Passivity preserving MOR

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Outline

- **Behavioral view**
- **Dissipativity, passivity**
- **Model order reduction problem formulation**

State space, transfer functions, behavioral approach

- **We have had input/output models (transfer function)**
- **Then, we have state space**
- **And now, behavioral approach**

Are they 'competing'?

Multiple views can only help

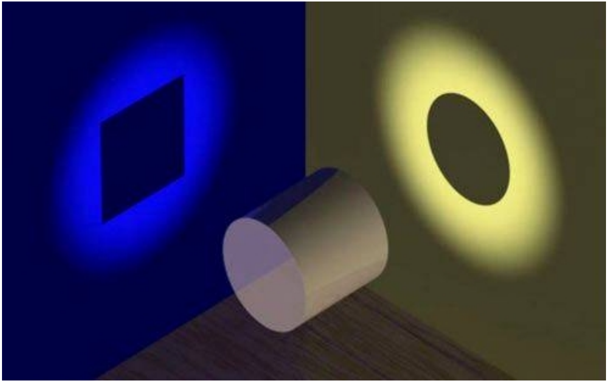

Figure : Source unknown, shared by Waghulde

- **Input/output classification of variables often un-natural. (Resistor, capacitor, spring, mass, damper)**
- **System $\equiv$ signal processor: input/output ideal**
- **Causality also helps classify**

Dissipativity

- **Energy exchange not necessarily linked to input/output classification**
- **Dissipativity studies: since early 1970s**
- **Behavioral approach: $\sim$ 1987**
- **Riccati equations: easier to follow**
- **Key work by Megretski and Rantzer on Integral Quadratic Constraints**

Throughout this lecture:

$\mathtt{m}$: **number of inputs,**	$\mathtt{p}$: **number of outputs**
$\mathtt{w}$: **number of 'manifest' variables:**	**typically** $\mathtt{m} + \mathtt{p}$
$\mathtt{n}$: **(minimum) number of states**	**(McMillan degree)**

$$G(s) \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}(s), \quad G(s) = P(s)^{-1}Q(s) = V(s)U(s)^{-1}$$

with $P, Q, U, V \in \mathbb{R}^{\bullet\times\bullet}[s]$**. More precisely,** $P, Q \in \mathbb{R}^{\mathtt{p}\times\bullet}[s]$ **and** $U, V \in \mathbb{R}^{\bullet\times\mathtt{m}}[s]$**.**

- **A 'system' is nothing but the set of trajectories that the system allows.**
- **The system 'behavior' is the set of allowed trajectories, i.e. those that the system laws allow. Suppose the system variables are w.**

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid w \text{ satisfies the system laws } \}.$$

- **$\mathfrak{C}^\infty$: trajectory is infinitely often differentiable: primarily for convenience.**
- **Some notions do depend on the signal space used. $\mathfrak{L}^1_{\text{loc}}$ is another frequently used space: this includes step, ramp and other such signals.**
- **For dissipativity-preserving model order reduction, $\mathfrak{C}^\infty$ is fine.**

Dissipative systems

For the rest of the talk, assume the system is controllable. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$

Power$(w) := w^T \Sigma w$, **with** $\Sigma = \Sigma^T \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$: **supply rate**

$\mathfrak{B}$ **is called dissipative with respect to supply rate** $w^T \Sigma w$ **if**

$$\int_{-\infty}^{\infty} w^T \Sigma w dt \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Along any system trajectory (starting from rest and ending at rest), 'net energy' is absorbed.

