Indian Institute of Technology Bombay

Dept of Electrical Engineering

Handout 5 EE 210 Signals and Systems

Homework 2 August 21, 2015

Question 1) For an odd function, i.e. $f_o(u) = -f_o(-u)$, defined for u in the range $[-\pi, \pi]$, show that

$$f_{odd}(u) = \sum_{m \geq 1} A_m \sin(mu)\,,\ -\pi \leq u \leq \pi. \tag{1}$$

Solution: We know that continuous functions admit a representation

$$f(u) = \sum_{m \geq 1} A_m \sin(mu) + \sum_{m \geq 0} A_m \cos(mu).$$

An oddfunction can be written as

$$\begin{aligned} f(u) &= \frac{1}{2}\left(f(u) - f(-u)\right) \\ &= \frac{1}{2}\Big(\sum_{m \geq 1} A_m \sin(mu) + \sum_{m \geq 0} A_m \cos(mu) - \sum_{m \geq 1} -A_m \sin(mu) - \sum_{m \geq 0} A_m \cos(mu)\Big) \\ &= \sum_{m \geq 1} A_m \sin(mu). \end{aligned}$$

Question 2) Let $f(t) = t$ for $-\frac{T}{2} \leq t \leq +\frac{T}{2}$. Find the FS expansion for $f(\cdot)$.
Solution: The FS coefficients are given by

$$\begin{aligned} a_m &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t \exp(-j\frac{2\pi}{T}mt)dt \\ &= \frac{1}{T}\frac{T}{j2\pi m}\left[-t\exp(-j\frac{2\pi}{T}mt)\right]_{-\frac{T}{2}}^{\frac{T}{2}} + 0 \\ &= \frac{1}{j2\pi m} - \left(\frac{T}{2}\exp(-j\pi m) + \frac{T}{2}\exp(j\pi m)\right) \\ &= -\frac{T}{j2\pi m}\cos(\pi m) \\ &= \frac{(-1)^m jT}{2\pi m}. \end{aligned}$$

An alternate (but equivalent) way is as follows. For any odd function we can write,

$$a_m = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t\exp(-j\frac{2\pi}{T}mt)dt \tag{2}$$

$$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t(\cos(\frac{2\pi}{T}mt) - j\sin(\frac{2\pi}{T}mt))dt \tag{3}$$

$$= -j\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} t\sin(\frac{2\pi}{T}mt)dt \tag{4}$$

$$= -2j\frac{1}{T}\int_{0}^{\frac{T}{2}} t\sin(\frac{2\pi}{T}mt)dt. \tag{5}$$

Thus, FS coefficients for an odd function boils down to evaluating $\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(\frac{2\pi}{T}mt))dt$, and multiplying the end result by $-j$. Thus

$$\begin{aligned} A_m &= -\frac{j}{T}2\int_0^{\frac{T}{2}} t\sin(\frac{2\pi}{T}mt)dt \\ &= \frac{j}{T}2\frac{T}{2\pi m}\left[t\cos\right]_0^{\frac{T}{2}} + 0 \\ &= \frac{(-1)^m jT}{2\pi m}. \end{aligned}$$

The take home lesson is that evaluating dotproducts with respect to sine functions will give Fourier series of an odd function. For even functions, we need to take dot-products on a single interval with cosine waves.

Question 3) Show that every continuous even function defined on $[-\pi,\pi]$ admits an expansion,

$$f_{even}(u) = \sum_{m\geq 0}\hat{A}_m\cos(mu) \tag{6}$$

Solution: Similar to the first question

$$\begin{aligned} f(u) &= \frac{1}{2}\left(f(u)+f(-u)\right) &(7)\\ &= \frac{1}{2}\left(\sum_{m\geq 1}A_m\sin(mu) + \sum_{m\geq 0}A_m\cos(mu) + \sum_{m\geq 1}-A_m\sin(mu) + \sum_{m\geq 0}A_m\cos(mu)\right) &(8)\\ &= \sum_{m\geq 0}B_m\cos(mu). &(9) \end{aligned}$$

