

1 Properties and Inverse of Fourier Transform

So far we have seen that time domain signals can be transformed to frequency domain by the so called Fourier Transform. We will introduce a convenient shorthand notation

$$x(t) \stackrel{FT}{\iff} X(f),$$

to say that the signal $x(t)$ has Fourier Transform $X(f)$. Observe that the transform is nothing but a mathematical operation, and it does not care whether the underlying variable stands for time, frequency, space or something else. Clearly it should remain true that

$$x(f) \stackrel{FT}{\iff} X(t).$$

This suggests that there should be a way to invert the Fourier Transform, that we can come back from $X(f)$ to $x(t)$. If the correspondence from $x(t)$ to $X(f)$ is a bijection, then we can uniquely invert $X(f)$. This is true for a wide class of functions, in particular, for those class of signals where both the signal and its Fourier transform are integrable. These can be extended to cover most signals that we deal with in this class. So we assume some regularity properties for the discussion here, that all the integrals are well defined etc, without explicitly stating these each time. Let us first list some properties of the Fourier Transform. Assume that

$$x(t) \stackrel{FT}{\iff} X(f)$$

Then

1. $x(t - \tau) \stackrel{FT}{\iff} \exp(-j2\pi f\tau)X(f)$.
2. $x(t) \exp(j2\pi f_0 t) \stackrel{FT}{\iff} X(f - f_0)$.
3. $x^*(t) \stackrel{FT}{\iff} X^*(-f)$ and $x^*(-t) \stackrel{FT}{\iff} X^*(f)$.
4. $x(at) = \frac{1}{|a|} X\left(\frac{f}{a}\right)$, $\forall a \neq 0$.
5. $x'(t) = j2\pi f X(f)$ if $f(t) \rightarrow 0$ with $|t| \rightarrow \infty$.

Let us prove the last statement, others are straightforward.

$$\begin{aligned} \int x'(t) \exp(-j2\pi ft) dt &= (-j2\pi f t)x(t) \Big|_{-\infty}^{+\infty} + j2\pi f \int \exp(-j2\pi ft)x(t) dt \\ &= j2\pi f \int x(t) \exp(-j2\pi ft) dt \\ &= j2\pi f X(f). \end{aligned}$$

In the study of Fourier Transforms, one function which takes a niche position is the Gaussian function. This is given by

$$g_\alpha(t) = \frac{1}{\sqrt{\alpha}} \exp\left(-\pi \frac{t^2}{\alpha}\right),$$

where $\alpha > 0$ is a parameter of the function. When $\alpha = 1$, we will denote the function as $g(t)$. Let us state a well known result.

$$\int g_\alpha(t) dt = 1.$$

This can be proved as follows.

$$\begin{aligned} \left(\int g_\alpha(t) dt\right)^2 &= \int g(u) du \int g(v) dv \\ &= \frac{1}{\alpha} \int \int \exp\left(-\frac{\pi(u^2 + v^2)}{\alpha}\right) dudv \\ &= \frac{1}{\alpha} \int_0^\infty \int_0^{2\pi} \exp\left(-\frac{\pi r^2}{\alpha}\right) r d\theta dr \\ &= \frac{2\pi}{\alpha} \int \exp\left(-\frac{\pi r^2}{\alpha}\right) r dr \\ &= \frac{\pi}{\alpha} \int_0^\infty \exp\left(-\frac{\pi t}{\alpha}\right) dt \\ &= 1. \end{aligned}$$

In Step 3 above, we converted to polar coordinates for solving the integral. The pivotal role of Gaussian functions follows from the fact that the Fourier transform of a Gaussian function is another Gaussian function. In particular,

$$\exp(-\pi t^2) \stackrel{FT}{\iff} \exp(-\pi f^2) \quad \text{i.e.} \quad g(t) \stackrel{FT}{\iff} g(f) \quad \text{when } \alpha = 1.$$

Exercise 1. Prove the above result.

