Uncontrollable Dissipative Dynamical Systems

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Outline

- **Dissipative system: definition**
- **Algebraic Riccati Equation (ARE), LMI & Hamiltonian matrix**
- **RLC circuit example**
- **Main result: ARE solvability**
- **Unobservable state: problem?**
- **Embeddability**
- **RLC realization: nullator**
- **Conclusion**

Intuitively: a dissipative system

- **has no source of energy,**
- **absorbs energy supplied,**
- **can store (previously supplied) energy.**

Power: $w^T \Sigma w$ with Σ: real, symmetric, nonsingular matrix
Quadratic in w: the 'manifest' variables: e.g.: v, i in power
$Q_\Psi(w, \ell)$: quadratic in w, ℓ, and their derivatives too
ℓ: extra/auxiliary variables, for e.g., 'state'

Storage function

Given a system and and a notion of power $w^T \Sigma w$:

storage function $Q_\Psi(w, \ell)$:⇔ $\frac{d}{dt} Q_\Psi(w, \ell) \leqslant w^T \Sigma w$ for all allowed system trajectories

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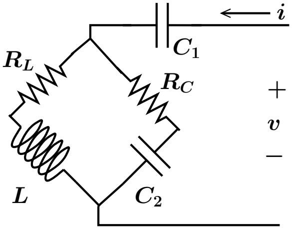
RLC Circuit example

$R_L = R_C = R$ and $L = R^2C \Rightarrow$ uncontrollable



One input, and one output
Variables $w = (v, i)$
Supply rate $= w^T \Sigma w =$
$vi = \frac{1}{2} \begin{bmatrix} v \\ i \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}$

Though the circuit contains no source, $v(t)i(t)$ can be negative at some time instants.
In any case (vi: of any sign)

$$\frac{d}{dt}\left(\frac{C_1 v_{C_1}^2}{2} + \frac{L i_L^2}{2} + \frac{C_2 v_{C_2}^2}{2}\right) \leqslant vi$$

Rate of increase of stored energy $\leqslant$ supplied power

Faster increase $\Rightarrow$ source

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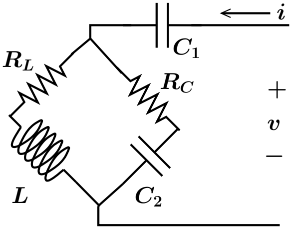

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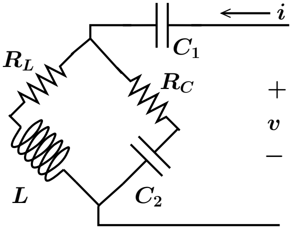

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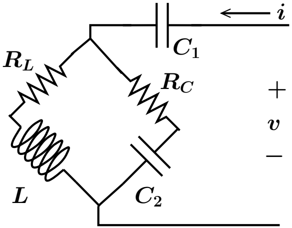

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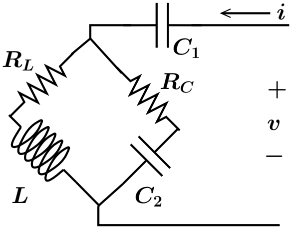

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What is general about $\boldsymbol{w}^T \boldsymbol{\Sigma} \boldsymbol{w}$*?*

- vi: voltage × current = physical power
- Fv: Force × velocity = physical power
- pressure & flow-rate, etc.
- $\gamma^2 u^2 - y^2$ disturbance attenuation: $\mathcal{H}_\infty$-norm
- $y = \phi(u)$, and ϕ is a 'sector' nonlinearity,
  $\phi \in$ sector $(\alpha, \beta) :\Leftrightarrow$
  $$0 \leqslant (y - \alpha u)(u - \tfrac{y}{\beta}) = \begin{bmatrix} u \\ y \end{bmatrix}^T \begin{bmatrix} -\alpha & \frac{(\alpha+\beta)}{2\beta} \\ \frac{(\alpha+\beta)}{2\beta} & \frac{-1}{\beta} \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}$$
- Popov criteria, involving dynamic notions of power
- A common framework for stability results: passivity result, small-gain theorem, circle criterion
- Stability results in nonlinear dynamical systems
- Lyapunov function: $\frac{d}{dt}$ storage function $\leqslant 0$.

