# Application of structured linearization for efficient $\mathcal{H}_\infty$-norm computation

Madhu N. Belur

Indian Institute of Technology Bombay

16th January, 2013
NLAA 2013, Guwahati

# Outline

1. $\mathcal{H}_\infty$-norm definition
2. Current method
3. New method (using 'loss of dissipativity' property)
4. Bezoutian matrix
5. Time improvement

# H-infinity norm

For a transfer matrix $G(s) \in \mathbb{R}(s)^{p \times m}$ (with $p$ rows and $m$ columns, entries from $\mathbb{R}(s)$):

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\lambda \in \mathbb{C},\, \text{Real}\,(\lambda) \geqslant 0} \sigma_{\max}(G(\lambda))$$

where $\sigma_{\max}(P)$ is the maximum singular value of a constant matrix $P$.

# H-infinity norm

For a transfer matrix $G(s) \in \mathbb{R}(s)^{p \times m}$ (with $p$ rows and $m$ columns, entries from $\mathbb{R}(s)$):

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\lambda \in \mathbb{C},\, \mathrm{Real}\,(\lambda) \geqslant 0} \sigma_{\max}(G(\lambda))$$

where $\sigma_{\max}(P)$ is the maximum singular value of a constant matrix $P$.
When the supremum exists: attained on the imaginary axis $i\mathbb{R} \cup \{\infty\}$.

# H-infinity norm

For a transfer matrix $G(s) \in \mathbb{R}(s)^{p\times m}$ (with $p$ rows and $m$ columns, entries from $\mathbb{R}(s)$):

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\lambda\in\mathbb{C},\, \text{Real}\,(\lambda)\geqslant 0} \sigma_{\max}(G(\lambda))$$

where $\sigma_{\max}(P)$ is the maximum singular value of a constant matrix $P$.
When the supremum exists: attained on the imaginary axis $i\mathbb{R} \cup \{\infty\}$.
(Assume stability, 'properness', for existence of $\|G\|_{\mathcal{H}_\infty}$ )

# H-infinity norm

For a transfer matrix $G(s) \in \mathbb{R}(s)^{p\times m}$ (with $p$ rows and $m$ columns, entries from $\mathbb{R}(s)$):

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\lambda\in\mathbb{C},\, \text{Real}\,(\lambda)\geqslant 0} \sigma_{\max}(G(\lambda))$$

where $\sigma_{\max}(P)$ is the maximum singular value of a constant matrix $P$.

When the supremum exists: attained on the imaginary axis $i\mathbb{R}\cup\{\infty\}$.

(Assume stability, 'properness', for existence of $\|G\|_{\mathcal{H}_\infty}$ )

$\|G\|_{\mathcal{H}_\infty}$ plays an important role in robust control and disturbance attenuation problems.

# H-infinity norm

For a transfer matrix $G(s) \in \mathbb{R}(s)^{p \times m}$ (with $p$ rows and $m$ columns, entries from $\mathbb{R}(s)$):

$$\|G\|_{\mathcal{H}_\infty} := \sup_{\lambda \in \mathbb{C},\, \text{Real}\,(\lambda) \geqslant 0} \sigma_{\max}(G(\lambda))$$

where $\sigma_{\max}(P)$ is the maximum singular value of a constant matrix $P$.
When the supremum exists: attained on the imaginary axis $i\mathbb{R} \cup \{\infty\}$.
(Assume stability, 'properness', for existence of $\|G\|_{\mathcal{H}_\infty}$ )
$\|G\|_{\mathcal{H}_\infty}$ plays an important role in robust control and disturbance attenuation problems.
This talk: about computation of $\|G\|_{\mathcal{H}_\infty}$