Question 4) A string is tied straight between two hinges at coordinates $(0,0)$ and $(\frac{L}{2},0)$ respectively. A point at a horizontal distance of p from origin is given a vertical displacement h initially. Let the initial position be described by the function $x(u)$. We know that the frequencies are multiples of $\frac{2\pi}{L}$, which is called the fundamental frequency or the first harmonic. The higher harmonics are now progressively counted as second, third etc.
a) Find the coefficients A_m if

$$g(u) = \sum_{m\geq 1}A_m\sin(\frac{2\pi}{L}mu).$$

Solution: We can extend the function to the interval $[-\frac{L}{2},\frac{L}{2}]$ by defining $g(u) = -g(-u)$, which makes it an odd function. Now, odd functions admit a sinusoidal series expansion as in Questions 1 and 2. Let a_m represent the FS coefficients of the odd function $g(u)$, then

$$\begin{aligned} g(u) &= \sum_{m\in\mathbb{Z}}a_m\exp(j\frac{2\pi}{L}mu) \\ &= \sum_{m\in\mathbb{Z}}a_m\left(\cos(\frac{2\pi}{L}mu) + j\sin(\frac{2\pi}{L}mu)\right) \\ &= j\sum_{m\in\mathbb{Z}}a_m\sin(\frac{2\pi}{L}mu) &(10)\\ &= j\sum_{m\geq 1}(a_m - a_{-m})\sin(\frac{2\pi}{L}mu), &(11) \end{aligned}$$

where the equality (10) is true since $f(u)$ is an odd function in $(-\frac{L}{2}, \frac{L}{2})$ (see Question 1). The expression (11) follows since $a_0 = \int g(u)du = 0$ for the odd function. From this, the coefficients A_m that we wish to evaluate are nothing but

$$A_m = j(a_m - a_{-m}), \forall m \geq 1.$$

We have already explained in Question 2 that the FS coefficient for a $T-$periodic odd function is given by

$$a_m = -j\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)\sin(\frac{2\pi}{T}mt)dt. \tag{12}$$

Some students were confused by the fact a_m is a complex number even when $g(t)$ is a real function. This can demystified as follows. When $g(t)$ is real, we know that the Fourier Transform (Series) is symmetric, i.e. $a_{-m} = a^{*}m$. Thus

$$A_m = j(a_m - a_{-m}) = j(a_m - a_m^{*}) = j^2 2\mathrm{Imag}(a_m) = -2\mathrm{Imag}(a_m).$$

From (12), we get

$$A_m = 2\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} g(t)\sin(\frac{2\pi}{T}mt)dt.$$

Take $L' = \frac{L}{2}$.

$$\begin{aligned}
A_m &= \frac{4}{L}\int_0^{\frac{L}{2}} g(t)\sin(\frac{2\pi}{L}mt)dt \\
&= \frac{4}{L}\int_0^{p} \frac{h}{p}t\sin(\frac{2\pi}{L}mt)dt + \frac{4}{L}\int_0^{p} h\frac{L'-t}{L'-p}\sin(\frac{2\pi}{L}mt)dt \\
&= \frac{4}{L}\frac{L}{2\pi m}\left[-\frac{h}{p}t\cos(\frac{2\pi}{L}mt)\right]_0^p - \frac{4}{L}\frac{L^2}{(2\pi m)^2}\left[-\frac{h}{p}\sin(\frac{2\pi}{L}mt)\right]_0^p \\
&\quad + \frac{4}{L}\frac{L}{2\pi m}\left[-h\frac{L'-t}{L'-p}\cos(\frac{2\pi}{L}mt)\right]_p^{L'} - \frac{4}{L}\frac{L^2}{(2\pi m)^2}\left[h\frac{1}{L'-p}\sin(\frac{2\pi}{L}mt)\right]_p^{L'} \\
&= -\frac{4}{L}\frac{L}{2\pi m}h\cos(\frac{2\pi}{L}mp) + \frac{4}{L}\frac{L^2}{(2\pi m)^2}\frac{h}{p}\sin(\frac{2\pi}{L}mp) \\
&\quad + \frac{4}{L}\frac{L}{2\pi m}h\cos(\frac{2\pi}{L}mp) + \frac{4}{L}\frac{L^2}{(2\pi m)^2}\frac{h}{L'-p}\sin(\frac{2\pi}{L}mp) \\
&= \frac{4}{L}\frac{1}{(\frac{2\pi}{L}m)^2}\sin(\frac{2\pi}{L}mp)\left(\frac{h}{p} + \frac{h}{L'-p}\right) \\
&= \frac{2h}{(\frac{2\pi}{L}m)^2}\frac{1}{p(L'-p)}\sin(\frac{2\pi}{L}mp).
\end{aligned}$$

