Indian Institute of Technology Bombay
Dept of Electrical Engineering

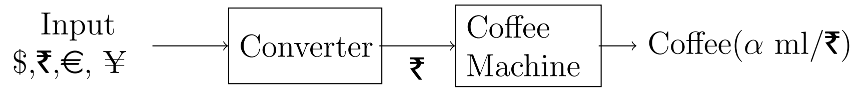
Handout 6 — EE 210 Signals and Systems
Lecture Notes 2 — August 24, 2015

Fourier Series

We have come across the term *Fourier Series* in the last chapter. This is a term so dear to Signal Processing, a panacea for many problems there. A natural question (often forgotten) here is "why Fourier Series/Analysis?". This is particularly significant to this class, where students are from different disciplines, and this is kind of a first exposure to signal processing.

0.1 Why Fourier?

A good example can answer this question logically. We don't have to search far, remember our coffee machine example from the last chapter (Example 6). Let us assume that the exchange-rates are fixed and the machine accepts any currency as input. Let us also not worry about conversion to numbers and available denominations, think that Indian currency (rupee) is a *real* currency (all real values are possible and available).



An automatic currency converter can make a dump device like a coffee machine appear intelligent. In signal processing parlance, we are searching for a converter which transforms input signals to some **currency signals**. The currency signals have the property that they can be manipulated quite easily by any given system, in particular linear systems. The Fourier's trail will lead the followers to such a signal representation.

Specifically, we like to write,

$$x(t) = \sum_{k \in \mathbb{Z}} \alpha_k \phi_k(t), \tag{1}$$

where $x(t)$ is the input signal and $\phi_k(t)$ are the **currency signals**. Furthermore, we insist that the currency signals obey

$$\phi_k(t) * h(t) = \beta_k \phi_k(t), \ \forall k \in Z, \tag{2}$$

for any given linear system $h(t)$.

In other words, convolution of $h(t)$ and $\phi_k(t)$ amounts to simply scaling $\phi_k(t)$ by some value β_k (typically a complex number). Notice that β_k depends on the system (or function) $h(t)$, while the scalars $\alpha_k, k \in \mathbb{Z}$ in (1) depend on the signal $x(t)$.

In a nut-shell, Fourier Analysis is an essential part of the Signal Processing toolbox, which helps us solve even seemingly complex problems, by transforming them to simple parallel problems with an additive structure, from which the global answers can be synthesised. Let us now study an example, which illustrates the origin and fundamentals of Fourier Series representation[1].

[1] Refer: Kreyszig: 'Engg Maths' or Stein and Schakarchi, 'Fourier Analysis'

1 Analysing the Vibrating String

Let us start with the equation of the string that we did in class. The string is of length L and placed along the line segment from origin to the point $(L, 0)$, with both ends fixed. It is plucked, or placed at an initial position. Let u be the horizontal distance and $s(u, t)$ be its displacement in the vertical direction. We know that $s(0, t) = s(L, t) = 0, \forall t$. We showed that the dynamics of a vibrating string can be approximated by

$$\frac{\delta^2 s}{\delta t^2} = \frac{\tau}{\rho}\frac{\delta^2 s}{\delta u^2}. \tag{3}$$

By taking $c^2 = \tau/\rho$,

$$\frac{1}{c^2}\frac{\delta^2 s}{\delta t^2} = \frac{\delta^2 s}{\delta u^2}. \tag{4}$$

By suitable scaling of the variables, we can equivalently solve

$$\frac{\delta^2 s}{\delta t^2} = \frac{\delta^2 s}{\delta u^2}. \tag{5}$$

Furthermore, let us take $L = \pi$, a suitable scaling can generalize the results to all $L \in \mathbb{R}^+$.