# Conventional method to compute $\|G\|_{\mathcal{H}_\infty}$

Obtain a state space realization of $G(s)$: $(A, B, C, D)$
Assuming $\gamma > \sigma_{\max}(D)$

$$\text{Define:} \quad H(\gamma) := \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^TS^{-1}C & -A^T + C^TDR^{-1}B^T \end{bmatrix}$$

with $S := (\gamma^2 I - DD^T)$ and $R := (\gamma^2 I - D^TD)$,

# Conventional method to compute $\|G\|_{\mathcal{H}_\infty}$

Obtain a state space realization of $G(s)$: $(A, B, C, D)$
Assuming $\gamma > \sigma_{\max}(D)$

$$\text{Define:} \quad H(\gamma) := \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^TS^{-1}C & -A^T + C^TDR^{-1}B^T \end{bmatrix}$$

with $S := (\gamma^2 I - DD^T)$ and $R := (\gamma^2 I - D^TD)$,
and use (Boyd & Balakrishnan):

# Conventional method to compute $\|G\|_{\mathcal{H}_\infty}$

Obtain a state space realization of $G(s)$: $(A, B, C, D)$
Assuming $\gamma > \sigma_{\max}(D)$

$$\text{Define:} \quad H(\gamma) := \begin{bmatrix} A - BR^{-1}D^TC & -\gamma BR^{-1}B^T \\ \gamma C^TS^{-1}C & -A^T + C^TDR^{-1}B^T \end{bmatrix}$$

with $S := (\gamma^2 I - DD^T)$ and $R := (\gamma^2 I - D^TD)$,
and use (Boyd & Balakrishnan):

$$\gamma \leqslant \|G\|_{\mathcal{H}_\infty} \Leftrightarrow H(\gamma) \text{ has } i\mathbb{R} \text{ eigenvalues}$$

to iterate over $\gamma$ to find $\gamma = \|G\|_{\mathcal{H}_\infty}$ to required accuracy.
For each $\gamma$ value: solve an eigenvalue problem

# New method

Obtain a right coprime factorization of $G(s) = N(s)(D(s))^{-1}$, with $N(s)$ and $D(s)$ polynomial matrices

# New method

Obtain a right coprime factorization of $G(s) = N(s)(D(s))^{-1}$, with $N(s)$ and $D(s)$ polynomial matrices
Dissipativity theory gives:

$$P(\gamma, \omega) := D(-j\omega)^T D(j\omega) - \gamma^2 N(-j\omega)^T N(j\omega) \geqslant 0 \quad \text{for all } \omega \in \mathbb{R}$$
$$\Updownarrow$$

# New method

Obtain a right coprime factorization of $G(s) = N(s)(D(s))^{-1}$, with $N(s)$ and $D(s)$ polynomial matrices

Dissipativity theory gives:

$$P(\gamma, \omega) := D(-j\omega)^T D(j\omega) - \gamma^2 N(-j\omega)^T N(j\omega) \geqslant 0 \quad \text{for all } \omega \in \mathbb{R}$$

$$\Updownarrow$$

$$\gamma \geqslant \|G\|_{\mathcal{H}_\infty}$$

# New method

Obtain a right coprime factorization of $G(s) = N(s)(D(s))^{-1}$, with $N(s)$ and $D(s)$ polynomial matrices
Dissipativity theory gives:

$$P(\gamma,\omega) := D(-j\omega)^T D(j\omega) - \gamma^2 N(-j\omega)^T N(j\omega) \geqslant 0 \quad \text{for all } \omega \in \mathbb{R}$$
$$\Updownarrow$$
$$\gamma \geqslant \|G\|_{\mathcal{H}_\infty}$$

Thus
$\|G\|_{\mathcal{H}_\infty} = \min\{\gamma \mid \gamma \text{ satisfies } P(\gamma,\omega) \geqslant 0 \text{ as above}\}$
Define $p_\gamma(\omega) := \det\, P(\gamma,\omega)$
Look for those $\gamma$'s that make $p_\gamma(\omega)$ and $\frac{\partial p}{\partial \omega} =: q_\gamma(\omega)$ noncoprime