Integral inequality insisted only on $\mathfrak{B} \cap \mathfrak{D}$**: denseness issues related to controllability.**

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Ignoring stability aspects (for this slide):

- $G(s)$ is positive real $\Leftrightarrow \Sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $w^T \Sigma w = 2uy$
- $G(s)$ has $\mathcal{L}_\infty$ norm at most $\gamma \Leftrightarrow w^T \Sigma w = \gamma^2 u^2 - y^2$.
- In LQ control, $w^T \Sigma w = x^T Q x + u^T R u$
- $y = \phi(u)$, and ϕ is a 'sector' nonlinearity, $\phi \in$ sector (α, β):

$$(y - \alpha u)(u - \frac{y}{\beta}) = \begin{bmatrix} u \\ y \end{bmatrix} \begin{bmatrix} -\alpha & \frac{(\alpha+\beta)}{2\beta} \\ \frac{(\alpha+\beta)}{2\beta} & \frac{-1}{\beta} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} \geqslant 0.$$

- Popov criteria, involving 'dynamic' notions of power

Interconnection of Σ-dissipative and $-\Sigma$-dissipative systems yields stability: Megretski & Rantzer: IQC paper

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A function $x^T K x$ is called a storage function if

$$\frac{d}{dt} x^T K x \leqslant w^T \Sigma w$$

$Q_\Psi(w)$ (if x can be expressed in terms of w and/or its derivatives)
All storage functions Q_Ψ satisfy

$$Q_{\Psi_{\min}}(w) \leqslant Q_\Psi(w) \leqslant Q_{\Psi_{\max}}(w).$$

The maximum and minimum storage functions satisfy a neat interpretation. Consider expressing the stored energy in terms of the state variable x. Accordingly, $Q_\Psi(w) = x^T K x$, say. Let $\mathfrak{B}_a$ denote all trajectories in $\mathfrak{B}$ such that at $t = 0$, the trajectory w has state $x = a \in \mathbb{R}^n$. Then,

$$Q_{\Psi_{\max}}(w)(0) = a^T K_{\max} a = \inf_{w \in \mathfrak{B}_a \cap \mathfrak{D}} \int_{-\infty}^{0} w^T \Sigma w dt \qquad (1)$$

and

$$Q_{\Psi_{\min}}(w)(0) = a^T K_{\min} a = \sup_{w \in \mathfrak{B}_a \cap \mathfrak{D}} \int_{0}^{\infty} -w^T \Sigma w dt \qquad (2)$$

$$A^T K + KA + Q - KBR^{-1}B^T K = 0$$

$$\int_{-\infty}^{\infty} w^T \Sigma w dt = 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

Several passivity preserving model order reduction methods.
Primary contribution: PRIMA: Odabasioglu, Celik, Pileggi (in 1988: IEEE Tran. CAD of Integrated Circuits and Systems)
SCL: Antoulas-05 and Sorensen-05: interpolation interpretation
'retain' (a lower dimensional subspace of) the set of trajectories of 'minimal dissipation' (Trentelman, Minh and Rapisarda)
Consider a nonsingular $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ and suppose $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ is $\boldsymbol{\Sigma}$-dissipative.
For a $w \in \mathfrak{B}$, consider the change $J_w(\delta)$ in dissipation [1]
If w is changed to $w+\delta$, for $\delta \in \mathfrak{B} \cap \mathfrak{D}$:

$$J_w(\delta) := \int_{-\infty}^{\infty} (Q_{\Delta}(w+\delta) - Q_{\Delta}(w))dt.$$

A trajectory $w \in \mathfrak{B}$ is said to be a trajectory of minimal dissipation if $J_w(\delta) \geqslant 0$ for all $\delta \in \mathfrak{B} \cap \mathfrak{D}$.
Any small change in w causes increase of net dissipated energy: in that sense, these are local minima.
The link between the set of trajectories (in a $\boldsymbol{\Sigma}$-dissipative behavior $\mathfrak{B}$) of minimal dissipation (denoted by $\mathfrak{B}^*$) and

Supply rate	LMI	P (the dissipation at ∞ frequency)
$\begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$	$\begin{bmatrix} A^TK+KA-Q & KB \\ B^TK & -R \end{bmatrix}$	R
$\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$	$\begin{bmatrix} A^TK+KA & KB-C^T \\ B^TK-C & -(D+D^T) \end{bmatrix}$	$(D+D^T)$
$\begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix}$	$\begin{bmatrix} A^TK+KA+C^TC & KB+C^TD \\ D^TC+B^TK & D^TD-\gamma^2 I \end{bmatrix}$	$(\gamma^2 I - D^TD)$

For behaviors $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^w)$

The orthogonal complement of $\mathfrak{B}$ (in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$): $\mathfrak{B}^\perp$
Adjoint system, dual system, co-state dynamics
(Dual Riccati: for the dual system)
Suppose $\Sigma \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ is symmetric and nonsingular. $\mathfrak{B}^{\perp_\Sigma}$ just 'normalization' w.r.t. Σ.

