Minimal Controller Structure for Generic Pole Placement

Rachel Kalpana K. & Madhu N. Belur.

Department of Electrical Engineering
Indian Institute of Technology Bombay

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Outline

- Introduction
- Minimal controller
- Structured systems
- Main results - 3 cases
- Interconnection of sub-systems with different input output structure.
- Conclusion

System is represented using higher order Differential Algebraic Equation (DAE).

$$\left[\; R_N \tfrac{\mathrm{d}^N}{\mathrm{d}t} + \cdots + R_1 \tfrac{\mathrm{d}}{\mathrm{d}t} + R_0 \;\right] w = 0$$

$$\text{(i.e) } R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w = 0, \text{ where } R(s) = \left[\; R_N s^N + \cdots + R_1 s + R_0 \;\right]$$

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Pole placement:

Plant laws : $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$

Controller laws: $K(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$

Control: Choose $K(s)$ such that $\begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$ is square, nonsingular and has determinant $d(s)$ as prescribed:

Roots of $d(s)$ = desired closed loop system poles.

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We seek only generic results.

Hence only structural aspects of the system are relevant.

This is captured in a bipartite graph.

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Plant matrix:
$$P(s) = \begin{bmatrix} s+1 & 1 & 0 \\ 0 & 1 & s+2 \end{bmatrix}$$

Controller matrix:
$$K(s) = \begin{bmatrix} a(s) & 0 & b(s) \end{bmatrix}$$

Closed loop poles: Determinant of $\begin{bmatrix} s+1 & 1 & 0 \\ 0 & 1 & s+2 \\ a(s) & 0 & b(s) \end{bmatrix} = a(s)(s+2) + b(s)(s+1)$.

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- Some entries of $K(s)$ could be zero.
- This is motivated by a minimum sensor-actuator network design issue.

Structured system

Given $P(s) \in \mathbb{R}^{n \times n}[s]$, associate an edge weighted bipartite graph $G = (\mathcal{R}, \mathcal{C}, E)$ as follows.

$\mathcal{R}$ and $\mathcal{C}$ denote the rows and columns of $P(s)$

An edge between vertex $v_i \in \mathcal{R}$ and $v_j \in \mathcal{C}$ exists if the $(i,j)th$ entry of $P(s)$ is non-zero.

Edges are classified as constant and nonconstant depending on corresponding entries in $P(s)$.

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Definition

Consider a system of LTI ODEs $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ with $P \in \mathbb{R}^{n\times m}[s]$.

- Classify the nonzero entries in $P(s)$ as constant and nonconstant and then associate the graph $G(\mathcal{R}, \mathcal{C}; E)$ to the polynomial matrix $P(s)$.
- Such association partitions the set of all polynomial matrices into equivalence classes and each class is identified by the corresponding graph.
- $G(\mathcal{R}, \mathcal{C}; E)$ captures the *structure of the LTI system.*

Henceforth a system will be described by a graph.

Example

System laws:

$$
\begin{aligned}
a_{11}\dot{w}_1 + b_{11}w_1 + a_{12}\dot{w}_2 + b_{12}w_2 &= 0 \\
a_{21}\dot{w}_1 + b_{21}w_1 + b_{22}w_2 &= 0 \\
b_{31}w_1 + a_{32}\dot{w}_2 + b_{32}w_2 + a_{33}\dot{w}_3 + b_{33}w_3 & \\
+ a_{34}\dot{w}_4 + b_{34}w_4 &= 0
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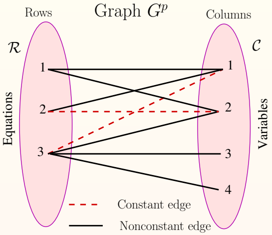

Figure: Graph for $P(s)$

Associated polynomial matrix:

$$P(s) = \begin{bmatrix} a_{11}s + b_{11} & a_{12}s + b_{12} & 0 & 0 \\ a_{21}s + b_{21} & b_{22} & 0 & 0 \\ b_{31} & a_{32}s + b_{32} & a_{33}s + b_{33} & a_{34}s + b_{34} \end{bmatrix}$$

Motivation

- Often, when uncontrollable, small perturbation → controllable.
- 'Generically controllable' ≡ controllable for almost all values for that structure.
- For example, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is generically nonsingular for real numbers a, b, c, d.
  (Singular only when $ad = bc$.)