# New method

Obtain a right coprime factorization of $G(s) = N(s)(D(s))^{-1}$, with $N(s)$ and $D(s)$ polynomial matrices
Dissipativity theory gives:

$$P(\gamma,\omega) := D(-j\omega)^T D(j\omega) - \gamma^2 N(-j\omega)^T N(j\omega) \geqslant 0 \quad \text{for all } \omega \in \mathbb{R}$$
$$\Updownarrow$$
$$\gamma \geqslant \|G\|_{\mathcal{H}_\infty}$$

Thus
$\|G\|_{\mathcal{H}_\infty} = \min\{\gamma \mid \gamma \text{ satisfies } P(\gamma,\omega) \geqslant 0 \text{ as above}\}$
Define $p_\gamma(\omega) := \det P(\gamma,\omega)$
Look for those $\gamma$'s that make $p_\gamma(\omega)$ and $\frac{\partial p}{\partial \omega} =: q_\gamma(\omega)$ noncoprime
Discriminant / Resultant results applicable now
Bezoutian matrix better than Sylvester matrix

# Bezoutian of two polynomials

Consider polynomials $p(\omega)$ and $q(\omega)$.
Bezoutian polynomial $b(\zeta,\eta)$ and matrix $B$ are defined as

$$b(\zeta,\eta) := \frac{p(\zeta)q(\eta) - p(\eta)q(\zeta)}{\zeta - \eta} = \begin{bmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^{n-1} \end{bmatrix}^T B \begin{bmatrix} 1 \\ \eta \\ \vdots \\ \eta^{n-1} \end{bmatrix}$$

with $n = \max(\deg p, \deg q)$.
$B$ is a symmetric matrix

# Bezoutian of two polynomials

Consider polynomials $p(\omega)$ and $q(\omega)$.
Bezoutian polynomial $b(\zeta, \eta)$ and matrix $B$ are defined as

$$b(\zeta, \eta) := \frac{p(\zeta)q(\eta) - p(\eta)q(\zeta)}{\zeta - \eta} = \begin{bmatrix} 1 \\ \zeta \\ \vdots \\ \zeta^{n-1} \end{bmatrix}^T B \begin{bmatrix} 1 \\ \eta \\ \vdots \\ \eta^{n-1} \end{bmatrix}$$

with $n = \max(\deg p, \deg q)$.
$B$ is a symmetric matrix
But not the Sylvester resultant matrix $S$

# Construction of $b(\zeta, \eta)$ and $B$

Let $p(\xi) = p_0 + p_1\xi + p_2\xi^2 \cdots + p_n\xi^n$ and
$q(\xi) = q_0 + q_1\xi + q_2\xi^2 \cdots + q_m\xi^m$ and
$S$, the Sylvester resultant matrix is $(n+m) \times (n+m)$

# Construction of $b(\zeta, \eta)$ and $B$

Let $p(\xi) = p_0 + p_1\xi + p_2\xi^2 \cdots + p_n\xi^n$ and
$q(\xi) = q_0 + q_1\xi + q_2\xi^2 \cdots + q_m\xi^m$ and
$S$, the Sylvester resultant matrix is $(n+m) \times (n+m)$
$S$ is singular if and only if $p$ and $q$ have a common root

# Construction of $b(\zeta, \eta)$ and $B$

Let $p(\xi) = p_0 + p_1\xi + p_2\xi^2 \cdots + p_n\xi^n$ and
$q(\xi) = q_0 + q_1\xi + q_2\xi^2 \cdots + q_m\xi^m$ and
$S$, the Sylvester resultant matrix is $(n+m) \times (n+m)$
$S$ is singular if and only if $p$ and $q$ have a common root
$S$ loses rank $d$ if and only if gcd of $p$ and $q$ has degree $d$.
But

# Construction of $b(\zeta, \eta)$ and $B$

Let $p(\xi) = p_0 + p_1\xi + p_2\xi^2 \cdots + p_n\xi^n$ and
$q(\xi) = q_0 + q_1\xi + q_2\xi^2 \cdots + q_m\xi^m$ and
$S$, the Sylvester resultant matrix is $(n+m) \times (n+m)$
$S$ is singular if and only if $p$ and $q$ have a common root
$S$ loses rank $d$ if and only if gcd of $p$ and $q$ has degree $d$.
But $S$ is quite 'abnormal'