(b) Find the coefficients B_m such that

$$g(t) = \sum_{m\geq 0} B_m \cos(\frac{2\pi}{L}mt).$$

Solution: While we can continue in the same fashion as above, let us do it by a different approach. Let us evaluate the Fourier Transform of $g(u)$ defined for $(0, L')$, call it $G(f)$. Then,

$$G(f) = \frac{h}{(2\pi f)^2}\left[\frac{\exp(-j2\pi fp) - \exp(-j2\pi fL')}{L'-p} - \frac{1-\exp(-j2\pi fp)}{p}\right].$$

Notice that the Fourier Transform of $g(-t)$ is $G(-f)$. Furthermore, we can construct an even function by $g_e(t) = g(t) + g(-t)$ and the Fourier Transform of $g_e(t)$ is $G(f) + G(-f)$. However

$$G_e(f) = G(f) + G^*(f) = 2.\text{Real}\left(G(f)\right).$$

Imagine repeating the waveform $g_e(t)$ over the entire real axis to obtain $g_p(t)$. This is like tiling over the $x-$axis with non-overlapping replicas of $g_e(t)$, see the Figure below.

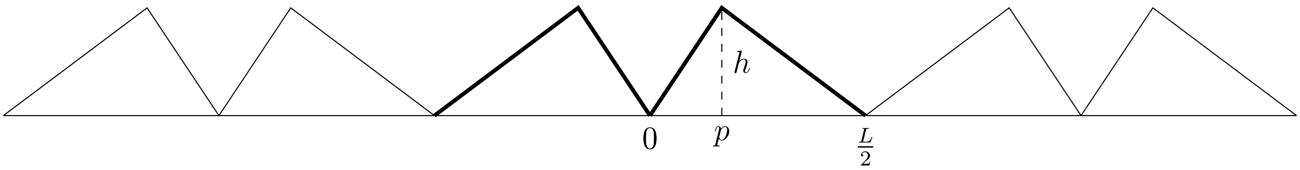


Figure 1: $g_p(t)$ obtained as periodic repetition of $g_e(t)$ (thick portion)

Now, the Poisson sum-formula will say that the FS coefficients of $g_p(t)$ is nothing but

$$\begin{aligned} a_m &= \frac{1}{L} G_e(\frac{m}{L}) = \frac{2}{L}\text{Real}\left(G(\frac{m}{L})\right) \\ &= \frac{2}{L}\frac{h}{(2\pi\frac{m}{L})^2}\left[\frac{\cos(2\pi\frac{m}{L}p) - \cos(2\pi\frac{m}{L}L')}{L'-p} - \frac{1-\cos(2\pi\frac{m}{L}p)}{p}\right] \\ &= \frac{2}{L}\frac{h}{(2\pi\frac{m}{L})^2}\left[\frac{\cos(2\pi\frac{m}{L}p) - (-1)^m}{L'-p} - \frac{1-\cos(2\pi\frac{m}{L}p)}{p}\right] \end{aligned}$$

The coefficients required in the question can now be found as $B_m = a_m + a_{-m}$ for $m > 0$ and $B_0 = a_0$.
Previous Question: Now that we introduced this technique, it can as well solve the previous question (part (a)).

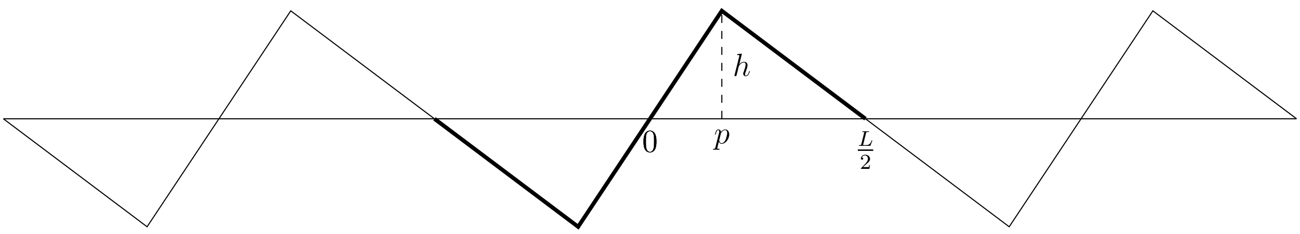


Figure 2: $g_p(t)$ obtained as periodic repetition of $g_{odd}(t)$ (thick portion)