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- For example, $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is generically nonsingular for real numbers a, b, c, d. (Singular only when $ad = bc$.)
- But, $\begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ is generically singular.
- If coefficients are any real numbers, two nonzero polynomials of any degree are 'generically' coprime. (Coefficients have to satisfy an equation for a common root.)
- For 'generic' situations, perhaps can conclude without numerical calculation.
- Useful in the analysis of large scale systems.
- Generic/structural conditions are necessary conditions in specific case.

Problem Statement

Problem 1: control perspective

Given : Plant structure:= $G^p(\mathcal{R}_p, \mathcal{C}; E_p)$.
Find a controller structure :=$G^k(\mathcal{R}_k, \mathcal{C}; E_k)$ which satisfies the following properties.

- Arbitrary pole placement is generically achievable with this controller structure.
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Problem 2: graph perspective

Given : a graph $G^p(\mathcal{R}_p, \mathcal{C}; E_p)$.
Find a graph $G^k(\mathcal{R}_k, \mathcal{C}; E_k)$ such that the following are satisfied.

1. In $G(\mathcal{R}_p \cup \mathcal{R}_k, \mathcal{C}; E_p \cup E_k)$ there exists a perfect matching.
2. Every edge $e \in E_p$ that is admissible in $G(\mathcal{R}_p \cup \mathcal{R}_k, \mathcal{C}; E_p \cup E_k)$ is in some cycle involving an edge e_k from E_k such that e_k is admissible in $G(\mathcal{R}_p \cup \mathcal{R}_k, \mathcal{C}; E_p \cup E_k)$.
3. $G^k(\mathcal{R}_k, \mathcal{C}; E_k)$ has the minimum number of edges amongst all graphs that satisfy conditions 1 and 2.

An edge e is admissible $:\equiv$ e is contained in some perfect matching.
Both the problems are the same.

- A set of edges in a graph $G = (\mathcal{R}, \mathcal{C}; E)$ is a matching M if no two edges have a common end vertex.

M is a perfect matching $\implies$ $|M| = |\mathcal{R}| = |\mathcal{C}|$.

- Let G be the bipartite graph associated to a square polynomial matrix $P(s)$.

A perfect matching M $\implies$ a non-zero term in the determinant expansion of P.

- The determinant expansion of P is the sum over all perfect matchings in G (with suitable signs).
- P is generically nonsingular $\Leftrightarrow$ G has at least one perfect matching.

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$$A_1 = \begin{bmatrix} 4 & 5 & -43 \\ 0 & 7 & * \\ 0 & 0 & 9 \end{bmatrix}$$

Pole placement for structured system

$G^p(\mathcal{R}_p, \mathcal{C}; E_p) :=$ Plant and $G^k(\mathcal{R}_k, \mathcal{C}; E_k) :=$ Controller.
$\mathfrak{L}_p$ and $\mathfrak{L}_k \rightarrow$ Equivalence classes of polynomial matrices with graphs G^p and G^k.
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Remove the inadmissible edges from G^{aut} to get G^{aut}_a.

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$A(s) := \begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$, for $P \in \mathfrak{L}_p$ and $K \in \mathfrak{L}_k$ and $\chi_{PK}(s)$:= determinant of $A(s)$.

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Then the following are equivalent.

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In our problem there is no controller structure to begin with.
Rather we propose a controller structure which is minimal and satisfies the above conditions.

- A path in a graph is assumed to be maximal, i.e. it is not properly contained in another path.
- Vertices with degree of incidence equal to one are referred as degree-one vertices.
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Lemma

Assume a plant, $G^p(\mathcal{R}_p, \mathcal{C}; E_p)$. Remove all inadmissible edges. Suppose there are no cycles. Then plant is structurally controllable if and only if every path whose terminal is in $\mathcal{R}_p$ has length one and is a constant edge.

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- Plant is controllable $\Rightarrow$ in G_p all paths containing at least one non-constant plant edge has both its terminals in $\mathcal{C}$.
- The next step to propose a controller is to complete all these paths to cycles using controller edges.

Graph with no cycles and only non-constant plant edges: main result-I

Assume (WLOG) that the graph of the plant after removing inadmissible edges is connected.

- $G^p(\mathcal{R}_p, \mathcal{C}; E_p)$:= Controllable plant after removing the inadmissible edges. Assume $G^p(\mathcal{R}_p, \mathcal{C}; E_p)$ is connected and has no cycles and only non-constant edges.