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$\textit{Dissipative} :\equiv \exists$ *a storage function*

Define a system to be Σ-dissipative if there exists a storage function $Q_\Psi(w, \ell)$, i.e.

$$\frac{d}{dt} Q_\Psi(w, \ell) \leqslant w^T \Sigma w \text{ for all allowed trajectories}$$

- w: **manifest variables: power is supplied through these variables:** $w^T \Sigma w$
- $Q_\Psi(w, \ell)$: **quadratic function of** (w, ℓ) **and their derivatives too: energy stored within the system**
- **for** $Q_\Psi(w) = x^T K x$ **with** K **as Algebraic Riccati Inequality (ARI) solutions**

{all allowed trajectories} =: system 'behavior'

Dissipative $:\equiv\ \exists$ a storage function

Define a system to be Σ-dissipative if there exists a storage function $Q_\Psi(w,\ell)$, i.e.

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- for $Q_\Psi(w) = x^TKx$ with K as Algebraic Riccati Inequality (ARI) solutions

{all allowed trajectories} =: system 'behavior'

Behavior

$P(s)$, $Q(s)$, $N(s)$ and $D(s)$ polynomial matrices of suitable size

Consider $G(s) = P(s)^{-1}Q(s)$, P and Q need not be left-coprime

System behavior $\mathfrak{B} := \{(u, y) \mid y = Gu\}$

$$\begin{bmatrix} Q(\frac{d}{dt}) & -P(\frac{d}{dt}) \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix} = 0 \text{ (Kernel representation)}$$

Consider $G(s) = N(s)D(s)^{-1}$

$$\mathfrak{B} := \{(u, y) \mid \text{ there exists an } \ell \text{ such that } \begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} D(\frac{d}{dt}) \\ N(\frac{d}{dt}) \end{bmatrix} \ell\}$$

Consider state space system $\dot{x} = Ax + Bu$ and $y = Cx + Du$

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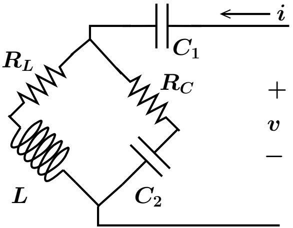

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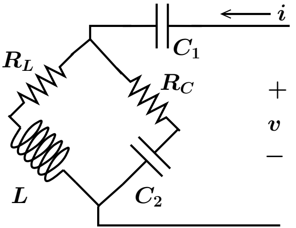

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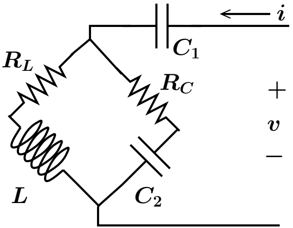

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For controllable 1D systems: key equivalence (Willems & Trentelman: 1998)

System is Σ-dissipative $\Leftrightarrow$ $\int_{\mathbb{R}} w^T \Sigma w \, dt \geqslant 0$ for all compactly supported system trajectories

- Behavior := all allowed system trajectories: ($\mathfrak{C}^\infty$)
- Controllable $\Leftrightarrow$ compactly supported trajectories: dense
- Compact support: start from rest, end at rest: (no 'initial/final energy' issues)
- Controllability $\Rightarrow$ there exist observable storage functions

Some extensions to controllable nD systems:
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Controllable systems

For controllable 1D systems: key equivalence (Willems & Trentelman: 1998)

System is Σ-dissipative	$\Leftrightarrow$	$\int_{\mathbb{R}} w^T \Sigma w \, dt \geqslant 0$ **for all compactly supported system trajectories**

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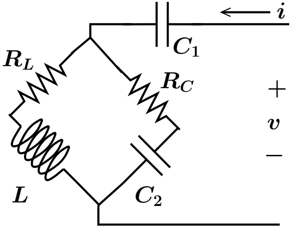
RLC circuit example

Consider again: $R_L = R_C = R$ and $L = R^2C$ (uncontrollable).
We expect system is dissipative.