The FS coefficients there are nothing but $j\frac{2}{L}(\text{Im}(G(\frac{m}{L}))$. Clearly,

$$\text{Im}(G(\frac{m}{L}) = -\frac{h}{(\frac{2\pi}{L}m)^2}\sin(\frac{2\pi}{L}mp)\left(\frac{1}{p} + \frac{1}{L'-p}\right).$$

We can now find A_m in the previous question as

$$\begin{aligned} A_m &= j\left(j\frac{2}{L}\text{Im}\left(G(\frac{m}{L})\right) - j\frac{2}{L}\text{Im}\left(G(-\frac{m}{L})\right)\right) \\ &= \frac{4}{L}\frac{h}{(\frac{2\pi}{L}m)^2}\sin(\frac{2\pi}{L}mp)\left(\frac{1}{p} + \frac{1}{L'-p}\right). \end{aligned}$$

We obtained the same formula by alternate means earlier. Figure illustrates the FS reconstuction.

(c) Find the coefficients C_m such that

$$g(t) = \sum_{m \in \mathbb{Z}} C_m \exp(j \frac{4\pi}{L} m t).$$

Solution: Can be done similar to the last two parts.

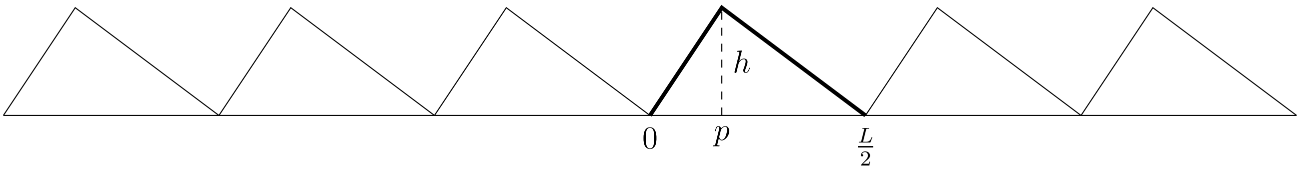


Figure 3: $g_p(t)$ obtained as periodic repetition of $g(t)$ (thick portion)

(d) We have learnt that FS coefficients can uniquely identify a continuous function. However, there seems to be three expansions given in parts $(a), (b)$ and (c). How do you reconcile these different expansions.
Solution: Certainly the three expansions correspond to three different functions in $-\frac{L}{2}, \frac{L}{2}$. In particular, (a) is an odd function, (b) is an even function and (c) is not even or odd. However, all these reconstructions agree on the same values in the interval $[0, \frac{L}{2}]$, which was our interest.

(e) Which of the expansions above is useful in identifying the harmonics of a vibrating string. Write the first harmonic frequency and suggest a position p for which all the even harmonics are missing.
Hint: 'Fourier Transform+Poisson sum-formula+Convolution-multiplication" can handle Fourier Series

Solution: Certainly part (a) meets the boundary conditions of the vibrating string (see derivation in class). Thus, the even harmonics can be suppressed by choosing

$$p = \frac{L}{4}.$$

Question 5) Find the Fourier Series expansion for

$$f(t) = \sin(\theta + 2\pi f_0 t) \text{ where } \theta \in \mathbb{R}. \tag{13}$$

Are the F.S. coefficients continuous in θ?
Solution: Now that we are familiar with the Fourier Transform, we can do it easily. Notice that

