Indian Institute of Technology Bombay

Dept of Electrical Engineering

Handout 9 EE 210 Signals and Systems

Home Work 3 August 28, 2015

Question 1) Parseval's Theorem: Show that

$$\int |x(t)|^2 dt = \int |X(f)|^2 df. \tag{1}$$

Solution: Recall that $x(t) \overset{FT}{\rightleftharpoons} X(f)$ will imply that $x^*(t) \overset{FT}{\rightleftharpoons} X^*(-f)$.

$$\int x^*(t)\exp(-j2\pi ft)dt = \left(\int x(t)\exp(j2\pi ft)dt\right)^* = X^*(-f)$$

By a change of variables

$$x^*(-t) \overset{FT}{\rightleftharpoons} X^*(f).$$

Consider $y(t) = x(t) \star x^*(-t)$, where the $\star$-operator is for convolution and $(\cdot)^*$ (superscript) is for complex conjugation. We know that

$$\begin{aligned} y(0) &= \int Y(f)df \\ &= \int X(f)X^*(f)df \\ &= \int |X(f)|^2 df. \end{aligned}$$

Also

$$\begin{aligned} y(0) &= \int x^*(-\tau)x(t-\tau)d\tau \text{ at } t=0 \\ &= \int |x(\tau)|^2 d\tau. \end{aligned}$$

Thus, we have proved Parseval's theorem.

Question 2) Find

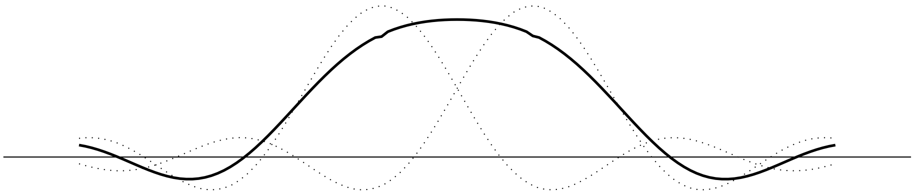
$$\int_{\mathbb{R}} \operatorname{sinc}^2(Tt)dt.$$

Solution: Using Parseval's theorem,

$$\int_{\mathbb{R}} \operatorname{sinc}^2(Tt)dt = \int \frac{1}{T}^2 (rect_T(t))^2 dt = \frac{1}{T}.$$

Parseval's theorem is thus extremely useful in converting difficult evaluations to almost trivial ones.

Question 3) In figure, the thick line $x(t)$ is the sum of the two dotted plots, and each dotted line is a $\alpha\operatorname{sinc}(\frac{2}{\pi}t)$ function, shifted by t_0 units to each side of origin.



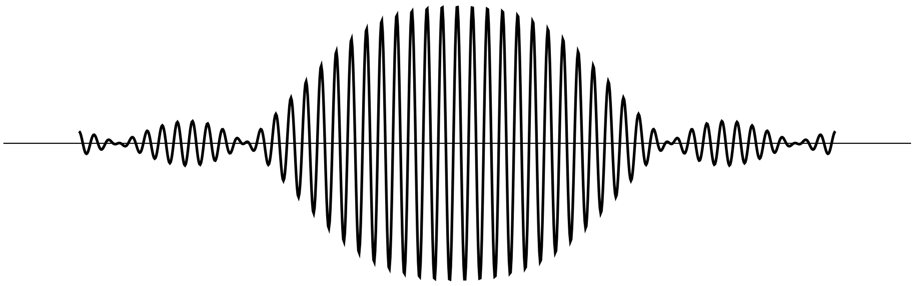
a) Compute the FT of this signal.
Solution: Clearly,

$$x(t) = \alpha \text{sinc}(\frac{2}{\pi}(t - t_0)) + \alpha \text{sinc}(\frac{2}{\pi}(t + t_0)) \qquad = \alpha \text{sinc}(\frac{2}{\pi}t) * (\delta(t - t_0) + \delta(t + t_0)),$$

where $*$ is for convolution. Now using convolution-multiplication theorem,

$$X(f) = \alpha \frac{\pi}{2} \text{rect}_{\frac{2}{\pi}}(f).2\cos(2\pi t_0 f)$$

b) Suppose now the plot is of the form,



What do you expect the FT to be.
Solution: Since the time domain waveform is multiplied by $\cos(2\pi f_0 t)$ to obtain $y(t)$, we have

$$Y(f) = X(f) * \left(\frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0)\right)$$

Question 4[*]) Recall the full wave rectifier that we discussed in the class. Let us design a filter with impulse response $h(t)$, which will convert the output of the bridge circuit to an ideal DC.