There are many ways to solve such partial differential equations. The initial conditions hold the key in resolving the coefficients of the general solution obtained by various means. The fundamental constraint imposed here is by the initial position of the string. We assume that the initial position is $f(u), 0 \le u \le \pi$ for some meaningful function $f(\cdot)$. It so happens that we need one more constraint or initial condition to completely characterize the solution. Let us assume that the initial velocity of the string is given by the function $g(u)$. Putting it all together,

Solve,

$$\frac{\delta^2 s}{\delta t^2} = \frac{\delta^2 s}{\delta u^2}.$$

subj to:

$$\begin{aligned} 0 \le u \le \pi \\ t \ge 0 \\ s(u,0) = f(u) \\ s'(u,0) = g(u) \end{aligned}$$

Fourier[2] in the early 1800s advocated the solution of the above problem using the **method of separation**. Notice that the left side of the p.d.e has derivative with respect to (*wrt*) t and the right side *wrt* x. It seems natural to look for solutions to (5) which factor into two functions, i.e.,

$$s(x, t) = w(t)\phi(u). \tag{6}$$

Substituting this in (5),

$$\frac{w''(t)}{w(t)} = \frac{\phi''(u)}{\phi(u)}. \tag{7}$$

[2]his interest was more in heat equations

Since the LHS is a function of t and RHS that of u, equality in above implies that both sides equal some real constant, say λ.

$$\begin{aligned} w''(t) - \lambda w(t) &= 0 \\ \phi''(u) - \lambda \phi(u) &= 0 \end{aligned} \tag{8}$$

Exercise 1. *Show that equation* (7) *implies the above equations.*

Notice that $\lambda > 0$ cannot be a solution to the wave equation, as the resulting w is not a wave at all.

Exercise 2. *Show that* $\lambda > 0$ *is not a solution to our wave equation.*

Thus assume that $\lambda = -m^2$ for some m. It is easy to verify that $\hat{A}_m \cos(mt), \hat{A}_m \in \mathbb{R}$ is a solution to the time equation, and so is $\hat{B}_m \sin(mt)$. So the general solutions take the form,

$$\phi(u) = \tilde{A}_m \cos(mu) + \tilde{B}_m \sin(mu) \tag{9}$$

$$w(t) = \tilde{\tilde{A}}_m \cos(mt) + \tilde{\tilde{B}}_m \sin(mt). \tag{10}$$

Substituting the initial condition that $\phi(0) = 0$ will imply

$$\tilde{A}_m = 0. \tag{11}$$

Further $\phi(\pi) = 0$ implies that m is an integer. We can then re-scale the solution as,

$$s(u,t) = (A_m \cos(mt) + B_m \sin(mt)) \sin(mu). \tag{12}$$

We know that if there are two solutions to the PDE, the sum of these solutions will also satisfy the PDE. We can then write,

$$s(u,t) = \sum_{m \in \mathbb{Z}^+} (A_m \cos(mt) + B_m \sin(mt)) \sin(mu). \tag{13}$$

How can we check that the above $s(u,t)$ covers all the possible solutions. One way to verify is to see whether $u(x,t)$ satisfies the initial condition. If all the solutions are included in (12), then we will have,

$$f(u) = \sum_{m \geq 1} A_m \sin(mu)\,,\ 0 \leq u \leq \pi. \tag{14}$$

Notice that the initial position of the string is arbitrary (depends on the way one holds it). So, if an arbitrary function (zero at both ends) can be expressed as a sum of sine waves as in (14), then we have some confidence in our solution to the wave equation.

Many didn't believe in the validity of this representation, and it was Fourier's die-hard belief which made the difference, and he applied this technique to some very important problems of his time.

The representation in (14) is valid in the range $[0, \pi]$, but we can extend this to $[-\pi, \pi]$ by making $f(u)$ an odd function. i.e. $f(-u) = -f(u)$. Since sine is an odd function, this extension is immediate. On the other hand, if the function is even in the interval $[-\pi, \pi]$, we expect the representation in (14) to take the form,

$$f_{even}(x) = \sum_{m \geq 0} \hat{A}_m \cos(mx). \tag{15}$$

Any function can be written as a sum of an odd function and even function.

