Some new links between graph theory and optimal charging/discharging control strategies

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Outline

- Optimal charging/discharging of RLC circuits
- Actual energy stored in state v_0 is geometric mean of
  - 'minimum required' energy for charging state v_0, and
  - 'maximum extractable' energy while discharging from v_0.
- Generalization to the multi-state case of the 'geometric' mean
- Positive real balancing in Model Order Reduction

Problem formulation

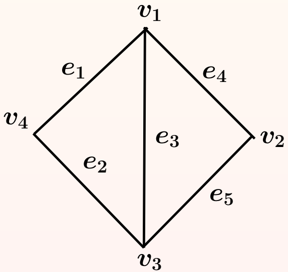
1. Generalize the intuitive notion: a port needed for controlling a circuit
2. Formulate port/capacitor/inductor relative locations which cause common eigenspaces in
   - state transition matrix A (of any state space realization)
   - required supply energy matrix K_{reqd}
   - available storage energy matrix K_{avail}
3. Generalize the geometric mean property to the matrix (multistate) case

Graph Laplacian

- For an undirected graph $G(V, E)$: with $|V| = n$ nodes, the Laplacian $L \in \mathbb{R}^{n \times n}$

$$L := D - N$$

D is the diagonal 'degree' matrix
(degree of a node: number of edges incident on that node)
N is the neighbourhood (adjacency) matrix:
$N_{ij} := 1$, if nodes v_i and v_j are neighbours, 0 otherwise.



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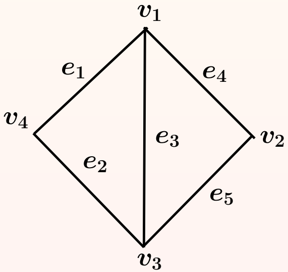

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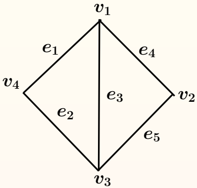

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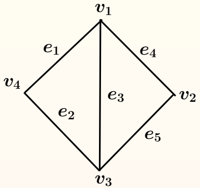

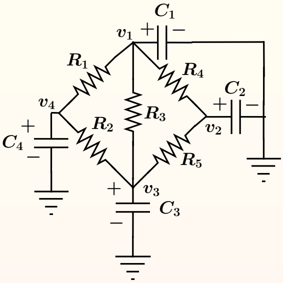

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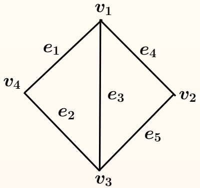

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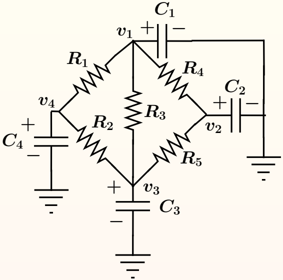

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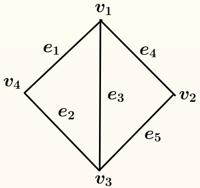

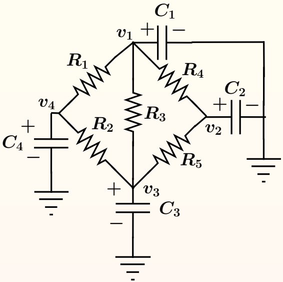

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- Edge conductances $\leftrightarrow$ edge weights $\leftrightarrow$ off-diagonal terms in L
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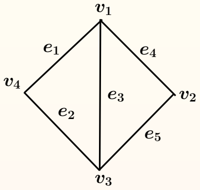

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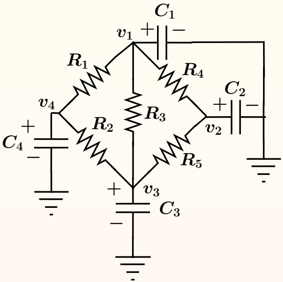

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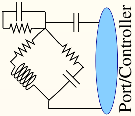
- With ports:
  - ⋆ optimal charging/discharging
  - ⋆ port location for controllability: general LTI systems

Hinged graphs: decoupled: uncontrollable?