# Construction of $b(\zeta, \eta)$ and $B$

Let $p(\xi) = p_0 + p_1\xi + p_2\xi^2 \cdots + p_n\xi^n$ and
$q(\xi) = q_0 + q_1\xi + q_2\xi^2 \cdots + q_m\xi^m$ and
$S$, the Sylvester resultant matrix is $(n+m) \times (n+m)$
$S$ is singular if and only if $p$ and $q$ have a common root
$S$ loses rank $d$ if and only if gcd of $p$ and $q$ has degree $d$.
But $S$ is quite 'abnormal' ($A$ is called normal if $AA^T = A^TA$)

# Construction of $b(\zeta, \eta)$ and $B$

Let $p(\xi) = p_0 + p_1\xi + p_2\xi^2 \cdots + p_n\xi^n$ and
$q(\xi) = q_0 + q_1\xi + q_2\xi^2 \cdots + q_m\xi^m$ and
$S$, the Sylvester resultant matrix is $(n+m) \times (n+m)$
$S$ is singular if and only if $p$ and $q$ have a common root
$S$ loses rank $d$ if and only if gcd of $p$ and $q$ has degree $d$.
But $S$ is quite 'abnormal' ($A$ is called normal if $AA^T = A^TA$)
At least the zero eigenvalue of $S$ (when $S$ is singular) is ill-conditioned.
(Left and right eigenvectors are same $\equiv$ that eigenvalue is well-conditioned.)

# Construction of $b(\zeta, \eta)$ and $B$

Let $p(\xi) = p_0 + p_1\xi + p_2\xi^2 \cdots + p_n\xi^n$ and
$q(\xi) = q_0 + q_1\xi + q_2\xi^2 \cdots + q_m\xi^m$ and
$S$, the Sylvester resultant matrix is $(n+m) \times (n+m)$
$S$ is singular if and only if $p$ and $q$ have a common root
$S$ loses rank $d$ if and only if gcd of $p$ and $q$ has degree $d$.
But $S$ is quite 'abnormal' ($A$ is called normal if $AA^T = A^TA$)
At least the zero eigenvalue of $S$ (when $S$ is singular) is ill-conditioned.
(Left and right eigenvectors are same $\equiv$ that eigenvalue is well-conditioned.)
But Bezoutian matrix: symmetric

# Bezoutian polynomial and matrix

Let degree of $p = n \geqslant$ degree of $q = m$.

# Bezoutian polynomial and matrix

Let degree of $p = n \geqslant$ degree of $q = m$.
Degree of $b(\zeta, \eta)$ is $n - 1$ in each of $\zeta$ and $\eta$ and

# Bezoutian polynomial and matrix

Let degree of $p = n \geqslant$ degree of $q = m$.
Degree of $b(\zeta, \eta)$ is $n - 1$ in each of $\zeta$ and $\eta$ and matrix $B$ is $n \times n$.

# Bezoutian polynomial and matrix

Let degree of $p = n \geqslant$ degree of $q = m$.
Degree of $b(\zeta, \eta)$ is $n-1$ in each of $\zeta$ and $\eta$ and matrix $B$ is $n \times n$.
$b(\zeta, \eta)$ explicit calculation quite easy to program (in Scilab):
Let $b(\zeta, \eta) = b_0(\zeta) + \eta b_1(\zeta) + \eta^2 b_2(\zeta) \cdots \eta^{n-1} b_{n-1}(\zeta)$
Equate terms with equal degree in $\eta$ in

$$(\zeta - \eta) b(\zeta, \eta) = p(\zeta) q(\eta) - q(\zeta) p(\eta) \text{ to get}$$

