Stationary trajectories, singular Hamiltonian systems and ill-posed Interconnection

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Outline

- Problem formulation: dual/adjoint system
- Main results: well-posed interconnection and the regular case
- Ill-posed interconnection & the singular case (descriptor system)
- Zeros at infinity, inadmissible initial conditions
- Necessary and sufficient conditions for no zeros at infinity (main result)

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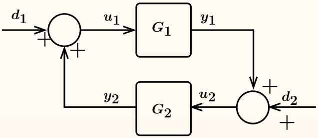
- Analogue of Hamiltonian matrix for ill-posed case?
- Link with other singular Hamiltonian pencils?
  Skew-Hermitian Hermitian pencil? (Mehl, Mehrmann, Meerbergen, Watkins)

Well-posed interconnection

Systems $\mathfrak{B}_1$ and $\mathfrak{B}_2$ (with transfer functions G_1 & G_2)



Call the interconnection of $\mathfrak{B}_1$ and $\mathfrak{B}_2$
well-posed if
for each d_1, $d_2 \in \mathfrak{L}_1^{\text{loc}}$,
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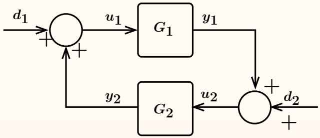

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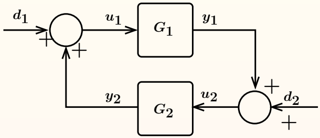

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Call the set of allowed trajectories of system 1 as behavior $\mathfrak{B}_1$

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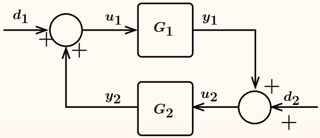

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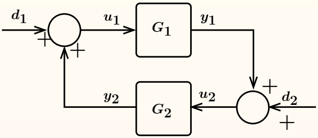

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Positive or negative feedback? Assume $d_1 = 0$ and $d_2 = 0$.

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$\mathfrak{B}_1$ and $\mathfrak{B}_2$ are called Σ-orthogonal if

$$\int_{-\infty}^{\infty} w_1^T \Sigma w_2 \, \mathrm{d}t = 0 \text{ for all } w_1 \in \mathfrak{B}_1 \text{ and } w_2 \in \mathfrak{B}_2.$$

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For a controllable system $\mathfrak{B}_1 \subseteq \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, define its Σ-orthogonal complement $\mathfrak{B}_1^{\perp_\Sigma}$ as:

$$\mathfrak{B}_1^{\perp_\Sigma} := \{ v \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \int_{\mathbb{R}} w^T \Sigma v \, dt = 0 \text{ for all } w \in \mathfrak{B}_1 \cap \mathfrak{D} \}.$$

Hamiltonian matrix

$$\mathfrak{B}: \begin{array}{l} \dot{x} = Ax + Bw_1 \\ w_2 = Cx + Dw_1 \end{array} \qquad \mathfrak{B}^{\perp_\Sigma}: \begin{array}{l} \dot{z} = -A^T z - C^T v_1 \\ v_2 = B^T z + D^T v_1 \end{array}$$

w_1, v_1: inputs, w_2, v_2: outputs.
Interconnect $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$: $w_2 = v_1$ and $w_1 = v_2$

$$\begin{bmatrix} \dot{x} \\ \dot{z} \\ 0 \end{bmatrix} = \begin{bmatrix} A & BB^T & BD^T \\ 0 & -A^T & -C^T \\ -C & -DB^T & I_{\mathtt{p}} - DD^T \end{bmatrix} \begin{bmatrix} x \\ z \\ v_1 \end{bmatrix}$$

$\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ is well-posed $\Leftrightarrow$ $(I_{\mathtt{p}} - DD^T)$ is nonsingular.

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Eliminating v_1, we get $I_{2\mathtt{n}} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = H \begin{bmatrix} x \\ z \end{bmatrix}$ with $H \in \mathbb{R}^{2\mathtt{n} \times 2\mathtt{n}}$ the Hamiltonian matrix:

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For LQ control, above trajectories are 'stationary': Willems, CDC-92.

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Nonsingularity of $(I_{\mathtt{p}} - DD^T)$ required for Riccati (in)equality.

Extreme case: $D = I$

We focus on the case when $(I_{\mathtt{p}} - DD^T)$ is singular: the ill-posed case.
To understand better the ill-posed case, assume G is square and $D = I$.
Also assume B has full column rank

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Also assume B has full column rank
(otherwise, can show, $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ is non-autonomous).
Under what conditions is $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$ autonomous?

Ill-posed interconnection $\mathfrak{B} \wedge \mathfrak{B}^{\perp_\Sigma}$

Assume $D = I$ and B is full column rank. Define

$$\widetilde{A} := \begin{bmatrix} A & BB^T \\ 0 & -A^T \end{bmatrix}, \widetilde{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}, \widetilde{C} := \begin{bmatrix} C & B^T \end{bmatrix} \quad (1)$$

Theorem

Consider the interconnection of the behaviors $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$.
Then the following are equivalent.