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Then a minimal controller $G^k(\mathcal{R}_k, \mathcal{C}; E_k)$, with $|\mathcal{R}_k| = |\mathcal{C}| - |\mathcal{R}_p|$, that generically achieves arbitrary pole placement has $e_p + |\mathcal{R}_k|$ edges.

Graph with cycles

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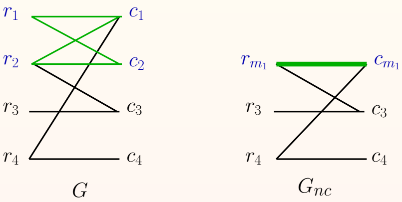
- Let edges, $e_i \subset E$ between vertices $r_i \subset \mathcal{R}$ and $c_i \subset \mathcal{C}$ form a cycle.
- Merge all vertices in r_i into one single vertex r_{m_i} and vertices in c_i to vertex c_{m_i}.
- The edge e_{m_i} between r_{m_i} and c_{m_i} is representative of all the edges in e_i.
- If at least one of the edges in e_i is a non-constant plant edge, then the edge e_{m_i} is also a non-constant plant edge.



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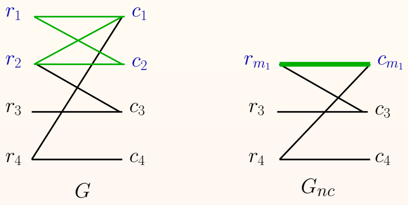

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The resulting graph is called the merged-cycles graph G_{nc}.
G_{nc} is independent of the sequence of merging the cycles.

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Proposition

Consider cycles $\mathscr{C}_1$, $\mathscr{C}_2$ in a bipartite graph $G(\mathcal{R}, \mathcal{C}; E)$. Let $e(\mathscr{C})$ denote the set of edges in $\mathscr{C}$. Then the set $e(\mathscr{C}_1) \cup e(\mathscr{C}_2) - e(\mathscr{C}_1) \cap e(\mathscr{C}_2)$ is also a cycle.

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Definition

In a graph G, the distance between two vertices v_1 and v_2 denoted as $\text{dist}(v_1, v_2)$ is defined as the minimum number of edges between v_1 and v_2.

Graph with cycles but only non-constant plant edges: main result-II

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Define γ through the sets $\mathcal{C}_m$ and A as follows:

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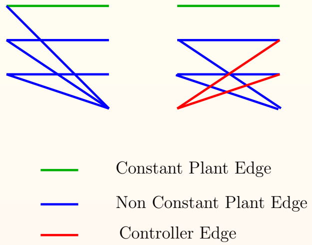

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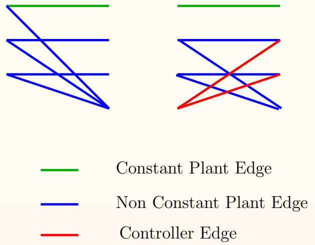

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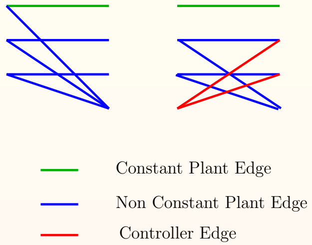

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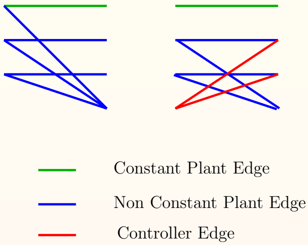

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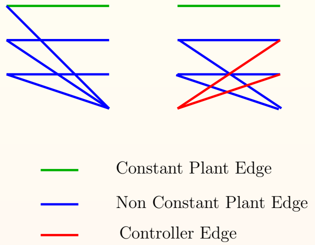

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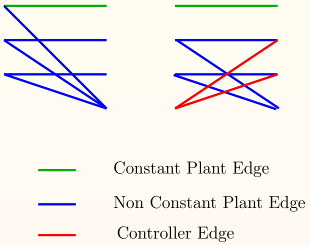

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Since there is a constant plant edge we cover only $\mathcal{R}_p^c$ vertices and hence $N_p = 1$ and consequently $k_{\min} = 2$.

