Generic degree structure of the minimal polynomial nullspace basis: a block Toeplitz matrix approach

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Motivation

Goal: *Computing the minimal polynomial basis (MPB) of a given polynomial matrix*

- Current state of the art:
  - Involves explicit knowledge of entries of polynomial matrix
  - *Examples*: matrix pencils, LQ factorization of Toeplitz matrices. Aim: numerically robust algorithms
- **This work:**
  - Generic case: use just degrees of entries to determine degrees of entries in MPB
  - For specific case, this gives upper bound on degree structure of MPB.
  - No numerical computation: we use degree-structure and block-Toeplitz structure

Minimal Polynomial Basis

- $\mathbb{R}[s]$: polynomials in s with real coefficients
- $\mathbb{R}^{m\times n}[s]$: $m \times n$ matrix with entries from $\mathbb{R}[s]$. (Suppose $m < n$)
- Suppose $R(s) \in \mathbb{R}^{m\times n}[s]$ and has rank m
- Consider matrix $M(s) \in \mathbb{R}^{n\times(n-m)}$ of rank $n-m$ and $R(s)M(s) = 0$
- Look for $M(s)$ with 'least column degrees'
- Sort columns of $M(s)$ to be increasing/nondecreasing degrees
- Find M with least total column degree $\equiv$

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- Sort columns of $M(s)$ to be increasing/nondecreasing degrees
- Find M with least total column degree $\equiv$ least individual column degrees
- When minimum, these columns $\equiv$: 'minimal polynomial basis'

Basis for the polynomial and/or rational nullspace of $R(s)$.
Degrees of $M(s)$ are unique, though $M(s)$ is not unique.

Minimal Polynomial Basis: why and where

- These minimum degrees also called: Forney indices: convolutional coding
- Helpful for calculating left/right coprime factorization of MIMO $G(s)$
- Linked to Kronecker canonical form of a matrix pencil: $sE - A$ and $[sI - A\ B]$
- Helps in 'parametrizing' all system trajectories: optimization/optimal control
- Examples: $R_1 = [s\ \ s^2 + 2s - 1]$, take $M_1 = \begin{bmatrix} s^2 + 2s - 1 \\ -s \end{bmatrix}$

  for $R_2 = \begin{bmatrix} sI - A & B \end{bmatrix}$ with $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ take $M_2 = \begin{bmatrix} 1 \\ s \end{bmatrix}$

Problem Formulation

Do the degrees of the polynomial matrix $M(s)$ depend on

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When there are 'common factors', then degrees drop.
For example: both $R_1 = \begin{bmatrix}(s+1) & (s^2+3s+2)\end{bmatrix}$ and $R_2 = [1 \;\; s+2]$ have minimal polynomial basis $M(s) = \begin{bmatrix} s+2 \\ -1 \end{bmatrix}$.

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For $R_3 = \big[(s+1.001) \quad (s^2+3s+2)]\big]$, Do the degrees of the polynomial matrix $M(s)$ depend on

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For $R_3 = \left[(s+1.001) \quad (s^2+3s+2)\right]$, MPB $M_3(s) = \begin{bmatrix} s^2+3s+2 \\ -s-1.001 \end{bmatrix}$.

Genericity of parameters

- *Algebraic variety* - set of solutions, $E_q \subseteq \mathbb{R}^n$ to a system of polynomial equations.
- Zero equation $\equiv$ variety *trivial*: variety is the whole of $\mathbb{R}^n$.
- Nontrivial algebraic variety is a thin-set (i.e. 'set of measure zero').

Genericity

Property P in terms of variables $p_1, p_2, \ldots, p_n \in \mathbb{R}$ is said to be satisfied generically if the set of values $p_1, p_2, \ldots, p_n$ that do NOT satisfy P form a nontrivial algebraic variety in $\mathbb{R}^n$.

- *Examples*
  - Two nonzero polynomials are generically coprime.
  - A square matrix with all entries generically from $\mathbb{R}$ is nonsingular.

Problem Formulation

- Given $R \in \mathbb{R}^{m \times n}[s]$, define $D \in \mathbb{Z}^{m \times n}$ such that
  - $[D]_{ij} := \deg[R]_{ij}$.
  - If $[R]_{ij} = 0$, then $[D]_{ij} := -\infty$ (Degree of the 0 polynomial)
- Define the sets:

$$\mathbb{Z}_+ = \{z \in \mathbb{Z} | z \geqslant 0\}$$
$$\bar{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{-\infty\}$$

- Given $R \in \mathbb{R}^{m \times n}[s]$, can construct unique $D \in \bar{\mathbb{Z}}_+^{m \times n}$
- Call D the ***degree structure*** of R.
- Given D, there exist many R with that degree structure.
- $D(R) := \{R \in \mathbb{R}^{m \times n}[s]$ with degree structure $D \in \bar{\mathbb{Z}}_+^{m \times n}\}$

Problem formulation and main observation

Problem 1

Consider $R \in \mathbb{R}^{m\times n}[s]$, with degree structure $D \in \bar{\mathbb{Z}}_{+}^{m\times n}$.
Suppose $M \in \mathbb{R}^{n\times(n-m)}[s]$ gives an MPB of R and let K is degree structure of M.
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Problem 2

Suppose $R_1, R_2 \in D(R)$ and $M_1, M_2 \in \mathbb{R}^{n \times (n-m)}[s]$ be their respective MPBs. Let K_1 and K_2 denote the degree structures of M_1 and M_2 respectively. Then, is $K_1 = K_2$?