Furthermore, applying the scaling property, we also have

$$g(t) \stackrel{FT}{\iff} \sqrt{\alpha} g(\alpha f) = \exp(-j\pi\alpha f^2), \forall \alpha > 0.$$

A cute aspect of the above formula is that both sides are Gaussians. Thus the formula from left to right can be traced in the reverse manner. In other words, we have obtained the Fourier inversion formula for Gaussian functions, essentially for free. More specifically, the inversion formula is

$$g_\alpha(t) = \int G_\alpha(f) \exp(j2\pi ft) df,$$

where $G_\alpha(f) := \exp(-\pi\alpha f^2)$. Can we extend this to other functions? This is possible if we make a seemingly simple, but appealing, observation. What happens to $g_\alpha(t)$ when $\alpha \downarrow 0$. We can take

$$\lim_{\alpha \downarrow 0} g_\alpha(t) = \delta(t).$$

Clearly, all the required properties of Dirac delta are satisfied by the LHS. In particular

$$\lim_{\alpha \downarrow 0} g_\alpha(t) * x(t) = x(t), \tag{1}$$

for all points of continuity for any integrable function $x(t)$ (*proof done in class*).

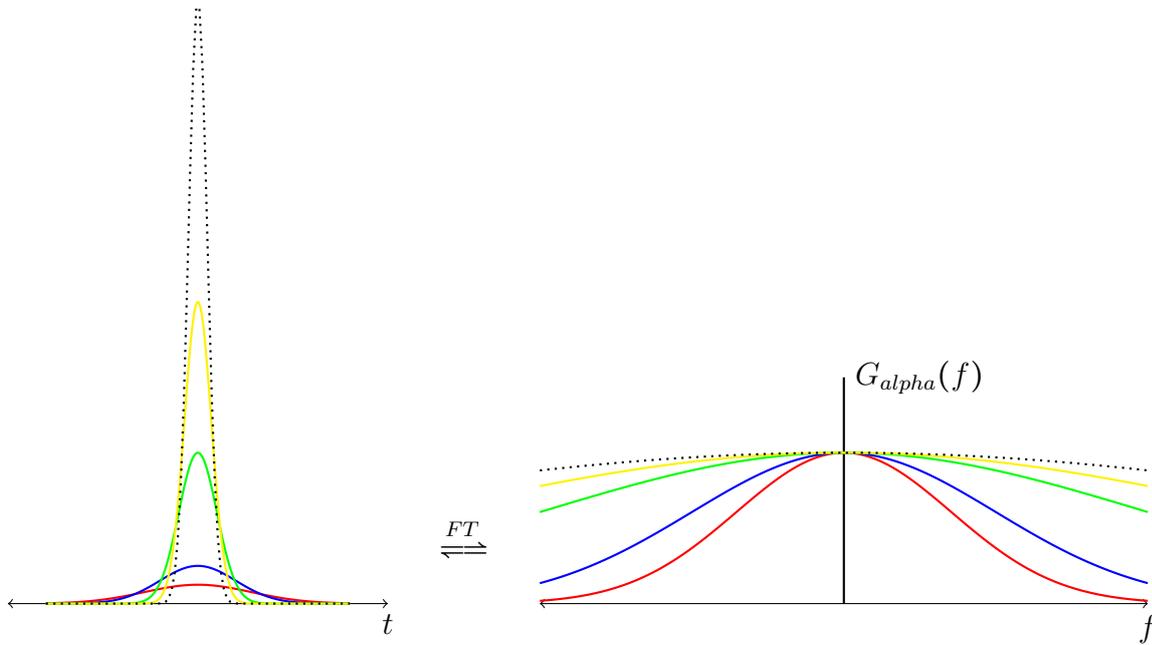


Figure 1: Illustrating $g_\alpha(t)$ and $G_\alpha(f)$ as $\alpha \downarrow 0$.

How about the Fourier Transform of $g_\alpha(t)$ as $\alpha \downarrow 0$. It is evident that $G_\alpha(f) = \exp(-\pi\alpha f^2)$ will end up approximating a constant function in any bounded interval, as $\alpha \downarrow 0$ (see Figure 1). We know that for integrable functions

$$\lim_{T \uparrow \infty} \int_{|t| \geq T} |x(t)| dt = 0.$$

Thus while dealing with the class of integrable functions, we can take

$$\lim_{\alpha \downarrow 0} g_\alpha(t) \stackrel{FT}{\iff} \lim_{\alpha \downarrow 0} G_\alpha(f).$$

The informality in the above equation is that we only know the Fourier Inverse formula when there are no limits on both sides. Nevertheless, we make a symbolic representation which will make life so much easier that all other jargon/mathematics so far can be nearly dispensed with. From now onward, we take the following as a fact, and feel free to apply this without much ado (If interested there are several treatments on the theory of distributions which will give a rigorous justification).

$$\delta(t) \stackrel{FT}{\iff} \mathbb{1}_{\{f \in \mathbb{R}\}}. \quad (2)$$

Notice that the right side is nothing but a constant function of unit amplitude. What is even more remarkable is that the formalism in (2) can also be applied while dealing with several classes of non-integrable signals. In fact, for all the engineering applications in this course, (2) will be treated as a theorem, or principle.