One input, and one output
Variables $w = (v, i)$
Supply rate $= w^T \Sigma w =$
$$vi = \frac{1}{2}\begin{bmatrix} v \\ i \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}$$

$\sigma_+(\Sigma) = 1$ and $\sigma_-(\Sigma) = 1$.
$\sigma_+(\Sigma)$ and $\sigma_-(\Sigma)$: # positive and negative eigenvalues of Σ

$$Q_\Sigma(w) = w^T \Sigma w,\ \Sigma = \begin{bmatrix} I_m & 0 & 0 \\ 0 & I_q & 0 \\ 0 & 0 & -I_r \end{bmatrix},\ J_{q,r} = \begin{bmatrix} I_q & 0 \\ 0 & -I_r \end{bmatrix}$$

$\frac{d}{dt}x = Ax + Bu, \quad y = Cx + Du$, with (C, A) observable, (A, B) possibly uncontrollable and $w = (u, y)$

Well-known result

$\exists$ a real symmetric solution K to the LMI

$$\begin{bmatrix} (KA + A^T K - C^T J_{q,r} C) & (KB - C^T J_{q,r} D) \\ (KB - C^T J_{q,r} D)^T & -(I_m + D^T J_{q,r} D) \end{bmatrix} \leqslant 0$$

Then, $x^T K x$ is a storage function ($\Rightarrow$ dissipativity)

m : number of inputs, $q + r$: number of outputs ($= p$)
$\sigma_+(\Sigma) = m + q$ and $\sigma_-(\Sigma) = r$.

Schur complement of this LMI[1]

Algebraic Riccati inequality

$$K\tilde{A} + \tilde{A}^*K + K\tilde{D}K - \tilde{C} \leqslant 0,$$

Define the Hamiltonian matrix $H := \begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{C} & -\tilde{A}^* \end{bmatrix}$

- ARE solutions $K \Leftrightarrow$ an n-dimensional H-invariant subspaces is a 'graph' subspace: image $\begin{bmatrix} I \\ K \end{bmatrix}$
- Controllability of $(\tilde{A}, \tilde{D})$: simplifies results
- 'Mixed'-sign: $\tilde{C}$ is not sign-definite (unlike LQ, $\mathcal{H}_\infty$-norm)
- Mixed-sign ARE: $\mathcal{H}_\infty$-control

[1] $(I_m + D^T J_{q,r} D) > 0$ and $\tilde{A} := (A - B(I_m + D^T J_{q,r} D)^{-1} D^T J_{q,r} C)$, $\tilde{D} := B(I_m + D^T J_{q,r} D)^{-1} B^T$ and $\tilde{C} = C^T (J_{q,r} + DD^T)^{-1} C$.

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Riccati Equation and Hamiltonian

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Main result

Suppose behavior $\mathfrak{B}$ satisfies:

1. **uncontrollable poles are unmixed, i.e. no two of them add to zero**
2. **The feed-through term D satisfies $(I_m + D^T J_{q,r} D) > 0$ (Controllable part of $\mathfrak{B}$ is strictly dissipative 'at infinity')**

Define $\mathfrak{B}_{cont}$ as the controllable part of $\mathfrak{B}$.

Then, $\mathfrak{B}$ is Σ-dissipative $\Leftrightarrow$ $\mathfrak{B}_{cont}$ is Σ-dissipative.

