Perron Frobenius result and its applications

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November, 2019

Take-aways and outline

Main take-aways: Perron Frobenius theorem

- Many applications involve square matrices A with all entries positive
- Eigenvalue farthest (from origin) is positive & that left/right eigenvectors are significant
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Outline

- Eigenvalue/eigenvector
- Positive matrices $\neq$ positive-definite matrices
- Perron Frobenius theorem
- Stochastic matrices and Markov chains and other applications
- Population dynamics
- Graph theory

This talk's content: standard: Perron/Frobenius: early 1900s, and from books/ppts: Bapat/Raghavan/Šiljak/Sternberg

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   then: look for v too in $\mathbb{C}^n$.

   View v as a direction that A does not 'change': but just scales it.
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Computational procedure

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- Can eigenvalue be repeated?

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- If $\lambda \in \mathbb{R}$, then $v \in \mathbb{R}^n$ (and $v \neq 0$). One can choose sign of one (nonzero) component.

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- Every other eigenvector (if real) has mixed signs. (Unique eigenvector in positive 'orthant'.)

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But: $A > 0 \not\Rightarrow A^{-1} > 0$.
(Will revisit later: M-matrices and N or Z matrices.)

More on $A > 0$

Find $v_P > 0$ such that $Av_P = \lambda_P v_P$.

- Easy (both conceptually and computationally) to obtain estimate of λ_P and v_P:
- Take any nonzero $x \geqslant 0$. Ax, $A \cdot Ax$, $A^k x$ remains in positive orthant.

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- Normalize each time to length one (and positive orthant).
- More generally, 'power method' helps for the dominant eigenvalue.
- Convergence 'rate': how quickly effect due to others dies down:

  Second farthest λ_2 decides: $\frac{|\lambda_2|}{\lambda_P} < 1$

  Spectral gap: $1 - \frac{|\lambda_2|}{\lambda_P}$
- How much within the circle $|z| = \lambda_P$ the other eigenvalues are?

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(a) Farthest need not be positive.

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(b) A can have multiple eigenvalues at $\lambda_P > 0$

(c) If multiple at λ_P, simple/semi-simple/defective?

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Relevant concepts: irreducible/strong-connectednes/primitive/ ergodic/Cesaro-limit/ stationarity (of Markov chain)/ ..more..

For $A \geqslant 0$: examples

$$A: \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

$$\lambda_i: \quad (0,0) \quad (0,0) \quad (1,1) \quad (1,1) \quad (1,-1) \quad (1, -\tfrac{1}{2} \pm \tfrac{\sqrt{3}}{2})$$

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- Prices of commodities: say a barter economy: 'pure' exchange
- Employee 'effort values' and corresponding distribution of net-profit

Above applications can be modelled: but we need certain quantities
to remain positive (for meaningful interpretation).

Markov chain

- A is called (row-) stochastic if $A \geqslant 0$ and each row sums to 1
- a_{ij} is probability that there is a transition from state i to state j.
- $\lambda_P = 1$, right eigenvector $= [1 \ \ 1 \ \ 1 \ \ \cdots \ \ 1]^T$, and left-eigenvector contains 'steady-state' distribution
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- Unique steady-state distribution?, linked to multiplicity of $\lambda_P = 1$: simple/semi-simple
- Large spectral gap $\equiv$ quickly mixing Markov chain

- Graph constructed from which webpage links to which
- Graph constructed gives A such that $\lambda_P = 1$ and v_P has webpage's ranks: highest value: most important webpage
- Need 'fast convergence' in the power method: trade-off between high spectral gap and relevance of the computed rankings

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See work of T.Saaty/S.Raju/N.Rangaraj/A.W.Date, for example

Consider a speci-population stratified into various age-groups: x_1, x_2, etc.
$x(k+1) = Ax(k)$, with A as the population 'transition matrix':
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  (Atlantic salmon die immediately after spawning.)
- Two or more $b_i >$, then primitive: oscillations stabilize to Perron vector

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Leslie matrix: population dynamics: single-speci (from Sternberg)

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- Only one eigenvector has all components positive/non-negative.

A different population model: multi-species

Suppose different species $1, 2, \ldots, n$ grow with growth-rates $a_i > 0$ as follows:

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Now assume off-diagonal terms of A are negative:
i.e. one species amount causes decrease in another
speci population (assuming $x_i(t) > 0$.)

- Will population of all species stabilize?
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This brings us to Z and M-matrices: closely linked to Perron-Frobenius theorem.

Z matrix and M matrix

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Inverse of eigenvalue nearest (to the origin) of A = farthest eigenvalue of A^{-1}
Conversely, if $P \geqslant 0$, and P is invertible, and suppose off-diagonal elements of P^{-1} are nonpositive, then P^{-1} is a (nonsingular) M-matrix.

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The Laplacian matrix $L \in \mathbb{R}^{n\times n}$ is defined as $L := D - A$:

Laplacian of graphs are (singular) M-matrices

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(degree of a node: number of edges incident on that node)
N is the neighbourhood (adjacency) matrix:
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- Known: L is singular, symmetric, and positive semi-definite
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For Laplacian matrix L, add $\epsilon > 0$ to any diagonal element
Perturbed L (say L_ϵ) is now invertible.
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Smallest eigenvalue of L is > 0 now. Check that smallest eigenvalue is $\approx \epsilon$
L_ϵ is a non-singular M-matrix $\Leftrightarrow (L_\epsilon)^{-1} \geqslant 0$

Exponential of a Laplacian matrix

- Recall, for a square matrix P: define $e^P = I + P + \frac{P^2}{2!} + \cdots$
- Consider Laplacian matrix L. Note $L = L^T$: (undirected graph)
- e^{-LT} is a (doubly) stochastic matrix for any Laplacian matrix L and time $T > 0$.

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- e^{-LT} is a (doubly) stochastic matrix for any Laplacian matrix L and time $T > 0$.
- View discretization of $\dot{x} = -Lx$ at sampling period $T > 0$ to get $x(k+1) = e^{-LT}x(k)$
- Stable in continuous time: eigenvalues in left-half-complex plane
- Stable in discrete time: eigenvalues in disc of radius $= 1$
- Eigenvalues of $e^A = e^{\lambda_i}$

Conclusion

Perron Frobenius theorem

- Saw some applications involve square matrices A with all entries positive
- Eigenvalue farthest (from origin) is positive & that left/right eigenvectors are significant
- When all entries of A are not positive, but non-negative under some graph-conditions, have some of these conclusions.

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