Generic arbitrary pole placement and structural controllability

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Outline

- Structural system of equations: plant and controller
- Arbitrary pole placement problem
- Known results
- Bipartite graphs
- Necessary and sufficient conditions
- Unimodular completion

Structured system of equations

- Often just structure specified for the equations of the plant (plant $\equiv$ the system to-be-controlled)
- Parameters not known precisely. (They vary slightly in practice.)
- If uncontrollable, sometimes slight perturbation in system parameters fetches controllability
- Structure: which variable occurs in which equation known
- This talk: only LTI systems
  Linear ordinary constant-coefficient differential equations
- Construct a polynomial matrix, and then a 'bipartite' graph

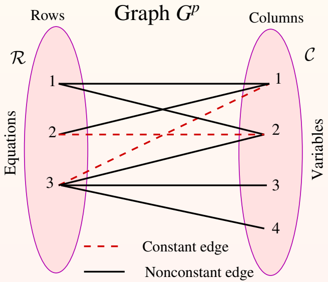
Example

Plant equations: 3 differential equations in 4 variables: w_1, w_2, w_3 & w_4.
System parameters: a_{ij} and b_{ij} are arbitrary real numbers.
Construct $P(s) \in \mathbb{R}^{3\times 4}[s]$:

$$a_{11}\dot{w}_1 + b_{11}w_1 + a_{12}\dot{w}_2 + b_{12}w_2 = 0$$

$$a_{21}\dot{w}_1 + b_{21}w_1 + b_{22}w_2 = 0$$

$$b_{31}w_1 + a_{32}\dot{w}_2 + b_{32}w_2 + a_{33}\dot{w}_3 + b_{33}w_3 + a_{34}\dot{w}_4 + b_{34}w_4 = 0$$



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More examples of structured system of equations

Large circuits involving 2-terminal devices:

- System variables: V and I: across-voltages and through-currents
- KCL involving I variables, KVL: V variables
- Device equations linking components of V and I vectors
- Only device parameters: not precise: ‘mixed’ formulation (Murota, van der Woude)

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Decentralized control:

- Local plant equations, across-subsystem-interconnection equations
- Each local controller can involve only local variables
- Similar sensor/actuator allocation constraints across subsystems

Bipartite graph

- Plant's structure captured by a bipartite graph
- Bipartite graph G having vertices $V = \mathcal{R} \cup \mathcal{C}$ (disjoint union)
- Each edge in G has one vertex in $\mathcal{R}$ and the other in $\mathcal{C}$
- Construct graph G from polynomial matrix $P(s)$ as follows.
- $\mathcal{R}$ is the set of rows of $P(s)$ and
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- Square $P(s) \equiv |\mathcal{R}| = |\mathcal{C}|$

Questions

- Is the system controllable? (Controllable: in ‘behavioral’ sense)
- Does the bipartite graph reveal this? ‘Structurally controllable’
- Dependence on values of a_{ij} and b_{ij}?
- Can we achieve arbitrary pole placement?
- What if the controller also has such constraints?
- Controller constraints $\equiv$ sensor/actuator allocation constraints

Important phrases

- Under-determined $\leftrightarrow$ wide, determined $\leftrightarrow$ square
- matchings $\leftrightarrow$ one-to-one assignment (from prespecified edges)
- perfect matching
- poles $\leftrightarrow$ roots of characteristic polynomial
- pole-placement $\leftrightarrow$ assign the (closed loop) poles

Behavioral definitions

- System $\equiv$ set of all 'allowed' trajectories: 'behavior'
- All solution-trajectories allowed by the system equations
- For LTI systems: System equations: $P(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, system variables: w
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Recall: 3 equations, 4 variables

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Behavioral controllability

- System controllable $:\equiv$ trajectories allow mutual 'patching'
- Controllability $\equiv P(\lambda)$ has full row rank for every $\lambda \in \mathbb{C}$
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- Call such a $P(s)$ left-prime
- System structurally controllable $:\equiv$ for ‘almost all’ coefficients a_{ij} and b_{ij} in $P(s)$, we have left-primeness
- Polynomial matrices allowed by that structure are ‘generically left-prime’

'Generic' $\equiv$ almost always

- $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is nonsingular (unless $ad - bc = 0$).
- Set of values a, b, c and d in $\mathbb{R}^4$ satisfying $ad - bc = 0$: 'thin set': unlikely that arbitrarily chosen real values of a, b, c and d would cause $ad - bc = 0$.
- We say B is generically nonsingular.
- Similarly, polynomials $p(s)$ and $q(s)$ with degrees m and $n \geqslant 1$ and arbitrary real coefficients generically do not have a common factor.

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- With some structure: $B = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is generically singular.
- Location of zero/nonzero entries in a bipartite graph reveals generic nonsingularity.