(a) Specify a filter with bounded support, i.e. $h(t) = 0$ when $|t| > B$, for some finite value $B < \infty$, such that the output of the filter is a steady (ideal) DC voltage.
Solution: In class, we showed that an ideal low pass filter which cuts all frequencies greater than or equal to double the supply frequency (f_s) will do the job, i.e. $H(f) = 0, |f| > 2f_s$. However, the time domain filter becomes a sinc$(\cdot)$ wavform, with unbounded support. In particular, any filter which has bounded support in frequency is to occupy an unbounded support in time, this is kind of an uncertainty principle. So, we need to look for filters with unbounded support in frequency. How can a $H(f)$ with unbounded support provide pure DC. This is possible only if $H(f) = 0$ at multiples of $2f_s$. In this case, all the ripple frequencies will fall at the zeros of $H(f)$, and $X_r(mf_s)H(mf_s) = 0, \forall |m| \geq 1, m \in \mathbb{Z}$. This visualization tells us that

$$H(f) = 2f_s \text{sinc}(2f_s f)$$

will do the job. Notice that the filter is a rectangle in the time domain. At the end of the day, we have take a rectangle which can contain an integer multiple of any of the frequencies $2mf_s, m \in \mathbb{Z}$, ensuring that they all average out to zero, when $m \neq 0$.

(b) From part (a), design a filter which additionally requires that $h(t) \geq 0, \forall t$ and $H(f) \geq 0, \forall f$.

Solution: The additional requirement can be met with

$$H(f) = 4f_s^2 \text{sinc}^2(2f_s f),$$

which corresponds to a triangle filter response in the time domain.

Question 5[*] Let us consider a half-wave recitifier, which acts like a closed switch for the positive cycles of the supply. On the other hand, negative inputs are blocked completely.

(a) If the supply is a cosine of frequency f_s Hz, give the frequency components present at the output of the rectifier.
Solution: Consider a unit amplitude cosine waveform of $f_s = \frac{1}{T}$ Hz, the output of the rectifier $x_r(t)$ can be visualized as

$$x_p(t) = x_r(t) * \sum_n \delta(t - nT),$$

where $x_r(t) = \cos(2\pi f_s t).rect_{\frac{T}{2}}$. Clearly, by Fourier Series analysis, there will be a dc voltage as well as ripples at multiples of f_s, as $x_r(t)$ is a T-periodic waveform.

(b) Consider a $20V$ peak, $50Hz$ supply feeding the half-wave rectifier. Design an RC filter circuit such that the variations due to ripple are limited to 20% around mean DC value.
Solution: Recall from the class that

$$H(f) = \frac{1}{1 + j2\pi f RC}$$

is the response of our RC filter. The magnitude of the response $H(f)$ dies down as $\frac{1}{|f|}$. Notice that the FS coefficients of $x_p(t)$ are $\frac{1}{T}X_r(\frac{m}{T})$ (by Poisson sum formula). $X_r(f)$ is nothing but the scaled superposition of two sinc waveforms, and it is evident that $X_r(f)$ also dies down as $\frac{1}{|f|}$ at higher frequencies, thus $X_p(\frac{m}{T})H(\frac{m}{T})$ dies down like $\frac{1}{|m|^2}$ at higher values of the frequencies. This points are mentioned to re-affirm that the effect of ripples at $2f_s$ or more will be automatically subdued if we ensure that the ripple at $\pm f_s$ is kept under control. Let us take

$$|H(f_s)| = \frac{1}{\sqrt{1 + 4\pi^2 f_s^2 R^2 C^2}} = 0.01$$

This can be realized by a $C = 1F$ and $R \approx \frac{1}{\pi}$. Typically, it is cheaper to get lower capacitance values and higher R, however the R here is most often dictated by the circuit internal impedances. Note that further ripples, i.e. at $\frac{m}{T}$ Hz will be attenuated by a factor of $\frac{1}{m}$ etc by this filter. However, the filter response as well as the input frequency components starts to decay like $\frac{1}{|m|}$ at higher order ripples. Thus the cumulative effect of all these ripples at the output remain nominal.