- **Number of inputs of $\mathfrak{B}$ (i.e. $\mathtt{m}(\mathfrak{B})$): column rank of $M(\xi)$ (Image representation $w = M(\frac{d}{dt})\ell$)**
- **$\mathfrak{B}$ is Σ-dissipative $\Rightarrow \mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$**
- **$\mathfrak{B} + \mathfrak{B}^\perp = \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, though not direct sum**
- **$\mathfrak{B} \cap \mathfrak{B}^\perp \neq \{0\}$, the intersection is 'thin'**
- **Intersection is autonomous, i.e. finite dimensional**
- **$\mathfrak{B} \cap \mathfrak{B}^\perp \cap \mathfrak{D} = \{0\}$**
- **Intersection has dynamics $\frac{d}{dt}x = Hx$, for a Hamiltonian matrix H.**
- **(H is called Hamiltonian if $H^T \sim -H$)**

Intersection: central role in model-order reduction (dissipativity preserving)

More generally, $\mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} =: \mathfrak{B}^*$

- $\mathfrak{B}^*$: trajectories in $\mathfrak{B}$ of 'minimal dissipation' (Trentelman, Minh & Rapisarda: MCSS 2009)
- Retain a lower dimension of $\mathfrak{B}^*$ into the reduced order model
- Restriction on $\mathfrak{B}$ (to $\mathfrak{B}^*$) can also be achieved by forcing ℓ to satisfy equations (instead of free/generic):

$$w = M(\frac{d}{dt})\ell \quad \text{and} \quad \partial\Phi'(\frac{d}{dt})\ell = 0.$$

- (In general, view additional laws as a controller: feedback? controller)
- Is intersection autonomous? Is $\det \partial\Phi'(\xi) \equiv 0$?

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- **Many papers in the literature: Feldman, Freund (1995, 1999 IEEE-TAC), Ober (1998, SIAM Con & Opt), PRIMA**
- **Based on positive real balancing**
- **'Simultaneous diagonalization': similarity transformation or congruence transformation?**
  $A \to S^{-1}AS$ or $P \to S^T PS$
- **Note that for storage $x^T Kx$, state space coordinate transformation due to S means storage $z^T(S^T KS)z$.**
- **Find coordinate transformation such that max/min of ARE/Dual-ARE solutions are 'balanced' (Simultaneously diagonalized: Antoulas, SIAM 2004 book)**
- **In this course, passivity preserving model reduction by**
  - **'interpolation at spectral zeros' (Antoulas/Sorensen: SCL 2005)**
  - **preserving trajectories of minimal dissipation (Minh, Trentelman, Rapisarda: MCSS 2009)**

Problem formulation

Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and symmetric nonsingular $\Sigma \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$
Suppose $\mathfrak{B}$ is strictly Σ-dissipative and suppose $\mathtt{n}$ is the McMillan degree of $\mathfrak{B}$
(McMillan degree: model order: minimum number of states)
Choose $k < \mathtt{n}$. Find $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that

1. **$\hat{\mathfrak{B}}$ has McMillan degree at most k**
2. $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$
3. **$\hat{\mathfrak{B}}$ is also strictly Σ-dissipative**
4. **$\hat{\mathfrak{B}}$ satisfies $\hat{\mathfrak{B}}^* \subset \mathfrak{B}^*$**

(Fourth point: trajectories in $\mathfrak{B}$ of minimal dissipation are retained into $\hat{\mathfrak{B}}$)
(Problem formulation correct except for stability aspect)