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- Matching theory: Plummer, Lovász

State space controllability

Kalman's state space controllability: $\frac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu$:

- (A, B) controllable $:\equiv$ for any arbitrary initial condition x_0 and arbitrary final condition x_f, there exist time $T \geqslant 0$ and an input $u : [0, T] \to \mathbb{R}^m$ such that $x(0) = x_0$ and $x(T) = x_f$
- (A, B) is controllable $\Leftrightarrow [B \mid AB \mid \cdots A^{n-1}B]$ is full row rank
- $\Leftrightarrow [sI - A \mid B]$ is 'left prime': $[\lambda I - A \mid B]$ has full row rank for every $\lambda \in \mathbb{C}$: Popov Belevitch Hautus (PBH) test.

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Characteristic polynomial of $(A + BF)$:= roots of det $(sI - A - BF)$
Feedback $u = Fx$ achieves desired poles: 'pole-placement'
Arbitrary pole placement possible $\Leftrightarrow$ (A, B) controllable

State space examples

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

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$u = f_1 x_1$. (State $x = (x_1, x_2)$)
View this control law as:

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Notice loop through controller edges
Feedback $\equiv$ loop

Problem formulation

- Left-prime: the only factors that can be pulled from 'left' side are those with polynomial inverse
- $[s+1 \;\; s] = a(\frac{1}{a}[s+1 \;\; s])$ (with any real $a \neq 0$) (left prime)
- $[s(s+1) \;\; s^2] = s([s+1 \;\; s])$ (not left prime)
- $[a(s) \;\; b(s)]$ is left-prime $\equiv$ a and b have no common roots
- 'Most state space systems are controllable' $\equiv$ $[sI - A \;\; B]$ is generically left-prime

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2. Given plant and controller structures: $G^p(\mathcal{R}_P, \mathcal{C}; E_p)$ and $G^k(\mathcal{R}_K, \mathcal{C}; E_k)$, find conditions on these graphs for ability to achieve arbitrary pole placement

Control as interconnection

- Suppose plant has laws $P(\frac{d}{dt})w = 0$ and controller has $K(\frac{d}{dt})w = 0$
- After interconnection, w has to satisfy both sets of laws
- Define $A(s) := \begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$.
  Controlled, i.e. closed loop system: $A(\frac{d}{dt})w = 0$
- Closed loop is autonomous[1]: $A(s)$ is square and nonsingular
- Pole placement: given desired polynomial d, construct K to get $\det A = d$
- For example, d has all roots sufficiently left (in the complex plane)

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Generic nonsingularity $\leftrightarrow$ perfect matchings

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Matchings and inadmissible edges

For a graph $G^p(\mathcal{R}_P, \mathcal{C}; E_p)$

- Matching M: subset $M \subseteq E$ such that each vertex is degree 1
- Maximum matching: maximum cardinality of M
- For square matrix $P(s)$: $|\mathcal{R}| = |\mathcal{C}|$.
  Maximum matching of size $|\mathcal{R}| \equiv$: perfect matching

Matching theory: very well-developed (Lovász, Plummer, Asratian, Denley, Häggkvist, Tassa)

In particular, elementary-bipartite-graphs

Perfect matchings

- Think of set of men M and set of women W.
- Each edge: man-woman 'compatibility' (don't mind marriage)
- Suppose equal number of men and women
- 'Perfect match' $\equiv$ all get matched
- Other examples (bipartite graph): College-students match, hospitals-patients match, Students-hostels match
- Also, preference possible: stable marriage
- Also, in male hostels, room-partner compatibility: non-bipartite graph
- In square matrix, take row set R and column set C: compatibility between some $r \in R$ and $c \in C \equiv r, c)$ entry is nonzero
- Nonzero terms in determinant expansion $\leftrightarrow$ perfect matching

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Link between structured matrices and graph theory:

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Link between structured matrices and graph theory:
Generically nonzero terms do not cancel.

Controller vertices $\mathcal{R}_K$ and $\mathcal{C}$

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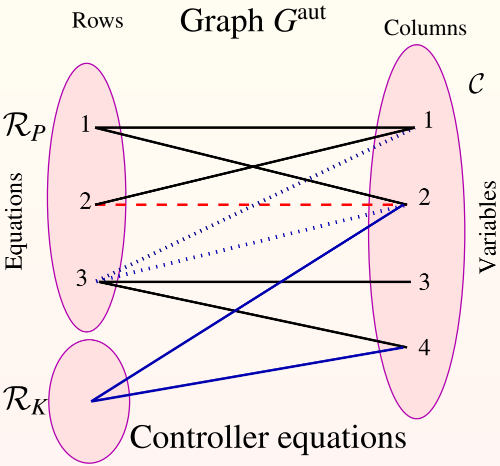
- E_k describes which variable can occur in which controller equation
- Controller no constraints $\equiv$ complete bipartite graph on $\mathcal{R}_K$ and $\mathcal{C}$
- Closed loop autonomous $\equiv$ $\quad |\mathcal{R}_P| + |\mathcal{R}_K| = |\mathcal{C}|$
  This is the interconnected system.