$$\begin{aligned} f(u) &= \frac{1}{2}\left(f(u) + f(-u)\right) + \frac{1}{2}\left(f(u) - f(-u)\right) \\ &= f_{even}(u) + f_{odd}(u) \\ &= \sum_{m\geq 0} \hat{A}_m \cos(mu) + \sum_{m\geq 1} A_m \sin(mu) \\ &= \hat{A}_0 + \sum_{m\geq 1} \frac{1}{2}(\hat{A}_m - jA_m)e^{jmu} + \sum_{m\geq 1} \frac{1}{2}(\hat{A}_m + jA_m)e^{-jmu} \end{aligned} \tag{16}$$

Exercise 3. *For an arbitrary function $f(\cdot)$, verify that $\frac{1}{2}(f(u) - f(-u))$ is an odd function.*

Defining,

$$a_m = \begin{cases} \frac{1}{2}(\hat{A}_m - jA_m), & m \geq 1 \\ \frac{1}{2}(\hat{A}_{-m} + jA_{-m}), & m \leq -1, \end{cases} \tag{17}$$

and $a_0 = \hat{A}_0$, we can rewrite

$$f(u) = \sum_{m=-\infty}^{\infty} a_m e^{jmx}. \tag{18}$$

Is this true for an arbitrary function?. "Oui", said Fourier. We will come back to the veracity of this statement shortly. Before that, we will show how to find the coefficients a_m if the above formula is true.

$$\int_{-\pi}^{\pi} f(u)e^{-jnu}du = \int \sum a_m e^{j(m-n)u}du = 2\pi a_n + \sum_{m\neq n} \int_{-\pi}^{\pi} e^{j(m-n)u}du = 2\pi a_n. \tag{19}$$

Thus

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(u)\exp(-jnu)du. \tag{20}$$

The coefficient a_n is called the n^{th} Fourier coefficient. Often we will replace a_n by $\hat{f}(n)$ to explicitly show its dependency on the function f. In signal processing scenarios, we generally deal with the interval $[-\frac{T}{2}, +\frac{T}{2}]$ as opposed to $[-\pi, \pi]$. Define

$$g(u) = f\left(\frac{2\pi}{T}u\right).$$

Now

$$\begin{aligned} \hat{f}_m &= \frac{1}{2\pi} \int_{-\pi}^{\pi} g(u\frac{T}{2\pi})e^{-jmu}du \\ &= \frac{1}{2\pi} \int_{-\frac{T}{2}}^{+\frac{T}{2}} g(v)e^{-j\frac{2\pi}{T}mv}dv\frac{2\pi}{T}. \end{aligned} \tag{21}$$

Fourier Series Coefficients

$$\hat{f}_m = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} f(u)e^{-j\frac{2\pi}{T}mu}du. \tag{22}$$

Fourier Series Expansion

$$f(u) = \sum_{m\in\mathbb{Z}} \hat{f}_m e^{j\frac{2\pi}{T}mu}. \tag{23}$$

In spite of Fourier being adamant that his representation holds true for arbitrary functions, it took another 150 years for the issue to get settled, mostly in Fourier's favor. We will adopt a particularly elegant formalism as a fact, but this is one of those things which we will not prove rigorously in this course.

2 Periodicity in Fourier Expansion

Consider a function $f(u)$ for which the Fourier Series expansion holds good in the interval $-\frac{T}{2} \leq u \leq +\frac{T}{2}$. What will happen if we simply consider u outside $[-\frac{T}{2}, +\frac{T}{2}]$. At $u' = u + kT$,

$$\begin{aligned}\sum_{m \in \mathbb{Z}} \hat{f}_m e^{j\frac{2\pi}{T}mu'} &= \sum_{m \in \mathbb{Z}} \hat{f}_m e^{j\frac{2\pi}{T}mu} e^{j\frac{2\pi}{T}kT} \\ &= \sum_{m \in \mathbb{Z}} \hat{f}_m e^{j\frac{2\pi}{T}mu} \end{aligned} \tag{24}$$

Thus the FS expansion defines a function $f(u)$ such that,

$$f(u + kT) = f(u). \tag{25}$$

In other words, $f(u)$ is $T-$ periodic. In general, text books and other material say Fourier Series is for periodic functions. The meaning is that the FS expansion gives a periodic function, even if the coefficients are evaluated for a function residing in $[-\frac{T}{2}, +\frac{T}{2}]$.