$b_0(\zeta) := (q_0 p(\zeta) - p_0 q(\zeta))/\zeta$

# Bezoutian polynomial and matrix

Let degree of $p = n \geqslant$ degree of $q = m$.
Degree of $b(\zeta, \eta)$ is $n-1$ in each of $\zeta$ and $\eta$ and matrix $B$ is $n \times n$.
$b(\zeta, \eta)$ explicit calculation quite easy to program (in Scilab):
Let $b(\zeta, \eta) = b_0(\zeta) + \eta b_1(\zeta) + \eta^2 b_2(\zeta) \cdots \eta^{n-1} b_{n-1}(\zeta)$
Equate terms with equal degree in $\eta$ in

$$(\zeta - \eta) b(\zeta, \eta) = p(\zeta)q(\eta) - q(\zeta)p(\eta) \text{ to get}$$

$b_0(\zeta) := (q_0 p(\zeta) - p_0 q(\zeta))/\zeta$
$b_1(\zeta) := (q_1 p(\zeta) - p_1 q(\zeta) + b_0(\zeta))/\zeta$

# Bezoutian polynomial and matrix

Let degree of $p = n \geqslant$ degree of $q = m$.
Degree of $b(\zeta, \eta)$ is $n-1$ in each of $\zeta$ and $\eta$ and matrix $B$ is $n \times n$.
$b(\zeta, \eta)$ explicit calculation quite easy to program (in Scilab):
Let $b(\zeta, \eta) = b_0(\zeta) + \eta b_1(\zeta) + \eta^2 b_2(\zeta) \cdots \eta^{n-1} b_{n-1}(\zeta)$
Equate terms with equal degree in $\eta$ in

$$(\zeta - \eta) b(\zeta, \eta) = p(\zeta) q(\eta) - q(\zeta) p(\eta) \text{ to get}$$

$b_0(\zeta) := (q_0 p(\zeta) - p_0 q(\zeta))/\zeta$
$b_1(\zeta) := (q_1 p(\zeta) - p_1 q(\zeta) + b_0(\zeta))/\zeta$
$\vdots$
$b_i(\zeta) := (q_i p(\zeta) - p_i q(\zeta) + b_{i-1}(\zeta))/\zeta$

# Bezoutian matrix

Notice that $b(\zeta, \eta) = b(\eta, \zeta)$

# Bezoutian matrix

Notice that $b(\zeta, \eta) = b(\eta, \zeta)$
Once $b(\zeta, \eta)$ is found, define $B_{ij}$ = coefficient of $\zeta^i \eta^j$

# Bezoutian matrix

Notice that $b(\zeta, \eta) = b(\eta, \zeta)$
Once $b(\zeta, \eta)$ is found, define $B_{ij}$ = coefficient of $\zeta^i \eta^j$
Due to constant term,
define $B_{ij}$ = coefficient of $\zeta^{i-1} \eta^{j-1}$

# Bezoutian matrix

Notice that $b(\zeta, \eta) = b(\eta, \zeta)$
Once $b(\zeta, \eta)$ is found, define $B_{ij} =$ coefficient of $\zeta^i \eta^j$
Due to constant term,
define $B_{ij} =$ coefficient of $\zeta^{i-1} \eta^{j-1}$

# Example

Let $p(s) = 2 + 3s + s^2$ and $q_a(s) = a + s$

Sylvester resultant matrix $S = \begin{bmatrix} 2 & 3 & 1 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$

det $S = a^2 - 3a + 2$ :

# Example

Let $p(s) = 2 + 3s + s^2$ and $q_a(s) = a + s$

Sylvester resultant matrix $S = \begin{bmatrix} 2 & 3 & 1 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$

det $S = a^2 - 3a + 2$ : $p$ and $q$ are noncoprime $\Leftrightarrow a = 1, 2$

# Example

Let $p(s) = 2 + 3s + s^2$ and $q_a(s) = a + s$

Sylvester resultant matrix $S = \begin{bmatrix} 2 & 3 & 1 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$

det $S = a^2 - 3a + 2$ : $p$ and $q$ are noncoprime $\Leftrightarrow a = 1, 2$

$b(\zeta, \eta) = \zeta\eta + a(\zeta + \eta) + 3a - 2$ and hence

# Example

Let $p(s) = 2 + 3s + s^2$ and $q_a(s) = a + s$

Sylvester resultant matrix $S = \begin{bmatrix} 2 & 3 & 1 \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$