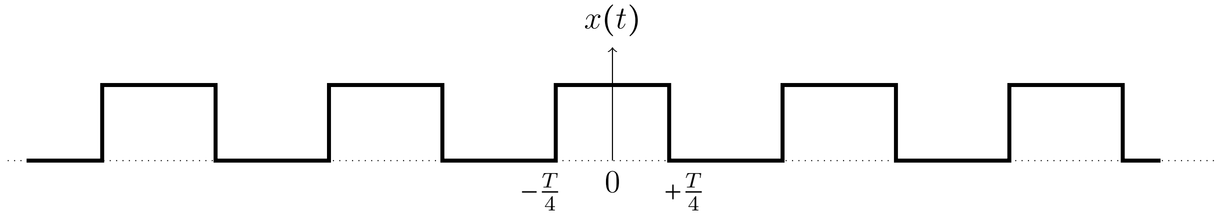
$$f(t) = \frac{1}{j2} \exp(j\theta) \exp(j2\pi f_0 t) - \frac{1}{j2} \exp(-j\theta) \exp(-j2\pi f_0 t)$$

Clearly the fundamental frequency is f_0, and the FS coefficients are

$$a_1 = \frac{1}{j2} \exp(j\theta); \; a_{-1} = -\frac{1}{j2} \exp(-j\theta)$$

Clearly, the coefficients are continuous in θ.

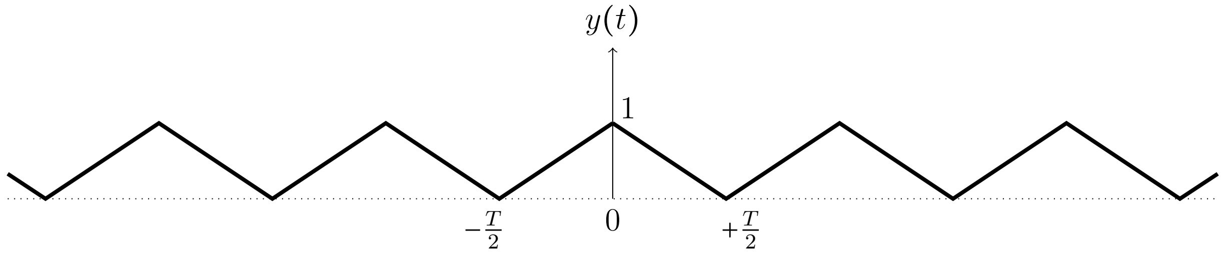
Question 6) Consider a T-periodic signal $x(t)$ shown in figure. This is known as the rectangular train, where the non-zero amplitude is unity.



a) Find the Fourier Series coefficients of this signal.
Solution: Since $rect_\tau(t)$ has $\tau\text{sinc}(f\tau)$ as the Fourier transform, a periodic repetition of the rectangle waveform will have a sampled and scaled sinc waveform in frequency domain. With $\tau = \frac{T}{2}$ the FS coefficients are

$$a_m = \frac{1}{T}\frac{T}{2}\text{sinc}(\frac{m}{T}\frac{T}{2}) = \frac{1}{2}\text{sinc}(\frac{m}{2}).$$

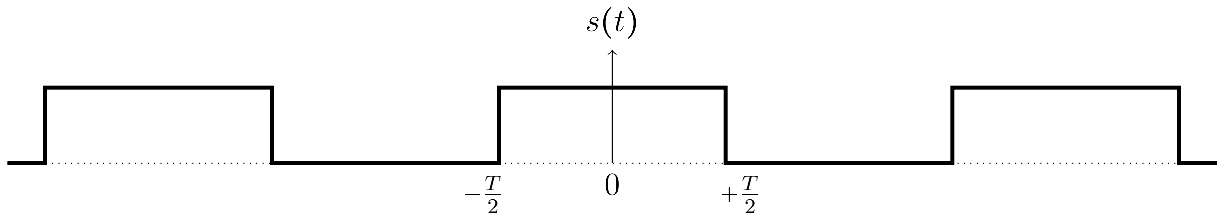
b) Can you find a system $h(t)$ such that $y(t) = x(t) * h(t)$ is the following signal,



Soution: $h(t) = \frac{2}{T}\text{rect}_{\frac{T}{2}}(t)$ will do.
c) Can you find the FS coefficients of $y(t)$ by using parts (a) –(b), and without explicitly performing an additional integration.
Solution: By convolution-multiplication theorem. Note that we are convolving $x(t)$ with $\frac{2}{T}\text{rect}_{\frac{T}{2}}(t)$ to obtain $y(t)$. Thus FS coefficients of $y(t)$ are

$$a_m = \frac{1}{2}\text{sinc}^2(\frac{m}{2}).$$

d) Consider the following $2T$-periodic rectangle train $s(t)$ of height 2 units.