3 Uniqueness of FS Expansion

Is the FS expansion unique. The following theorem positively answers it except for some number of possible discontinuities in the signal.

Theorem 1 (Stein and Shakarchi 2003)**.** *Suppose f is a T−periodic function which is integrable (locally), with $\hat{f}_n = 0, \forall n \in \mathbb{Z}$. Then $f(t) = 0$ whenever f is continuous at t.*

Proof: The proof uses contradiction arguments. We will start with the assumption that $\hat{f}_n = 0$. If the assertion of the theorem does not follow, i.e. $f(t) \neq 0$ at some instant t where $f(\cdot)$ is continuous, then we will show that $\hat{f}_n \neq 0$ for some n, thus violating our assumption. In other words, the assumption and the assertion has to co-exist. These arguments also show a glimpse of the beauty of proofs in functional analysis. Consider the function

$$p(t) = \epsilon + \cos(\frac{2\pi}{T}t)\,,\ \epsilon > 0 \tag{26}$$

and let

$$p_k(t) = (p(t))^k\,. \tag{27}$$

Since $\hat{f}_n = 0, \forall n \in \mathbb{Z}$, we have

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) p_k(t) = 0, \forall k. \tag{28}$$

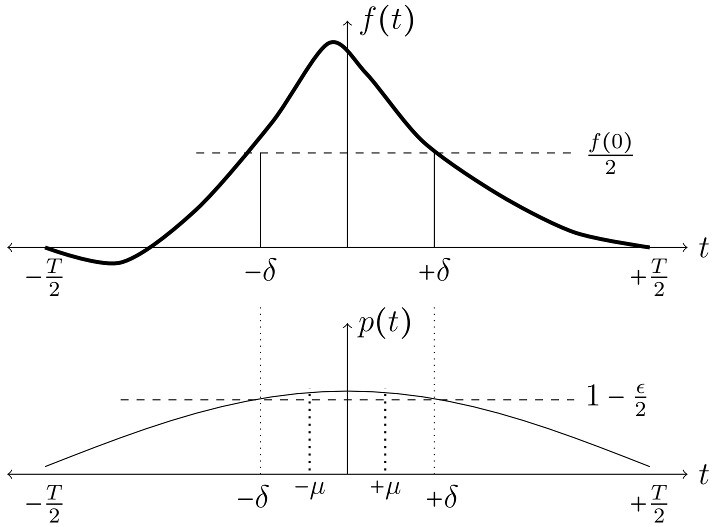
This can be seen by applying binomial expansion on $\left(\epsilon + \frac{e^{j\frac{2\pi}{T}t} + e^{-j\frac{2\pi}{T}t}}{2}\right)^k$, and noticing that each term in the expansion will correspond to a multiple of $\hat{f}_n$ for some $n \in \mathbb{Z}$. Take any point in $t \in [-\frac{T}{2}, +\frac{T}{2}]$, w.l.o.g assume $t = 0$. Otherwise we can shift our arguments to any t. Also let $f(t)$ be continuous at $t = 0$.

Suppose, contrary to the assertion of theorem, $f(0) \neq 0$. Then assuming $f(0) > 0$ entails no loss of generality since we can have the same arguments for $f(t)$ or $-f(t)$. We will now show that $\hat{f}_n > 0$ for some $n \in \mathbb{Z}$, thus achieving the desired violation of the assumption.