New bipartite graph: with controller



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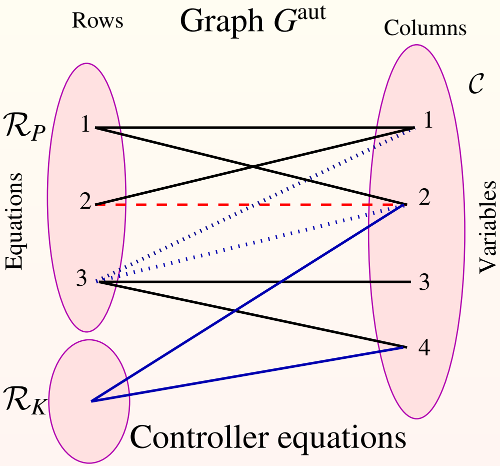

Too many colours!
Plant non-constant edges, plant constant edges,
inadmissible edges, controller edges

Main result: pole placement

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1. Arbitrary pole placement is possible generically using controllers having structure G^k.
2. There do not exist subsets $r \subseteq \mathcal{R}_P$ and $c \subset \mathcal{C}$ that satisfy the following three conditions
   (a) $|r| = |c|$,
   (b) there is a nonconstant plant edge in G_a^{aut} incident on r,
   (c) every perfect matching M of G_a^{aut} matches r and c.

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1. Arbitrary pole placement is possible generically using controllers having structure G^k.
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   - (a) $|r| = |c|$,
   - (b) there is a nonconstant plant edge in G_a^{aut} incident on r,
   - (c) every perfect matching M of G_a^{aut} matches r and c.
3. Every nonconstant plant edge in G_a^{aut} is in some cycle containing controller edges in G_a^{aut}.

Main result: structural controllability

Consider $G^p(\mathcal{R}_P, \mathcal{C}; E_p)$ with $|\mathcal{R}| < |\mathcal{C}|$ and remove all inadmissible edges from G^p to obtain G_a^p.
Let $g_1, g_2, \ldots\ g_t$ be the connected components of G_a^p.
Then the following are equivalent.

1. The plant is structurally controllable.
2. The graph G^p represents an equivalence class of generically left-prime polynomial matrices.

[2] The subgraph of G_a^p on the symmetric difference between M and N. The symmetric difference between two sets A and B, denoted as $A\Delta B$, is defined as $(A \cup B) \backslash (A \cap B)$.

Main result: structural controllability

Consider $G^p(\mathcal{R}_P, \mathcal{C}; E_p)$ with $|\mathcal{R}| < |\mathcal{C}|$ and remove all inadmissible edges from G^p to obtain G^p_a.
Let $g_1, g_2, \ldots\ g_t$ be the connected components of G^p_a.
Then the following are equivalent.

1. The plant is structurally controllable.
2. The graph G^p represents an equivalence class of generically left-prime polynomial matrices.
3. Each component g_i that contains a nonconstant plant edge satisfies $|\mathcal{R}(g_i)| < |\mathcal{C}(g_i)|$.
4. For each nonconstant plant edge e in G^p_a, there exist $\mathcal{R}$-saturating matchings M and N such that e is in a path in $G^p_a[M \Delta N]$.[2]

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- Call a polynomial matrix $U(s) \in \mathbb{R}^{g\times g}[s]$ unimodular if $\det U(s) \in \mathbb{R}\backslash 0$
- $P(s)$ is left-prime $\equiv P(s)$ can be completed to a unimodular matrix
- $P(s)$ is left-prime $\Leftrightarrow$ there exists $K(s)$ such that $A(s) := \begin{bmatrix} P(s) \\ K(s) \end{bmatrix}$ has determinant equal to 1.

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- Completion $K(s)$ could have its constraints/structure too

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- Hogben: Completion problems: constant matrices (‘$>$’, Hicks, many more)

Main take-aways

- For square matrices, bipartite graph between rows and columns
- Each perfect matching : a term in determinant expansion
- Some entries don't occur in any term in determinant
- Some edges don't occur in any perfect matching: inadmissible edges
- Autonomous system (no inputs) : square system of equations
- Autonomous : at least one perfect matching
- Pole-placement $\Leftrightarrow$ all (nonconstant) admissible plant edges through some controller loop

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Questions. Thank you