$\det S = a^2 - 3a + 2$ : $p$ and $q$ are noncoprime $\Leftrightarrow a = 1, 2$

$b(\zeta, \eta) = \zeta\eta + a(\zeta + \eta) + 3a - 2$ and hence

$B_a = \begin{bmatrix} 1 & a \\ a & (3a - 2) \end{bmatrix}$ with determinant $-(a^2 - 3a + 2)$

# $\mathcal{H}_\infty$ norm computation: Bezoutian polynomial

In our case,

# $\mathcal{H}_\infty$ norm computation: Bezoutian polynomial

In our case, coefficients of $p(\omega)$ and $q(\omega)$ are polynomials in $\gamma$.

# $\mathcal{H}_\infty$ norm computation: Bezoutian polynomial

In our case, coefficients of $p(\omega)$ and $q(\omega)$ are polynomials in $\gamma$. The polynomial matrix $B(\gamma)$ has each coefficient: a symmetric constant matrix:

$$B(\gamma) = B_0 + \gamma B_1 + \cdots + \gamma^m B_m, \text{ say.}$$

# $\mathcal{H}_\infty$ norm computation: Bezoutian polynomial

In our case, coefficients of $p(\omega)$ and $q(\omega)$ are polynomials in $\gamma$. The polynomial matrix $B(\gamma)$ has each coefficient: a symmetric constant matrix:

$$B(\gamma) = B_0 + \gamma B_1 + \cdots + \gamma^m B_m, \text{ say.}$$

Roots of det $B(\gamma)$ cause $p_\gamma$ and $q_\gamma$ to have a common root: $\omega_0$

# $\mathcal{H}_\infty$ norm computation: Bezoutian polynomial

In our case, coefficients of $p(\omega)$ and $q(\omega)$ are polynomials in $\gamma$. The polynomial matrix $B(\gamma)$ has each coefficient: a symmetric constant matrix:

$$B(\gamma) = B_0 + \gamma B_1 + \cdots + \gamma^m B_m, \text{ say.}$$

Roots of det $B(\gamma)$ cause $p_\gamma$ and $q_\gamma$ to have a common root: $\omega_0$
$\omega_0$ is that eigenvalue of the Hamiltonian matrix at the maximum $\gamma$ that satisfies

# $\mathcal{H}_\infty$ norm computation: Bezoutian polynomial

In our case, coefficients of $p(\omega)$ and $q(\omega)$ are polynomials in $\gamma$.
The polynomial matrix $B(\gamma)$ has each coefficient: a symmetric constant matrix:

$$B(\gamma) = B_0 + \gamma B_1 + \cdots + \gamma^m B_m, \text{ say.}$$

Roots of det $B(\gamma)$ cause $p_\gamma$ and $q_\gamma$ to have a common root: $\omega_0$
$\omega_0$ is that eigenvalue of the Hamiltonian matrix at the maximum $\gamma$ that satisfies
Hamiltonian matrix $H(\gamma)$ has imaginary axis eigenvalues.

# $\mathcal{H}_\infty$ norm computation: Bezoutian polynomial

In our case, coefficients of $p(\omega)$ and $q(\omega)$ are polynomials in $\gamma$.
The polynomial matrix $B(\gamma)$ has each coefficient: a symmetric constant matrix:

$$B(\gamma) = B_0 + \gamma B_1 + \cdots + \gamma^m B_m, \text{ say.}$$

Roots of det $B(\gamma)$ cause $p_\gamma$ and $q_\gamma$ to have a common root: $\omega_0$
$\omega_0$ is that eigenvalue of the Hamiltonian matrix at the maximum $\gamma$ that satisfies
Hamiltonian matrix $H(\gamma)$ has imaginary axis eigenvalues.
Recall that

$$\gamma \leqslant \|G\|_{\mathcal{H}_\infty} \Leftrightarrow H(\gamma) \text{ has } i\mathbb{R} \text{ eigenvalues}$$