1.1 Fourier Inverse

It turns out that (2) is all that we need to find the Fourier inverse, whenever both the function and its transform are integrable. However, we have defined a Dirac delta in an

operational manner, and for (2) to be true, both the function as well as its Fourier transform should decay sufficiently fast when the respective arguments to these functions are large enough in magnitude. For the time being, let us throw caution to the wind, and press forward.

Inversion Formula

$$x(t) = \int X(f) \exp(j2\pi ft) df. \quad (3)$$

To prove this using (2), let us write

$$\begin{aligned} \int X(f) \exp(j2\pi ft) df &= \int \int x(\tau) \exp(-j2\pi f\tau) d\tau \exp(j2\pi ft) df \\ &= \int x(\tau) \int \exp(-j2\pi f(t - \tau)) df d\tau \\ &= \int x(\tau) \delta(t - \tau) d\tau \\ &= x(t). \end{aligned}$$

The real power of this theorem from an engineering standpoint is perhaps not in having an integral formula, but in realizing that we can simply identify the Fourier Inverse of $X(f)$ as that function $x(t)$ which gives the required Fourier Transform. Thus, we can identify that $\text{sinc}(f\tau)$ has Fourier inverse $\frac{1}{\tau} \text{rect}_\tau(t)$. More generally, we chose notation $x(t) \xLeftrightarrow{FT} X(f)$ to clearly indicate that you can go in both directions, i.e. the RHS is the Fourier Transform of the LHS, and conversely, the LHS is the Fourier Inverse of the RHS.

2 Transform or Series

We have made some progress in advancing the two concepts of Fourier Series and Fourier Transform. Which of them to use, we do not have such a freedom as of now. Some students were confused about this aspect, with the following comment.

“ Sometimes the teacher uses the Fourier series representation, and some other times the Fourier Transform ”

Our lack of freedom has more to do with our mind-set. Fourier Series representation is for periodic signals while Fourier Transform is for aperiodic (or non-periodic) signals. Consider an integrable signal which is non-zero and bounded in a known interval $[-\frac{T}{2}, \frac{T}{2}]$, and zero elsewhere. This signal will have a Fourier Transform. Will it have a Fourier Series? Yes!, but with the underlying imagination that we have to repeat the signal in time by some interval greater than or equal to T , to make it a periodic one. A T -periodic repetition has the effect of *removing* all frequency components from the Fourier Transform other than those which are multiples of $\frac{1}{T}$. This is consistent with our interpretation of the Fourier Series.

This section is aimed at providing a unified view to Fourier Series and Fourier Transform. We will argue that everything can be viewed as Fourier Transform, in a generalized sense. A key tool-kit which can be of great use is called the Dirac Formalisms, which defines symbolic/formal rules by which we can seamlessly move from Fourier Transform to Series. The formalisms allow the identification of Fourier Series as nothing but scaled samples of

the Fourier Transform. This result can also be alluded to a well known formula called Poisson's sum formula.

By the way, picking certain components from a continuous signal is called **sampling**, a very important tool for signal processing. Sampling and corresponding signal reconstruction allow us to move between continuous valued signals and discrete ones in a seamless fashion, which is often necessary in today's digital/discrete world. By using Dirac formalisms, it will become evident that the laws governing sampling are same as the ones that connect Fourier Series and Transform.

3 Sampling

Sampling literally means picking up the values from a few *places*. In discrete countable situations it means choosing a few objects out of the available ones. Unfortunately the situation is not that simple when it comes to sampling a continuous time signal to get a discrete-time one. Let us go through an example.