First: Lyapunov equation solvability (autonomous part)

After all, Lyapunov functions: storage functions for autonomous systems

Second: $(I_m + D^T J_{q,r} D) > 0$:

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Proof techniques

- **Hamiltonian matrix has $j\mathbb{R}$ eigenvalues (unlike controllable/observable LQ)**
- **Used results on Indefinite Linear Algebra: Gohberg-Lancaster-Rodman**
- **$\exists$ an observable storage function, state function: $x^T K x$**
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Unobservable inevitable?

Suppose some uncontrollable poles are on the imaginary axis (i.e. periodic solutions).

Then, unobservability of the ARE solution K is inevitable

Autonomous behavior $\mathfrak{B}_{\text{aut}}$

Consider $\frac{d}{dt}x = Ax, \quad w = Cx$, with $\sigma(A) \cap i\mathbb{R} \neq \emptyset$ and power $= -w^T w$.
Suppose $\exists$ a storage function $x^T K x$ satisfying

$$\frac{d}{dt}x^T K x \leqslant Q_\Sigma(w) \qquad \text{for all } w \in \mathfrak{B}_{\text{aut}}.$$

Then every $\lambda \in \sigma(A) \cap i\mathbb{R}$ is unobservable.

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Some non-autonomous $\mathfrak{B}$ too

Need unobservable variables in storage function.

More precisely, for periodic uncontrollable trajectories,
$\exists$ storage functions $\Rightarrow$ unobservable.

$\mathfrak{B}$ with a 'static' controllable part

Consider $\frac{d}{dt}x = Ax$, $w_2 = Cx + Dw_1$ with (C, A) observable.
Assume $(I_m + D^T J_{q,r} D) > 0$ and $\sigma(A) \subset i\mathbb{R}$.
Then, $\nexists$ a symmetric solution to the corresponding ARE.

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Can unobservable variables cause a problem?

Every autonomous system is orthogonal to $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$
Willems (CDC-2004):

$\mathfrak{B}_1$: any autonomous system $\quad$ $\mathfrak{B}_2$: full $= \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^n)$

System $\mathfrak{B}_1$ in variable w

$$\frac{d}{dt}z = Az$$
$$w = Cz$$

System $\mathfrak{B}_2$ in variable v

$$\frac{d}{dt}x = -A^T x + C^T v$$

Storage function $x^T z$ (unobservable from w and v).
$\mathfrak{B}_1$ and $\mathfrak{B}_2$ are 'orthogonal'[2]
Consider $S = \frac{1}{2}\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$. We get $\frac{1}{2}\begin{bmatrix} w \\ v \end{bmatrix}^T S \begin{bmatrix} w \\ v \end{bmatrix} = w^T v$.
The system $\mathfrak{B}_1 \times \mathfrak{B}_2$ is S-lossless.

[2] For controllable behaviors $\mathfrak{B}_1$ and $\mathfrak{B}_2$, call them orthogonal if $\int_\mathbb{R} w^T v dt = 0$ for all $w \in \mathfrak{B}_1$ and $v \in \mathfrak{B}_2$ of compact support.

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Recall
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- ~~**Compactly supported trajectories definition**~~
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Embeddability definition

A behavior $\mathfrak{B}$ is called Σ-dissipative if

there exists $\mathfrak{B}^{\text{sup}}$ such that

$\mathfrak{B} \subseteq \mathfrak{B}^{\text{sup}}$ $\quad$ $\mathfrak{B}^{\text{sup}}$ is Σ-dissipative $\quad$ $\mathfrak{B}^{\text{sup}}$ is controllable

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$\mathtt{m}(\mathfrak{B})$: number of inputs of a system
$\sigma_+(\Sigma)$: number of positive eigenvalues of Σ

Then, $\mathtt{m}(\mathfrak{B}) \leqslant \sigma_+(\Sigma)$

Consider

Σ	M_1	M_2	M_3	N
$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$	$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$
	Yes	Yes	Yes	No