# Structured linearization

Define symmetric matrices $E$ and $A$ as

$$E := \begin{bmatrix} B_m & & & & \\ & -B_{m-2} & \cdots & -B_1 & -B_0 \\ & \vdots & \iddots & \iddots & 0 \\ & -B_1 & \iddots & & \\ & -B_0 & 0 & & \end{bmatrix} \qquad A := \begin{bmatrix} -B_{m-1} & \cdots & -B_1 & -B_0 \\ \vdots & \iddots & \iddots & 0 \\ -B_1 & \iddots & & \\ -B_0 & 0 & & \end{bmatrix}.$$

# Structured linearization

Define symmetric matrices $E$ and $A$ as

$$E := \begin{bmatrix} B_m & & & & \\ & -B_{m-2} & \cdots & -B_1 & -B_0 \\ & \vdots & \iddots & \iddots & 0 \\ & -B_1 & \iddots & & \\ & -B_0 & 0 & & \end{bmatrix} \qquad A := \begin{bmatrix} -B_{m-1} & \cdots & -B_1 & -B_0 \\ \vdots & \iddots & \iddots & 0 \\ -B_1 & \iddots & & \\ -B_0 & 0 & & \end{bmatrix}.$$

Find generalized eigenvalues of the pair $(E, A)$: the sought $\|G\|_{\mathcal{H}_\infty}$ is one of these eigenvalues.

# Structured linearization

Define symmetric matrices $E$ and $A$ as

$$E := \begin{bmatrix} B_m & & & & \\ & -B_{m-2} & \cdots & -B_1 & -B_0 \\ & \vdots & \iddots & \iddots & 0 \\ & -B_1 & \iddots & & \\ & -B_0 & 0 & & \end{bmatrix} \qquad A := \begin{bmatrix} -B_{m-1} & \cdots & -B_1 & -B_0 \\ \vdots & \iddots & \iddots & 0 \\ -B_1 & \iddots & & \\ -B_0 & 0 & & \end{bmatrix}.$$

Find generalized eigenvalues of the pair $(E, A)$: the sought $\|G\|_{\mathcal{H}_\infty}$ is one of these eigenvalues.

Note:

- due to Bezoutian matrix reasons:
- $B_m$ is nonsingular
- Pencil is a regular pencil

# Structured linearization

Define symmetric matrices $E$ and $A$ as

$$E := \begin{bmatrix} B_m & & & & \\ & -B_{m-2} & \cdots & -B_1 & -B_0 \\ & \vdots & \iddots & \iddots & 0 \\ & -B_1 & \iddots & & \\ & -B_0 & 0 & & \end{bmatrix} \qquad A := \begin{bmatrix} -B_{m-1} & \cdots & -B_1 & -B_0 \\ \vdots & \iddots & \iddots & 0 \\ -B_1 & \iddots & & \\ -B_0 & 0 & & \end{bmatrix}.$$

Find generalized eigenvalues of the pair $(E, A)$: the sought $\|G\|_{\mathcal{H}_\infty}$ is one of these eigenvalues.
Note:

- due to Bezoutian matrix reasons:
- $B_m$ is nonsingular
- Pencil is a regular pencil
- But need to check that common root is on $i\mathbb{R}$, maximum such $\gamma$, etc.