By the definition of continuity, there exists $\delta > 0$ such that

$$f(t) \geq \frac{1}{2} f(0)\,,\ t \leq |\delta|. \tag{29}$$

Fix the above parameter δ. We can now choose the parameter ϵ in (26) such that $p(\delta) \leq 1 - \frac{\epsilon}{2}$. Since $p(0) = 1 + \epsilon$ and $p(\delta) = 1 - \frac{\epsilon}{2}$, we can choose a parameter $\mu \in (0, \delta)$ such that $p(t) \geq 1 + \frac{\epsilon}{2}$ whenever $|t| \leq \mu$. See the illustration to get the meaning of these parameters.



$$\begin{aligned}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)p_k(t)dt &= \int_{|t|\leq\mu} f(t)p_k(t)dt + \int_{\mu<|t|<\delta} f(t)p_k(t)dt + \int_{|t|\geq\delta} f(t)p_k(t)dt \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{30}$$

We enumerate each of these integrals.

1. $\underline{|I_3| < \infty}$

   By straightforward computations and since $p(t) \leq 1 - \frac{\epsilon}{2}$ when $|t| \geq \delta$,

$$|I_3| \leq \int_{|t|\geq\delta} |f(t)p_k(t)|dt \tag{31}$$

$$\leq \left(1 - \frac{\epsilon}{2}\right)^k \int_{|t|\geq\delta} |f(t)|dt \tag{32}$$

$$\to 0 \text{ as } k \uparrow \infty, \tag{33}$$

   where we used the assumption that $f(\cdot)$ is locally integrable ($\int_a^b |f(t)|dt \leq \infty$).

2. $\underline{I_2 \geq 0}$

   Since $f(t) \geq \frac{f(0)}{2} > 0$ and $p_k(t) > 0$ when $|t| \leq \delta$.

3. $\underline{I_1 \to \infty}$

$$\int_{|t|\le\mu} f(t)p_k(t)dt \ge \left(1+\frac{\epsilon}{2}\right)^k \int_{|t|\le\mu} f(t)dt \tag{34}$$

$$\ge \left(1+\frac{\epsilon}{2}\right)^k \frac{f(0)}{2} \tag{35}$$

$$\to \infty\,, \text{ as } k \uparrow \infty. \tag{36}$$

Collecting the results together,

$$I_1 + I_2 + I_3 \to \infty.$$

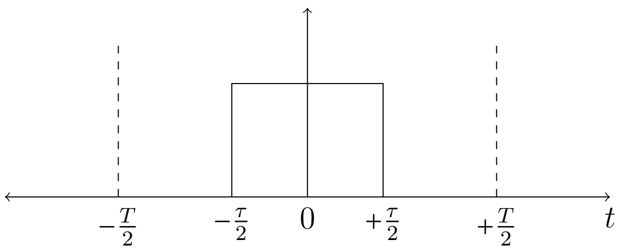
Comparing with (28), $I_1 + I_2 + I_3 = 0$ by assumption, which is invalid if we assume $f(t) \neq 0$. Hence the theorem is proved.

4 Examples

It seems that there has been too much talk and it is time to get wet. Our first example is to find the Fourier Series of a train of rectangular signals. Imagine a goods train *whistling* past. Let us find its frequency components.

Example 1. *A rectangular signal, we will call it* $\mathbf{rect}_\tau(\mathbf{t})$ *is a symmetric waveform which is defined as,*

$$x(t) = \begin{cases} 1.0 \text{ if } -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0 \text{ otherwise} \end{cases} \tag{37}$$



Consider $T-$periodic repetitions of $rect_\tau(t)$. i.e. $f(t) = \sum_n rect_\tau(t - nT)$. Let us note down a few points about $f(t)$.

1. $f(t)$ is periodic: we expect it to have a Fourier Series representation.

2. $f(t)$ is $T-$periodic: only multiples of $\frac{1}{T}$ are the frequencies present in Fourier Series (we know this from the vibration of a string).