$M_i^T \Sigma M_i \geqslant 0 \Rightarrow \text{rank}(M_i) \leqslant 2 (= \sigma_+(\Sigma))$
Controllable systems: images of $M(\frac{d}{dt})$ with $\mathtt{m}(\mathfrak{B}) = \text{rank } M(\xi)$

Dissipativity $\Rightarrow$ *not too many inputs*

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$$\Sigma = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad M_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad M_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.5 & 0 \\ 0 & 0 \end{bmatrix} \quad N = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

M_1: Yes; M_2: Yes; M_3: Yes; N: No

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‘Fallacious example’ resolved

Recall $\mathfrak{B}_1$ and $\mathfrak{B}_2$:

System $\mathfrak{B}_1$ in variable w	System $\mathfrak{B}_2$ in variable v
$\frac{d}{dt}z = Az$, $w = Cz$	$\frac{d}{dt}x = -A^T x + C^T v$

$\mathfrak{B}_1$, $\mathfrak{B}_2$ not embeddable in controllable & orthogonal $\mathfrak{B}_1^{\text{sup}}$, $\mathfrak{B}_2^{\text{sup}}$.

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An independent result that deduces:
enough inputs, uncontrollable $\Rightarrow$ cannot be lossless (w.r.t. observable storage function).

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Suppose $\mathfrak{B}$ is lossless ($:\equiv \exists$ an observable storage function) with power $= w^T \Sigma w$, and $\Sigma = \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$ and $\mathfrak{B}$ has n-inputs. Then, $\mathfrak{B}$ is controllable.

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Embeddability: pros/cons

- **Embeddability definition removes fallacious examples**
- **Concludes independently an observable-storage-function-based dissipativity result**

Drawbacks?

Can construct nonzero $\mathfrak{B}$ that is both strictly Σ-dissipative and strictly anti Σ-dissipative!

Theorem

Suppose $\Sigma = \Sigma^T$ is nonsingular and indefinite.
Then, $\exists$ nonzero behavior $\mathfrak{B}$ such that

- *can construct controllable $\mathfrak{B}_+$ and $\mathfrak{B}_-$ with $\mathfrak{B} = \mathfrak{B}_+ \cap \mathfrak{B}_-$,*
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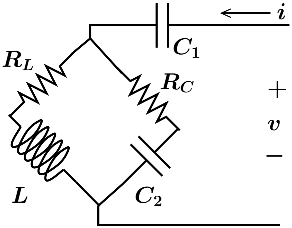

 # inputs = # outputs = 1
$\sigma_+(\Sigma) = \sigma_-(\Sigma) = 1$
Variables $w = (v, i)$
Supply rate $= w^T \Sigma w =$
$vi = \frac{1}{2} \begin{bmatrix} v \\ i \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix}$

System has a (different) observable storage function.

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(Uncontrollable) RLC circuit revisited

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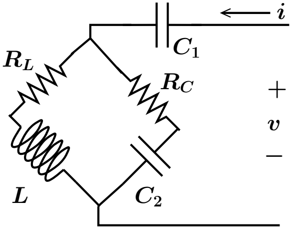

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Maximum input cardinality

 # system inputs = positive signature of supply rate
For example,

$$w = (v, i) \text{ and power } = vi, \text{ then } \Sigma = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

- Are there physical systems where
  number of inputs < positive signature? Yes
- Consider a one-port network: $v = 0$, $i = 0$: 'Nullator'
  Both open and short: controllable, 'passive'
- Studied extensively by Carlin, Tellegen in 1960s
- Cannot be realized using just RLC components (Carlin, 1964, IEEE Circuit Theory)
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 # system inputs = positive signature of supply rate
For example,

$$w = (v, i) \text{ and power } = vi, \text{ then } \Sigma = \tfrac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

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  number of inputs < positive signature? Yes
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  Both open and short: controllable, 'passive'
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  - rules out fallacious examples,
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Questions? Thank you

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Conclusions

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