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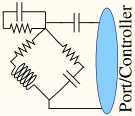

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- ⋆ Bipartite graphs: edges between equation set and variable set

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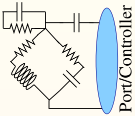

- ⋆ Electrical networks: much structure
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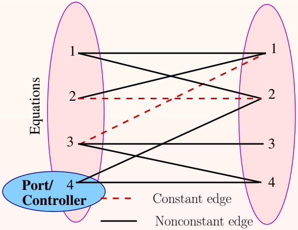
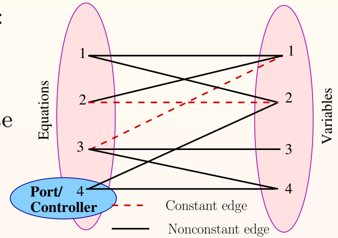
- ⋆ Bipartite graphs: edges between equation set and variable set
  - Two types of equations (left nodes):
    - plant equations (left-top nodes): given to us
    - controller equations (left-bottom nodes): what we connect at the port: to-be-designed
      - ⋆ feedback controller: additional laws
  - Plant structure: equation-variable interaction
  - Controller structure: sensor/actuator interaction constraint
  - Controlled system: square polynomial matrix: plant and controller/port equations
  - # equations = # variables: (determined system of equations)
  - One-to-one correspondence ≡ 'marriage' ≡ matching
  - All nodes get matched ≡ perfect matching
  - Terms in determinant ↔ perfect matchings



Pole-placement, structural controllability

- Given equation-variable graph structure of the plant and controller: find conditions on controller structure for arbitrary pole-placement

- Those edges that occur in some perfect matching: 'admissible'.
- Discard the rest
- Resulting graph if connected: elementary bipartite graph: well-studied (Lovasz & Plummer)
- Classify edges into plant/controller edges (plant edges: constant and non-constant)



Theorem

Given plant and controller structure, following are equivalent

- arbitrary pole-placement possible
- every admissible plant edge occurs in some loop containing controller edges

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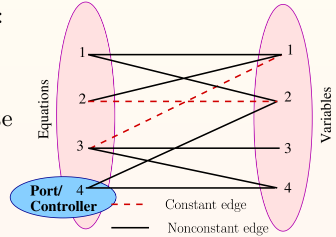

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(with R. Kalaimani and S. Krishnan (Math): Linear Algebra & its Applications: 2013)

Techniques from matching theory aspects of graph theory

Optimal charging and discharging

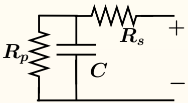
Consider the RC circuit

Initially (at $t = -\infty$) discharged: $v_C(-\infty) = 0$

Finally (at $t = +\infty$) discharged: $v_C(+\infty) = 0$

Actual energy at $t = 0$: $\frac{1}{2}Cv_0^2$ with $v_C(0) = v_0$ V

Energy supplied/extracted from the port



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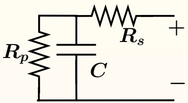

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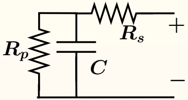

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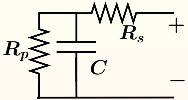

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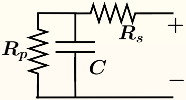

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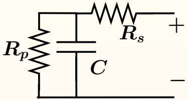

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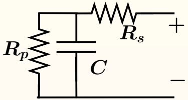

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In general, suppose $v_C(0) = v_0$: available and required energies: quadratic in v_0

$$\frac{1}{2}K_{\text{avail}}\ v_0^2 \leqslant \frac{1}{2}Cv_0^2 \leqslant \frac{1}{2}K_{\text{reqd}}\ v_0^2$$

Optimal charging/discharging: single state

- $\frac{1}{2}Cv_0^2$ is geometric mean of $\frac{1}{2}K_{\text{reqd}}v_0^2$ and $\frac{1}{2}K_{\text{avail}}v_0^2$
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For n number of 'memory elements' (L and C),

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- ⋆ K_{avail} and K_{reqd} symmetric $n \times n$ matrices
- ⋆ Standard Riccati equation procedures (from a state space representation of RLC network) to compute K_{avail} and K_{reqd}
- ⋆ Geometric mean of matrices??

$\dot{x} = Ax + Bi$ and $v = Cx + Di$: minimal state space realization of biproper impedance matrix $Z(s)$
x is vector of capacitor voltages and inductor currents and $x(0) = a \in \mathbb{R}^n$

Multiple memory elements? Multiports?: Riccati equations

$\dot{x} = Ax + Bi$ and $v = Cx + Di$: minimal state space realization of biproper impedance matrix $Z(s)$

x is vector of capacitor voltages and inductor currents and $x(0) = a \in \mathbb{R}^n$

Suppose $x(0) = a \in \mathbb{R}^n$ is a given voltage/current configuration

- Required supply is $a^T K_{\text{reqd}}\, a$ and available storage is $a^T K_{\text{avail}}\, a$: K_{reqd} and K_{avail} are maximum and minimum solutions of the Algebraic Riccati Equation (ARE):

$$A^T K + KA + (KB - C^T)(D + D^T)^{-1}(B^T K - C) \leqslant 0$$

- Quadratic in unknown $K \in \mathbb{R}^{n \times n}$ (we seek symmetric solutions)
- Multi-state: symmetric solutions of ARE/ARI form a 'poset' (partially ordered set): K_{avail} and K_{reqd} are minimum/maximum elements
- My work so far: uncontrollable case (with D. Pal, S. Karikalan, C. Athalye: IJC 2014, SIAM 2008)
- Today: network topology link