3. The FS coefficients are

$$a_m = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\exp(-j\frac{2\pi}{T}mt)\,dt. \tag{38}$$

For the rectangle train in consideration, using 22

$$\begin{aligned}
a_m &= \frac{1}{T}\int_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} \exp(-j\frac{2\pi}{T}mt)\,dt \\
&= -\frac{1}{T}\left|\frac{\exp(-j\frac{2\pi}{T}mt)}{j\frac{2\pi}{T}m}\right|_{-\frac{\tau}{2}}^{+\frac{\tau}{2}} \\
&= \frac{1}{T}\frac{1}{j\frac{2\pi}{T}m}\left(\exp(j\frac{2\pi}{T}m\frac{\tau}{2}) - \exp(-j\frac{2\pi}{T}m\frac{\tau}{2})\right) \\
&= \frac{1}{T}\frac{1}{\frac{\pi}{T}m}\sin(\frac{2\pi}{T}m\frac{\tau}{2}) \\
&= \frac{\tau}{T}\frac{\sin\pi m\frac{\tau}{T}}{\pi m\frac{\tau}{T}} \\
&= \frac{\tau}{T}\,\mathrm{sinc}(m\frac{\tau}{T}). \qquad (39)
\end{aligned}$$

The second to last equality was obtained by multiplying both numerator and denominator by τ. The last equality uses the definition of **cardinal sine** function.

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \qquad (40)$$

Example 2. *Consider a periodic train of impulses, denoted as* $\mathbf{\Delta_T(t)}$.

$$\Delta_T(t) = \sum_n \delta(t - nT) \qquad (41)$$

The Fourier series coefficients are given by

$$a_m = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\exp(-j\frac{2\pi}{T}mt)\,dt \qquad (42)$$

$$= \frac{1}{T}. \qquad (43)$$

The FS coefficients corresponds to a periodic sequence of weighted delta measures, at multiples of the interval $\frac{1}{T}$.

5 A Fourier Currency on Signals

Our search initially was a currency for the signals, with a countable number of designations and easy manipulations. Now we do have a representation of signals, particularly periodic ones, in denominations which are integral multiples of the fundamental frequency. This is good news. Since a signal can be represented by its frequency components, let us study the effect of the system on a signal. Specifically, we need to analyze the effect of each input frequency, represented as $\exp(j\omega t), \omega = 2\pi f$, when passed through the system $h(t)$. Then by using linearity of the system, we can add the effect of each frequency to find the system output.

6 Fourier Transform

Consider a system $h(t)$. We assume that

$$\int_t |h(t)| dt < \infty. \qquad (44)$$

This means that the system is *integrable* and we keep on with this definition for the rest of the section. We have already learnt that an input $x(t)$ to an LTI system $h(t)$ will generate an output,

$$y(t) = h(t) * x(t). \tag{45}$$

By using the formula for convolution and since $x(t) = \exp(j\omega t)$,

$$y(t) = \int h(\tau)e^{j\omega(t-\tau)}d\tau \tag{46}$$

$$= e^{j\omega t}\int_{\tau} h(\tau)e^{-j\omega\tau}d\tau. \tag{47}$$

Notice that for a given input frequency $\omega = 2\pi f$, the output is a constant multiple of the input. That is surprising, but looks elegant. It also says that when we pass a frequency through an LTI system, no new frequencies can appear at the output. Thus the input frequency, unless filtered out, will come out of the LTI system. The effect of the system on each frequency ω is given by,

$$H(f) = \int_{\tau} h(\tau)e^{-j2\pi f\tau}d\tau. \tag{48}$$

In other words, with the input $x(t) = \exp(j\omega t)$,

$$y(t) = H(f)e^{j\omega t}. \tag{49}$$

We will call $H(f)$ as the **Fourier Transform** of $h(\tau)$. This may ring a bell (maybe alarm) to many. Equation (49) bears no similarity to the Fourier Series that we have learnt, at least in the first inspection. Why should we call this Fourier Transform, since we already felicitated Fourier by naming a series after him. On the other hand, equation (48) has a striking resemblance to the Fourier Series expansion in (22). It looks like replacing f in (49) by $2\pi m/T$ will make it similar to (22), but not precisely the same.