Graph with cycles and constant plant edges: Main Result-III

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$\Longrightarrow$

1. SISO (Series cascade)
2. MISO
3. SIMO

- Consider three subsystems of the plant that are connected in each of the above cases.
- Assume each subsystem S_i has transfer function $\frac{n_i(s)}{d_i(s)}$. The differential equation for each S_i is $d(\frac{\mathrm{d}}{\mathrm{d}t})y_i = n(\frac{\mathrm{d}}{\mathrm{d}t})u_i$.
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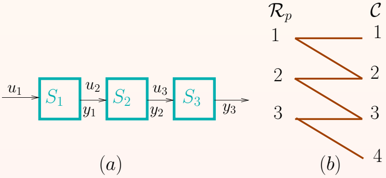

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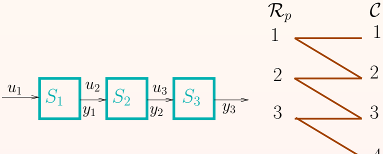

(a) (b)

$$P = \begin{bmatrix} * & * & & \\ & * & * & \\ & & * & * \end{bmatrix}$$

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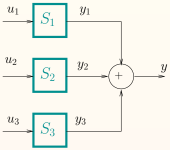
MISO



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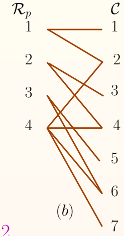

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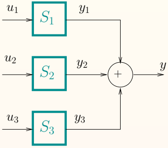
$e_{\text{mimo}} = 3 \, k_{\text{min}} = 6 \, N_p = 2$

MISO



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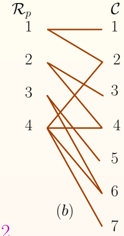

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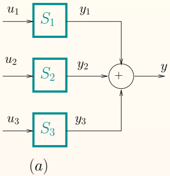

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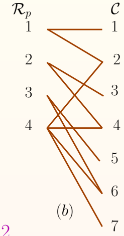

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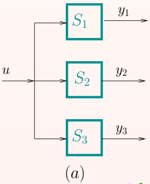
SIMO



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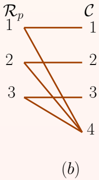

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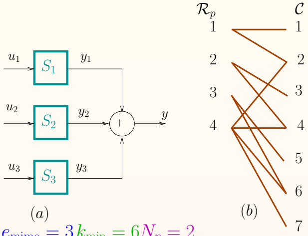

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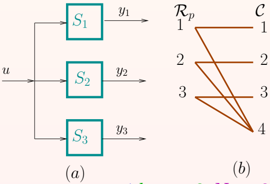

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Following lemma provides an upper and lower bounds for the index e_{mimo}.

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Then $1 \leqslant e_{\text{mimo}} \leqslant \min\,(d, 2p-1)$.

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- In the SISO and SIMO case, the lower bound of e_{mimo} is achieved.
- In the MISO case, the upper bound of e_{mimo} is achieved.
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  $\uparrow e_{\text{mimo}} \Rightarrow$ Prominence of MISO input-output structure.
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Effect of e_{mimo} and N_p on $k_{\min}$

Following lemma provides an upper and lower bounds for the index e_{mimo}.

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$G^p(\mathcal{R}_p, \mathcal{C}; E_p) :=$ controllable plant. Assume G^p is connected, has no cycles and has no constant plant edges. $d := |\mathcal{C}| - |\mathcal{R}_p|$.
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- No of controller edges for SIMO is more than SISO as N_p is more.
  Note e_{mimo} is same for both cases.

Main Result-IV

Assume a controllable plant with n subsystems.
Suppose they are interconnected with the input-output structure as series cascade: SISO, MISO and SIMO.

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Table I: SISO, MISO, SIMO: key parameters

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SISO	1	1		2
MISO	n	n even: $n/2$	n odd: $(n+1)/2$	$2n$
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- $N_p \uparrow \;\Rightarrow\; k_{\text{min}} \uparrow$. Due to need to 'feed back' more number of plant outputs or assign larger number of plant inputs.
- e_{mimo} is higher if the plant is more under-determined, i.e. more number of controller equations are required in order to make the closed loop system autonomous.
- In this sense, e_{mimo} is the extent of Multi-Input-Multi-Output structure within a system.

Conclusion

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- We related e_{mimo} and N_p with the input-output structure within the plant.
- Arbitrary pole placement is same as ensuring the polynomial matrix corresponding to the closed loop is square, nonsingular, and, in fact, unimodular. Thus we addressed the question of unimodular completion using the least number of nonzero entries in the completion.

Thank you!

Please write to us for further queries:

belur@ee.iitb.ac.in
rachel@ee.iitb.ac.in