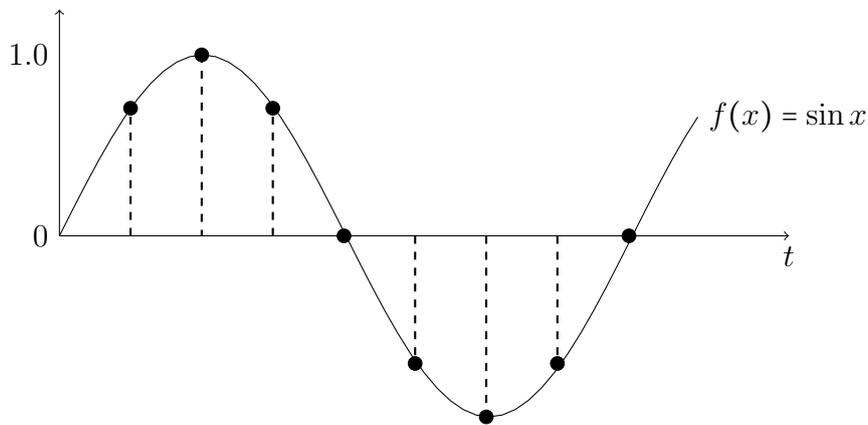
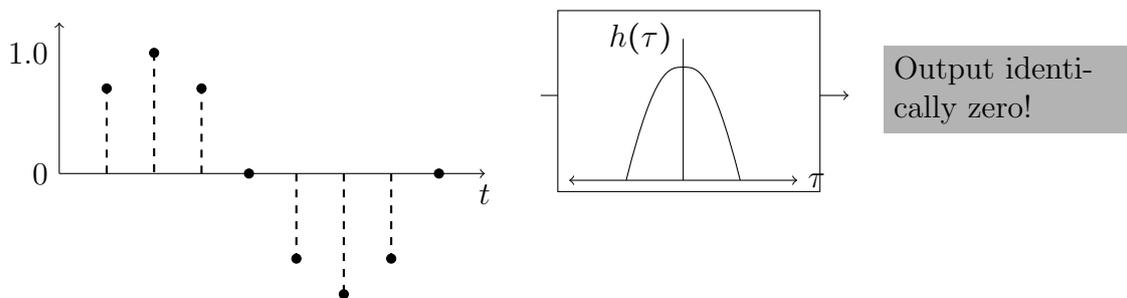


Figure 2: Sampling a sine wave

In Figure 2, it is evident that sampling at $\frac{4}{\pi}$ samples per second will yield pictorially the points represented by the black dots. These values are convenient to store and manipulate. But they present a difficulty when we wish to convert them back to a continuous time signal. Why?. To understand this, imagine a continuous time filter (LTI system) which takes the samples as input and outputs some waveform (for example, an interpolator in continuous time.)

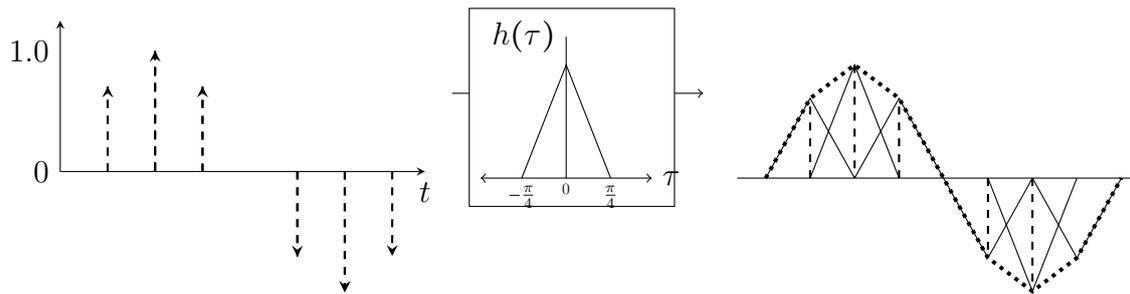


The convolution integral will not recognize the discrete samples as such.

$$\int_{\tau} h(\tau)x(t - \tau)d\tau = 0. \tag{4}$$

We need a formalism to side-step such difficulties. We were in a similar situation a while back, when we introduced the Dirac formalism to define an impulse, so that the convolution of any continuous time signal with a Dirac is well defined. We can extend that formalism to sampling too. Instead of samples, we will be dealing with sample and hold. i.e, the sampled value represents an impulse with area equal to the value of the function at that point. With this additional formalism, we can seamlessly move between continuous and discrete time in almost all physical situations and engineering problems.

Under this new formalism for samples, Figure 3 shows the output when the waveform $\sin(x)$ sampled at intervals of $\frac{\pi}{4}$ are fed into a linear interpolator.