Exercise 4. *Write down 3 visible differences between the formulas for Fourier Series and Transform.*

The issue of nomenclature is one thing, but being partial is unbearable. Why do we define just the Fourier Transform of the system $h(t)$, and not for the signals. So in similar vein, if $x(t)$ is *integrable*, we define its Fourier Transform as

$$X(f) = \int_{t} x(t)\exp(-j2\pi ft)dt. \tag{50}$$

This probably needs no explanation, we have learnt that a system and signal can be interchanged in the convolution formula. However, the Fourier Transform of a system had a physical meaning, i.e. the multiplicative gain when we pass a sinusoid through the system. In case of signals, one such strategy is to visualize $x(t)$ as a system for a moment and imagine that we are passing various sinusoids through the system, and measuring the effect at the output. $X(f)$ is precisely the gain for frequency f.

To make matters more clear, imagine our system being an impulse. Any sensible input will come out unruffled through the system. We input a sinusoid of amplitude α, a sinusoid of α comes out. So what is the Fourier Transform of an impulse (or Dirac delta). Though the guess is correct, please verify by doing this exercise.

Exercise 5. *Evaluate the Fourier Transform of a Dirac delta from its definition.*

In retrospect, we defined the impulse that way to have the same response at all frequencies.

In the previous section we have computed the Fourier series coefficients for a rectangular waveform. Recall that it was important to take a fundamental interval $[-\frac{T}{2}, \frac{T}{2}]$ while evaluating the coefficients. This ensured that the frequencies involved are integer multiples of $\frac{2\pi}{T}$. On the other hand, we can also compute the Fourier Transform of a rectangular waveform. In this case, there is no parameter T, as the integration is over all real numbers. Consequently, we can expect the Fourier Transform to be defined for all reals, not just on integer multiples of the fundamental harmonic. However, the Fourier series and transform are intimately connected.

Example 3. *Recall the* $\mathrm{rect}_\tau(t)$ *function that we saw in Example 1.*

$$x(t) = \begin{cases} 1.0 \text{ if } & -\frac{\tau}{2} \le t \le \frac{\tau}{2} \\ 0 \text{ otherwise} & \end{cases} \tag{51}$$

What is the Fourier Transform $X(f)$ of this signal. Let us note some points.

1. Since the function seems to be well behaved and defined in a bounded time interval, we expect it to have a Fourier Transform.

2. The lack of periodicity points that the frequency components can be continuous, unlike the Fourier Series expansion.

3. The Fourier Transform can be obtained by the formula,

$$X(f) = \int_t x(t) \exp(-j2\pi f t)\, dt. \tag{52}$$

In fact,

$$\begin{aligned} X(f) &= \int_{\frac{\tau}{2}}^{\frac{\tau}{2}} \exp(-j2\pi f t)\, dt && (53) \\ &= \frac{1}{j2\pi f} \exp(j2\pi f \frac{\tau}{2}) - \exp(-j2\pi f \frac{\tau}{2}) \\ &= \tau \frac{\sin(\pi f \tau)}{\pi f \tau} \\ &= \tau \,\mathrm{sinc}(f\tau). && (54) \end{aligned}$$

It is now quite clear that the Fourier Series is nothing but the Fourier Transform sampled at multiples of $\frac{1}{T}$ and scaled by $\frac{1}{T}$. We will generalize this result below.

Example 4. ***Constant Function****: We have seen that the Fourier Transform of an impulse is a constant function. What about the Fourier Transform of a constant function. Do we expect it to be an impulse.*

Consider the rectangular function $\mathrm{rect}_\tau(t)$ in the previous example. If we take the width τ to infinity, we can approximate a constant function. Intuitively, we expect the Fourier Transform to be

$$X(f) = \lim_{\tau \to \infty} \tau \,\mathrm{sinc}(f\tau). \tag{55}$$

For the strict minds, this is not an impulse for any value of τ, however arbitrarily large. The function $\mathrm{sinc}(\cdot)$ is not even integrable. Nevertheless, in the limit, we can intuitively consider the waveform as an impulse, in a generalized sense. This intuition is illustrated in Figure 1.