Program implemented in Scilab (and code on github)

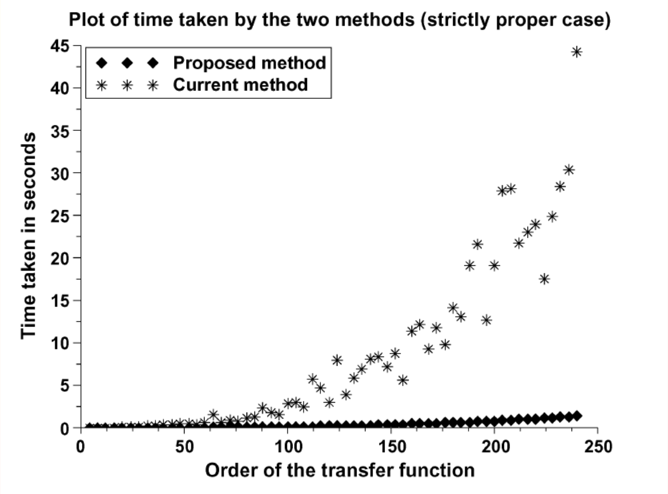


Figure: Plot from Belur & Praagman, IEEE-TAC, 2011

# Comparison of two methods (small order)

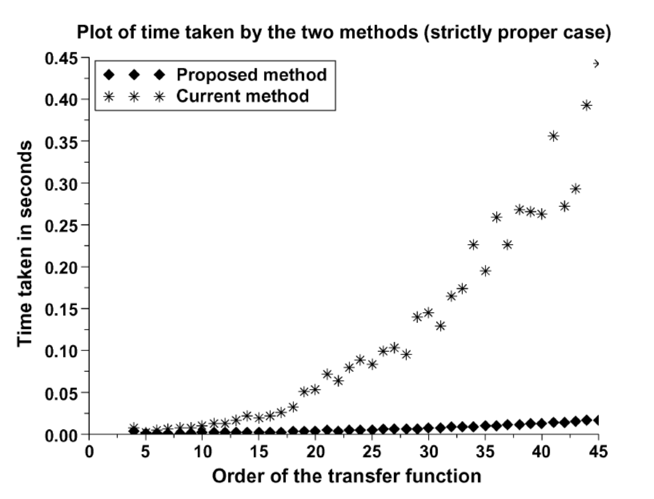


Figure: Plot from Belur & Praagman, IEEE-TAC, 2011

# Conclusion

- $\mathcal{H}_\infty$ norm computation is important in Systems & Control
- Hamiltonian matrix arguments have proven quadratic convergence rates (Bruinsma & Steinbuch)

# Conclusion

- $\mathcal{H}_\infty$ norm computation is important in Systems & Control
- Hamiltonian matrix arguments have proven quadratic convergence rates (Bruinsma & Steinbuch)
- But requires an eigenvalue iteration within $\gamma$ iteration

# Conclusion

- $\mathcal{H}_\infty$ norm computation is important in Systems & Control
- Hamiltonian matrix arguments have proven quadratic convergence rates (Bruinsma & Steinbuch)
- But requires an eigenvalue iteration within $\gamma$ iteration
- Sylvester matrix resultant (and block companion linearization): marginally better

# Conclusion

- $\mathcal{H}_\infty$ norm computation is important in Systems & Control
- Hamiltonian matrix arguments have proven quadratic convergence rates (Bruinsma & Steinbuch)
- But requires an eigenvalue iteration within $\gamma$ iteration
- Sylvester matrix resultant (and block companion linearization): marginally better
- Proposed method is much faster due to symmetric linearization:

# Conclusion

- $\mathcal{H}_\infty$ norm computation is important in Systems & Control
- Hamiltonian matrix arguments have proven quadratic convergence rates (Bruinsma & Steinbuch)
- But requires an eigenvalue iteration within $\gamma$ iteration
- Sylvester matrix resultant (and block companion linearization): marginally better
- Proposed method is much faster due to symmetric linearization:
  20 to 40 times faster.

# Conclusion

- $\mathcal{H}_\infty$ norm computation is important in Systems & Control
- Hamiltonian matrix arguments have proven quadratic convergence rates (Bruinsma & Steinbuch)
- But requires an eigenvalue iteration within $\gamma$ iteration
- Sylvester matrix resultant (and block companion linearization): marginally better
- Proposed method is much faster due to symmetric linearization:
  20 to 40 times faster. Improvement further better for higher orders

# Acknowledgements


Dr. Bibhas Adhikari
Dr. Swanand Khare
Dr. Cornelis Praagman