Half-line dissipativity: $\Sigma = \Sigma^T \in \mathbb{R}^{w \times w}$

- **Recall a behavior $\mathfrak{B} \in \mathfrak{L}^{w}_{\text{cont}}$ was called Σ-dissipative if**

$$\int_{\mathbb{R}} w^T \Sigma w dt \geqslant 0 \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

- **Call $\mathfrak{B}$ dissipative on $\mathbb{R}_-$ if for all $w \in \mathfrak{B} \cap \mathfrak{D}$ and for all T**

$$\int_{-\infty}^{T} w^T \Sigma w dt \geqslant 0.$$

('bounded from below')
(like physical storage)

- and on $\mathbb{R}_+$ if $\int_T^{\infty} w^T \Sigma w dt \geqslant 0$.
- dissipative $\Leftrightarrow$ $\exists$ storage function $Q_\Psi(w)$
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When supply rate Σ equals $\gamma^2 u^T u - y^T y$ and for system with input u and output y (Case of maximal input cardinality: $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$)

- **dissipativity on $\mathbb{R}_-$ $\Leftrightarrow$ transfer matrix is stable (no poles in CRHP)**
- **$\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ $\Leftrightarrow$ $\mathfrak{B}^{\perp_\Sigma}$ is $-\Sigma$-dissipative on $\mathbb{R}_+$**

Dissipativity on $\mathbb{R}_-$ $\Leftrightarrow$ maximum storage function
$Q_{\Psi_{\max}}(w) \geqslant 0$ (i.e. $K_{\max} \geqslant 0$)
($Q_{\Psi_{\max}}(w)$: 'required supply')

Dissipativity on $\mathbb{R}_+$ $\Leftrightarrow$ minimum storage function
$Q_{\Psi_{\min}}(w) \leqslant 0$ (i.e. $K_{\min} \leqslant 0$)
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Stability and half-line dissipativity

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McMillan degree

Again assume controllable $\mathfrak{B}$

- Corresponding to $w = (y, u)$, also partition $M(\xi) = \begin{bmatrix} Y(\xi) \\ U(\xi) \end{bmatrix}$, $G(s) = -P(s)^{-1}Q(s) = Y(s)U(s)^{-1}$ (left/right (polynomial) coprime factorization of $G(s)$)
- Amongst all maximal nonsingular minors P in $R(\xi) = [P(\xi) \;\; Q(\xi)]$, find one with maximum determinantal degree: $\mathtt{n}(\mathfrak{B})$: McMillan degree
- Ensures $G(s)$ is proper: $\det U(s)$ has same degree, and is also maximum
- $\mathtt{n}(\mathfrak{B})$: least number of 'states' (defined using a 'concatenability' axiom)

Model order reduction

Given $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and symmetric nonsingular $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$
Suppose $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and suppose $\mathtt{n}$ is the McMillan degree of $\mathfrak{B}$
Choose $k < \mathtt{n}$. Find $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ such that

1. $\hat{\mathfrak{B}}$ has McMillan degree at most k
2. $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$
3. $\hat{\mathfrak{B}}$ is also strictly Σ-dissipative on $\mathbb{R}_-$
4. $\hat{\mathfrak{B}}$ satisfies $(\hat{\mathfrak{B}}^*)_{\text{anti-stab}} \subset \mathfrak{B}^*$

(Fourth point: trajectories in $\mathfrak{B}$ of minimal dissipation are retained into $\hat{\mathfrak{B}}$)
$\mathfrak{B}^* = M(\frac{d}{dt}) \ker \partial\Phi'(\frac{d}{dt})$ and
strict dissipativity $\Leftrightarrow$ no $j\mathbb{R}$ roots of $\det \partial\Phi'(\xi)$

Algorithm

Proposed by Sorensen, SCL 2005, and as interpreted in Minh, Trentelman & Rapisarda (MCSS, 2009)