Find the FS coefficients of $y(t)s(t)$, where the multiplication is point-wise for every t.
Solution: The waveform $y(t)s(t)$ can also be obtained in the following way. Consider the pulse train $x_p(t)$ shown in Figure 4.

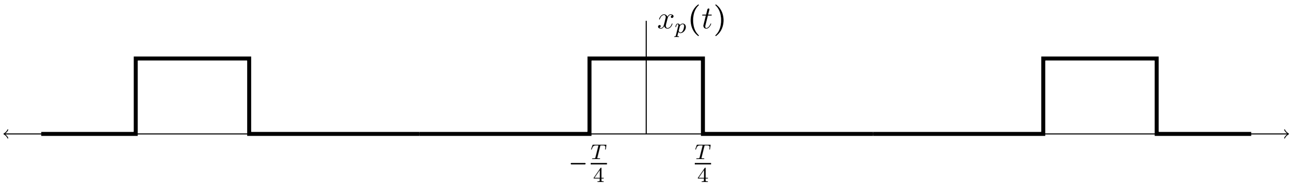


Figure 4: Rectangular train $x_p(t)$

Notice that the difference between $x(t)$ and $x_p(t)$ is only in the time period, i.e. the latter is $2T$ periodic, while the former is $T-$periodic. Furthermore, notice that

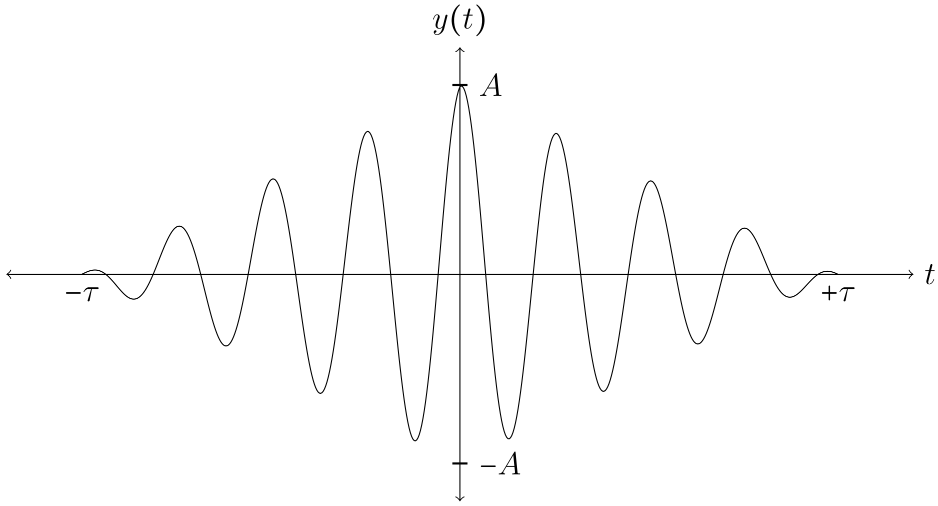
$$y(t).s(t) = x_p(t) * 2\frac{2}{T}rect_{\frac{T}{2}}(t),$$

where the factor 2 comes because of the height of $s(t)$. The FS coefficients of $y(t)s(t)$ can now be easily identified as

$$\begin{aligned} a_m &= 2.\frac{1}{2T}\frac{T}{2}\text{sinc}(\frac{m}{2T}\frac{T}{2}).\text{sinc}(\frac{m}{2T}\frac{T}{2}) \\ &= \frac{1}{2}\text{sinc}^2(\frac{m}{4}). \end{aligned}$$

e) Plot the FS coefficients for parts (c) and (d), assuming $T = 10$.
Solution: to appear soon.

Question 7) Find the Fourier Transform of the following signal $y(t)$.



Solution: The shown waveform is a product of a cosine waveform with a triangle waveform. This will results in a Fourier transform of the form,

$$Y(f) = \frac{A}{2}\text{sinc}^2(f - f_0) + \frac{A}{2}\text{sinc}^2(f + f_0),$$

where f_0 is the frequency of the cosine waveform.