \|A^{-1}\| =: \kappa(A)$
- Maximum singular value $\sigma_{\max}$: induced 2-norm (maximum amplification/gain when measured in Euclidean/2-norm)
- For induced 2-norm of matrix A, large $\kappa_2(A) \Rightarrow$ singular values very separated: $\sigma_{\max} \gg \sigma_{\min}$
- Well-conditioned A: columns of A are roughly same length and quite mutually orthogonal.
- $\kappa_2(A) = 1 \Leftrightarrow A = cQ$ (for any constant c and orthogonal Q).
- Recall a square matrix Q is called orthogonal if $Q^{-1} = Q^T$.
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- Not just rotations, but 'reflections' allowed too.
- Givens rotators and Householder reflectors

How to find $\hat{G}$

Moment matching method: overall plan

- Compute moments far more easily by changing basis
- Change of basis using orthogonal transformation
- Bring A to 'Hessenberg' form
- If A was symmetric, then, in fact, tridiagonal form
- Very scalable methods:
  - memory-wise,
  - computational effort-wise,
  - computational floating-point-error-wise
- Computational procedure: Arnoldi (for unsymmetric A) and Lanzos (for symmetric A)

Hessenberg form

- Not Heisenberg, of the Heisenberg uncertainty principle, but Hessenberg
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- Upper triangular matrix ⇒ upper Hessenberg
- Upper triangular: diagonal and super-diagonals can be nonzero
  All sub-diagonals have to be zero
- Upper Hessenberg: upper triangular and first sub-diagonal can be nonzero

$$H = \begin{bmatrix} \star & \star & \star & \star & \star & \dots \\ \star & \star & \star & \star & \star & \dots \\ 0 & \star & \star & \star & \star & \dots \\ 0 & 0 & \star & \star & \star & \dots \\ 0 & 0 & 0 & \star & \star & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \star & \star \end{bmatrix}$$

Orthogonal similarity transform

- Given (A, B, C) (recall that we assumed $D = 0$), find change of coordinates: $x = Qz$ such that
  - Q is orthogonal
  - $Q^T AQ$ is upper Hessenberg
  - correspondingly find B and C
- By $x = Tz$ (with T nonsingular),

$$A \to T^{-1}AT, \qquad B \to T^{-1}B, \qquad C \to CT$$

- For simplicity consider single input: B is just one column: b.
- Well-known: if first column of Q is b (normalized to length one), then all columns Q are just 'orthonormal' basis of the Krylov subspace
- We will see Krylov subspace $K(A, b, j)$ for a matrix $A \in \mathbb{R}^{n \times n}$, vector $b \in \mathbb{R}^n$ and index $j \leqslant n$.

Slides kept at:

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