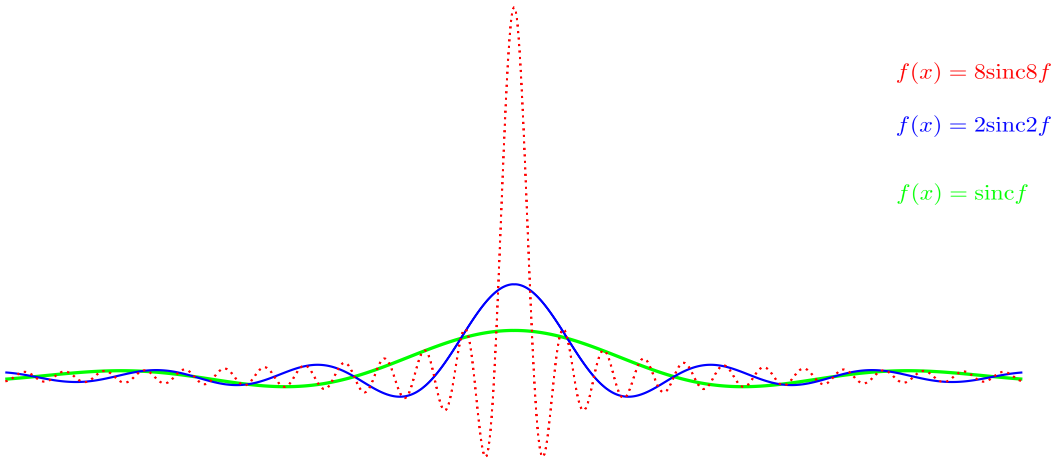


Figure 1: $\tau\,\text{sinc}(f\tau)$ approaching an Impulse

7 Energy of Signals

Upto this we didn't worry about how much energy a signal has. This question so often relates to parameters like battery life, transmission range etc in practical communication/signal processing situations. The rate of energy spent/received will give the power. We found that an impulse or Dirac delta is a *powerful* concept. Is it full of energy too?. The energy of a signal is given by

$$E(f) = \int_t |f(t)|^2 dt. \tag{56}$$

Exercise 6. *What is the energy of a Dirac delta measure?. Do you find the answer surprising?. Can you imagine a physical interpretation.*

One concept that we should keep in mind when transforming to frequency is that, Fourier Transform preserves the energy of the signals.

$$\int_t |x(t)|^2 dt = \int_f |X(f)|^2 df. \tag{57}$$

This has a straight forward proof, which we will see as we go on.

8 LTI Output: Convolution-Multiplication Theorem

If signals and systems have Fourier Transform, why not do it for the output too. After all, the output is just another signal.

$$Y(f) = \int_t y(t) \exp(-j2\pi f t) dt \tag{58}$$

Since $y(t) = x(t) * h(t)$,

$$Y(f) = \int_t \int_\tau h(\tau)x(t-\tau)d\tau \exp(-j2\pi ft)dt \tag{59}$$
$$= \int_\tau h(\tau) \int_t x(t-\tau)\exp(-j2\pi f(t-\tau+\tau))dtd\tau \tag{60}$$
$$= \int_\tau h(\tau)\exp(-j2\pi f\tau) \int_t x(t-\tau)\exp(-j2\pi f(t-\tau))dtd\tau \tag{61}$$
$$= \int_\tau h(\tau)\exp(-j2\pi f\tau)d\tau \int_t x(t-\tau)\exp(-j2\pi f(t-\tau))dt \tag{62}$$
$$= H(f)X(f) \tag{63}$$

In writing this, we assumed that $x(\cdot)$ and $h(\cdot)$ are integrable functions [3].

Exercise 7. *Justify the steps above*

Equation (63) is so important to us, that it warrants a statement as a separate theorem.

Convolution-Multiplication Theorem
Convolution of two time domain signals corresponds to multiplication of their respective Fourier Transforms in the frequency domain.

[3] The interchange of integrals is justified by Fubini's theorem in analysis