1. The interconnected system is autonomous.
2. $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ is nonsingular for some t.
3. $\ker(\widetilde{C}\widetilde{B}) \cap \ker(\widetilde{C}\widetilde{A}\widetilde{B}) \cap \cdots \cap \ker(\widetilde{C}\widetilde{A}^{2\mathtt{n}-1}\widetilde{B}) = 0$

Inadmissible initial conditions

For a descriptor system $E\frac{\mathrm{d}}{\mathrm{d}t}x = Ax$, with det $(sE - A) \neq 0$, some initial conditions $x(0^-)$ could result in impulsive solutions $x(t)$.

Call these initial conditions inadmissible.

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No inadmissible initial conditions $\Leftrightarrow$ $(sE - A)$ has no 'zeros at infinity'.

(Like a polynomial matrix $P(s)$ can have finite zeros, $P(s)$ can also have zeros at $s = \infty$.)

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When does an autonomous singular Hamiltonian system have inadmissible initial conditions?

Main result

$$\widetilde{A} := \begin{bmatrix} A & BB^T \\ 0 & -A^T \end{bmatrix}, \widetilde{B} := \begin{bmatrix} B \\ -C^T \end{bmatrix}, \widetilde{C} := \begin{bmatrix} C & B^T \end{bmatrix} \tag{2}$$

Theorem

Assume the singular Hamiltonian system is autonomous. The following are equivalent:

1. There are no inadmissible initial conditions.
2. $\widetilde{C}e^{\widetilde{A}t}\widetilde{B}$ is nonsingular at $t = 0$.
3. $\ker(\widetilde{C}\widetilde{B}) = 0$.
4. $\text{rank}\,(CB - B^TC^T) = \mathtt{p}$

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Recall our theorem for the interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$.
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Relation with all pass MIMO transfer matrix

- det $(CB - (CB)^T) \neq 0$ is 'opposite' to requirement for being all-pass.
- $G(s)$ is all-pass $\Leftrightarrow I - G(-j\omega)^T G(j\omega) = 0$ for each $\omega \in \mathbb{R}$.
- Assumption $D = I$ and G is all-pass $\Rightarrow$

$$CB - B^T C^T = 0,\ CAB + (CAB)^T - B^T C^T CB = 0,\ \cdots$$

- Notice that CB is the first moment of $G(s)$ about $s = \infty$.
  Thus a necessary condition on the first moment for G to be all-pass is:

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- On the other hand,
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- Odd number of inputs (and autonomous, ill-posed) $\Rightarrow$ there exist inadmissible initial conditions.

Example

$G(s) := \frac{s+1}{s+2}$ with input u and output y.
State space realization :
$(A, B, C, D) = (-2, 1, -1, 1)$.

Dual system $:= \frac{s-1}{s-2}$.
State space realization :
$(A, B, C, D) = (2, 1, 1, 1)$.

The interconnection of G and its dual:

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \\ y \end{bmatrix}$$

- Above $Es - A$ has a zero at infinity.
- The differential equation in just x and z has initial conditions that have impulsive solutions.
- From our result too: $CB - B^T C^T = 0$.

Conclusion

- For the Σ we considered: $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$ is an 'all-pass' subsystem (possibly autonomous)
- In the context of Σ-dissipativity, this is set of 'stationary' trajectories with respect to $w^T \Sigma w$.
- The stationary trajectories are interconnection of $\mathfrak{B}$ and $\mathfrak{B}^{\perp_\Sigma}$.
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  no inadmissible initial conditions $\Leftrightarrow$ G's first moment has its skew-symmetric part nonsingular.
- Note: square MIMO all-pass G has first moment symmetric.
- No obvious way to have E as skew-Hamitonian and H as Hamiltonian (as considered by Mehl, Mehrmann, et al.)
- But: for our pencil (E, H) also, generalized eigenvalues occur in quadruplets $(\lambda, \bar{\lambda}, -\lambda, -\bar{\lambda})$: like Mehl, Mehrmann and others.

Questions, thank you!

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Inadmissible initial condition

- Consider an autonomous system $P(\frac{\mathrm{d}}{\mathrm{d}t})w(t) = 0$, with $P(\xi) \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ nonsingular.
- A vector $\overline{w}(0) \in \mathbb{R}^{\mathtt{zw}}$ is said to be an *inadmissible* initial condition vector if the corresponding solution $w(t)$ contains the Dirac impulse $\delta(t)$ and/or its distributional derivatives.
- There exist no inadmissible initial conditions for $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0 \Leftrightarrow P$ has no zeros at infinity.

Stationary trajectories

- Consider a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and a symmetric nonsingular matrix $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$.
  A trajectory $w \in \mathfrak{B}$ is Σ-*stationary* if

$$\int_{-\infty}^{\infty} w^T \Sigma v \, \mathrm{d}t = 0 \text{ for all } v \in \mathfrak{B} \cap \mathfrak{D}.$$

- Assume (wlog) that $\Sigma := \begin{bmatrix} I_{\mathtt{m}} & 0 \\ 0 & -I_{\mathtt{p}} \end{bmatrix}$

Definition

[a] Given a controllable behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$, *the* Σ*-orthogonal complement of* $\mathfrak{B}$, denoted by $\mathfrak{B}^{\perp_\Sigma}$ is the set of all the trajectories $v \in \mathfrak{L}_1^{\text{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ such that $\int_{-\infty}^{\infty} v^T \Sigma w \, \mathrm{d}t = 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$.

[a] J.C. Willems and H.L. Trentelman, "On quadratic differential forms, SIAM Journal on Control and Optimization", 1998.

- The set of Σ-stationary trajectories is equal to $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma}$.