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Key observation (using Scilab)

Given degree structures K_1, K_2 of minimal polynomial bases corresponding to the same degree structure D, $K_1 = K_2$.

MPB computation: Block Toeplitz Matrices (Henrion, et al)

- Given $R \in \mathbb{R}^{m \times n}[s]$ with degree d,

$$R = R_0 + R_1 s + \cdots + R_d s^d$$

where $R_i \in \mathbb{R}^{m \times n}$ for $i = 0, 1, \ldots, d$.

- Construct a sequence of real structured matrices from the given polynomial matrix as:

$$A_0 := \begin{bmatrix} R_0 \\ R_1 \\ \vdots \\ R_d \end{bmatrix}, A_1 := \left[\begin{array}{c|c} A_0 & 0 \\ \hline 0 & A_0 \end{array}\right], A_2 := \left[\begin{array}{c|c|c} A_0 & 0 & 0 \\ & & 0 \\ \hline 0 & A_0 & \\ 0 & 0 & A_0 \end{array}\right], \cdots \quad (1)$$

Stop when $(d + i + 1)m \geqslant (i + 1)n$.

- Right nullspaces of constant matrices A_i yield polynomial nullspace of $R(s)$.

Degree Structure of MPB of generic 1×3 $R(s)$ case

- Let $R \in \mathbb{R}^{1\times 3}[s]$ have degree structure $D = \begin{bmatrix} a & b & c \end{bmatrix}$ and $M \in \mathbb{R}^{3\times 2}[s]$ with deg struct K form an MPB of $R(s)$.
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Theorem 2

For even c, the degree structure of the MPB is:

$$K = \begin{bmatrix} c/2 & c/2 \\ c/2 & c/2 \\ b - c/2 & b - c/2 \end{bmatrix} \tag{2}$$

and for odd c is

$$K = \begin{bmatrix} (c-1)/2 & (c+1)/2 \\ (c-1)/2 & (c+1)/2 \\ (c-1)/2 - (c-b) & (c+1)/2 - (c-b) \end{bmatrix} \tag{3}$$

- When $c = 2b + k$, the MPB will contain the zero polynomial, corresponding to a $-\infty$ term in its degree structure.

Example

Given $D = \begin{bmatrix} 0 & 1 & 2 \end{bmatrix}$, find $K = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$, such that $DK = 0$.

- $A_0 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$; $A_1 = \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & * \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{bmatrix}$
- A_1 is a *wide* matrix. We have $(D_0 + D_1 s + D_2 s^2)K = 0$, where D_i corresponds to the coefficients of the degree i terms in the given polynomial matrix.
- *Note*: If a particular A_i yields only some columns of the MPB, the remaining columns can be got by constructing A_{i+1}.

Example

- *Need*: Constant matrix P such that $A_1 P = 0$. For last row of A_1 to be annihilated, corresponding element(s) in K must be zero. This effectively eliminates the last column of A_1, as shown below:

$$\begin{matrix} * & * & * & 0 & 0 & 0 \\ 0 & * & * & * & * & * \\ 0 & 0 & * & 0 & * & * \\ 0 & 0 & 0 & 0 & 0 & * \end{matrix}$$

- $P = \begin{bmatrix} * & * \\ * & * \\ * & * \\ \hline * & * \\ * & * \\ 0 & 0 \end{bmatrix}$, yielding $K = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$.

Saturation

- *Observe*: degree structure of K independent of a.
- Same algorithm can be used to determine degree structure of minimal left indices of K, D_1.
- If $D \geqslant D_1$ (component wise), and they have an MPB with the same degree structure, K, then D is said to be *saturated*.
- 'Unsaturated' $D \Rightarrow$ some degree of freedom to 'change' one or more coefficients from 0 to a nonzero value, $*$.
- *Saturation*: degree of freedom offered to replace zeros by nonzeros in degree structure of D while maintaining that of K.

Proposition

When $D = \begin{bmatrix} a & b & c \end{bmatrix} \in \bar{\mathbb{Z}}_+^{1\times 3}$, and $c \leqslant 2b$, $D_{sat} = \begin{bmatrix} b & b & c \end{bmatrix}$.

- D_{sat} for higher dimensions of D? *Not (yet) known*.

Conclusions and future work

- Degree structure of MPB of a given polynomial matrix depends only on its degree structure, and not its coefficients: Scilab observation.
- Closed form degree structure of MPB obtained for generic case for 1×3. Upper bound for specific case: we used block Toeplitz methods.
- Genericity of parameters ensured matrices had full rank.
- Saturation of a degree structure examined in the context of freedom of making some zero coefficients nonzero.

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- Results for $(n-1) \times n$ is easy.
- Need to generalize to other cases: future work.

Thank You