New results

Assume biproper impedance $Z(s)$ and admittance $Y(s)$: construct
$\frac{\mathrm{d}}{\mathrm{d}t}x = A_z x + B_z i, \quad v = C_z x + D_z i$ and $\frac{\mathrm{d}}{\mathrm{d}t}x = A_y x + B_y v, \quad i = C_y x + D_y v.$

- Construct $\hat{x}$ has $2n$ components: consists of x and 'dual state' λ:
  - ⋆ original state x (capacitor voltage/inductor current), and
  - ⋆ the dual/adjoint/Lagrange multiplier λ
  - ⋆ Admittance $Y(s)$ gives $2n \times 2n$ matrix H_y
  - ⋆ Impedance $Z(s)$ gives $2n \times 2n$ matrix H_z

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Optimal trajectories dynamics: independent of impedance/admittance state-space realization:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x} = H_y \hat{x}, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\hat{x} = H_z \hat{x} \quad \text{then} \quad H_y = H_z$$

Common eigenspaces

Suppose some capacitors form cutsets with ports: those states

- ⋆ result in imaginary axis eigenvalues of A_z

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$\frac{\mathrm{d}}{\mathrm{d}t}x = A_z x + B_z i, \quad v = C_z x + D_z i$ and $\frac{\mathrm{d}}{\mathrm{d}t}x = A_y x + B_y v, \quad i = C_y x + D_y v.$

- Construct $\hat{x}$ has $2n$ components: consists of x and 'dual state' λ:
  - ⋆ original state x (capacitor voltage/inductor current), and
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Optimal trajectories dynamics: independent of impedance/admittance state-space realization:

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Common eigenspaces

Suppose some capacitors form cutsets with ports: those states

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Also: for L forming loops with ports, and 'LC tanks' forming cutsets/loops with ports

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Recall: $K_{\text{reqd}} \times K_{\text{avail}} = c^2$. Generalize to many states?

- Assume multiport RC network
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$\Leftrightarrow (K_{\max})^{-1} = K_{\min}$

Why positive real balancing: MOR

- Balancing useful in Model Order Reduction (MOR)
- Balanced truncation:
  discard those states that are 'very' uncontrollable and 'very' unobservable
- Retain states only if:
  - ⋆ very controllable (low $x^T P^{-1} x$) and
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- In general, positive real balancing: simultaneous diagonalization (of K and L)
  Involves Riccati equation solution: then Cholesky factorization
- We showed[1] :

for RC circuits

capacitor voltage normalizing results in balancing.

- Also true for: RL, and 'symmetric' realizations $A = A^T$ (of the state space)
  (Recall Laplacian $L = L^T$ in RC example: though no ports there)

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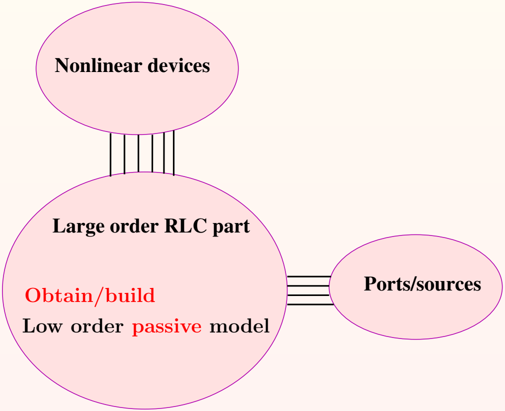
Passive reduced-order LTI models: important for

- circuit simulation and analysis,
- controller design



Other RC like systems

- RL systems
- Multi-agents:
  - Network of many agents to achieve a task collectively
  - Surveillance pursuit-evasion games
  - Controlled through port $\equiv$ one or more agents are 'leaders'
  - Laplacian matrix arises for single-integrator dynamics
- Network of Voltage-Source-Inverters (VSI) forming a microgrid (after neglecting sufficiently quick dynamics): with Iyer, Chandorkar: IEEE-Trans-PEL 2010 & IEEE-Trans-Energy Conv 2011
- Mass-damper systems
- Other so-called ZIP (Zero-Interlacing-Pole) systems

Summary

- Controller locations that achieve controllability/pole-placement
- Optimal charging/discharging matrix relations with RLC locations
- Positive real balancing and state variable normalization

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