$\boldsymbol{w^T \Sigma w = u^T y}$, $\boldsymbol{w = (u, y)}$

$\frac{d}{dt}x = Ax + Bu$, and $y = Cx + Du$ for $\mathfrak{B}$, and hence

$\mathfrak{B}^{\perp_\Sigma}$ represented by $\frac{d}{dt}z = -A^T z + C^T u$, $y = B^T z - D^T u$

(Try $\frac{d}{dt}x^T z \stackrel{?}{=} u^T y$)

Interconnecting (& assuming strict passivity $\Rightarrow D + D^T > 0$)

$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = H \begin{bmatrix} x \\ z \end{bmatrix}$ and $\begin{bmatrix} u \\ y \end{bmatrix} = L \begin{bmatrix} x \\ y \end{bmatrix}$ with H and L respectively as

$$\begin{bmatrix} A-B(D+D^T)^{-1}C & B(D+D^T)^{-1}B^T \\ -C^T(D+D^T)^{-1}C & -A^T+C^T(D+D^T)^{-1}B^T \end{bmatrix}, \begin{bmatrix} -(D+D^T)^{-1}C & (D+D^T)^{-1}B^T \\ C-D(D+D^T)^{-1}C & D(D+D^T)^{-1}B^T \end{bmatrix}$$

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Algorithm: continued

Choose anti-Hurwitz $R \in \mathbb{R}^{k \times k}$ (from ORHP spectral zeros) and corresponding real X and Y such that

$$H \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} R.$$

Strict dissipativities $\Rightarrow$ X and Y are both full column rank. They are 'part' of maximal ARE solution (known to be symmetric), same argument helps $X^T Y \in \mathbb{R}^{k \times k}$ being symmetric and positive definite.

- Obtain $X^T Y = QS^2Q^T$ with $Q^T = Q^{-1}$, and S diagonal.
- Define $V := XQS^{-1}$ and $W := YQS^{-1}$,
- $\hat{A} := W^T AV$, $\hat{B} := W^T B$, $\hat{C} := CV$ and $\hat{D} := D$
- Define reduced order system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$.

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- **Define reduced order system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$.**

$W^T V$ is identity matrix and
WV^T satisfies $(WV^T)^2 = WV^T$
?? $X^T Y$ is the largest ARE solution of the reduced system??
Recall: we sought $\hat{B}$:

1. $\hat{\mathfrak{B}}$ has McMillan degree at most k
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3. $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}_-$
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With $\hat{X} := \hat{Y} := SQ^T$ (Sorensen, SCL-'05), Minh, et al gets

$$\hat{H} \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} = \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix} R$$

Further, $\hat{L}\hat{X} = LX$ and $\hat{L}\hat{Y} = LY$ give $(\hat{\mathfrak{B}}^*)_{\text{anti-stab}} \subset \mathfrak{B}^*$

Pick and Löwner matrices: Antoulas, SCL, 2005

- **Lagrange interpolating polynomials**
- **Rational interpolant with degree constraint → 'Löwner' matrices**
- **Link with Nevanlinna Pick interpolation problem**
- **Given N pairs $(x_i, y_i) \in \mathbb{C}^2$, find p.r. interpolant $G(s)$**
- **Pick matrix Π with Π_{ij} defined as**

$$\frac{y_i + y_j^*}{x_i + x_j^*} \quad \text{and} \quad \frac{1 - w_i w_j^*}{x_i + x_j^*} \quad \text{and} \quad \frac{1 - w_i w_j^*}{1 - z_i z_j^*}$$

depending on P.R., B.R. (OLHP), B.R. ($|z| = 1$), with

$$w_i := \frac{1 - y_i}{1 + y_i} \quad \text{and} \quad z_i := \frac{1 - x_i}{1 + x_i}$$

"Model reduction by interpolating at (some) spectral zeros"
"Pick matrix ≡ minimum energy required across trajectories in ker $A(\frac{d}{dt})$" (QDF, Willems & Trentelman, SIAM 1998)

Thank you

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