The output is something that we expect and logical. Can we always treat samples as impulses. Yes, if we apply adequate care. We will write this symbolically as the second Dirac formalism.

Multiplication by an impulse corresponds to sampling

$$f(t)\delta(t - a) = f(a)\delta(t - a) \quad (5)$$

Here we assume that $f(t)$ is locally continuous at a .

This formalism was introduced by physicists, in particular by Laurent Schwartz in the theory of distributions. Keep this formalism in mind while we go forward: viewing samples as impulses will enable us to treat Fourier Transform and Fourier Series as the two faces of a single entity. Essentially formulation of a sample as an impulse is like treating the discrete-time signal as a continuous time one, and do all the operations relevant to the class \mathcal{C}^0 .

“ Once this is understood, we will start calling all frequency conversions by the name Fourier Transform, without explicitly mentioning that sometimes its samples are being used to account for periodicity. ”

4 Series is Transform

“ The best way to deal with Fourier Series now is to get rid of it” :-WT

The key step is to identify that the Fourier Series are nothing but weighted samples of the Fourier Transform at periodic intervals in frequency. This result is so important that it warrants a separate statement, and indeed it is well known as the Poisson sum formula.

4.1 Poisson's Sum Formula

Given $x(t) \stackrel{FT}{\rightleftharpoons} X(f)$,

$$\sum_{m \in \mathbb{Z}} x(t + nT) = \frac{1}{T} \sum_{m \in \mathbb{Z}} X\left(\frac{m}{T}\right) \exp(j2\pi \frac{m}{T}t) \quad (6)$$

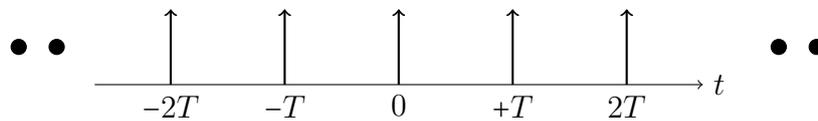
Observe that both sides have T -periodic waveforms. Assuming well behaved functions, the sum-formula can be proved by showing that both sides have the same FS coefficients. We already learned in last chapter that FS coefficients can uniquely identify periodic functions at the points of continuity. The RHS has FS coefficients $\frac{1}{T}X(\frac{m}{T})$, by identification. It is now straightforward to evaluate the FS coefficients of the LHS, and thus prove the formula.

Exercise 2. Find the FS coefficients of the LHS of (6).

Poisson sum-formula provides a unified view of the Fourier Series and Fourier Transform. Specifically, the former is the scaled and sampled version of the latter at appropriate sampling points. While we defined the sum-formula for nice well behaved functions, one can stretch this further to include symbolic waveforms like the impulse. Let us illustrate this by finding the Fourier Series coefficients of a periodic impulse train.

Example 1. Consider the signal

$$\Delta_T(t) = \sum_{n \in \mathbb{Z}} \delta(t - nT). \quad (7)$$



The FS coefficients are given by,

$$a_m = \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} \delta(t) \exp(-j2\pi ft) dt \quad (8)$$

$$= \frac{1}{T}. \quad (9)$$

Since all the coefficients are $\frac{1}{T}$, they constitute another train in the frequency domain, consisting of weighted discrete Kronecker deltas. However, in line with our Dirac formalism, it is better to consider each frequency sample present as an impulse. Thus the Fourier transform of a time impulse train is a frequency impulse train. We repeat this important result which is also known as the third Dirac formalism.

The Fourier Transform of an impulse train is another impulse train.

$$\sum_n \delta(t - nT) \stackrel{FT}{\rightleftharpoons} \frac{1}{T} \sum_m \delta(f - \frac{m}{T}) \quad (10)$$

By reading the same formula from right to left, we see that it is consistent irrespective of the way we look at it. So this seems to be a right step. In fact, the above formalism contains all the ingredients of the Poisson sum-formula. To illustrate this, let us show that FS series is nothing but scaled and sampled Fourier Transform values, this time using the Dirac formalism.

What will happen if we pass a Dirac train of period T through a LTI system $h(t)$ (assume $h(\cdot)$ is integrable). The output at time t is,

$$y(t) = \sum_{n=-\infty}^{\infty} h(t - nT). \tag{11}$$

Exercise 3. Prove the above equation.

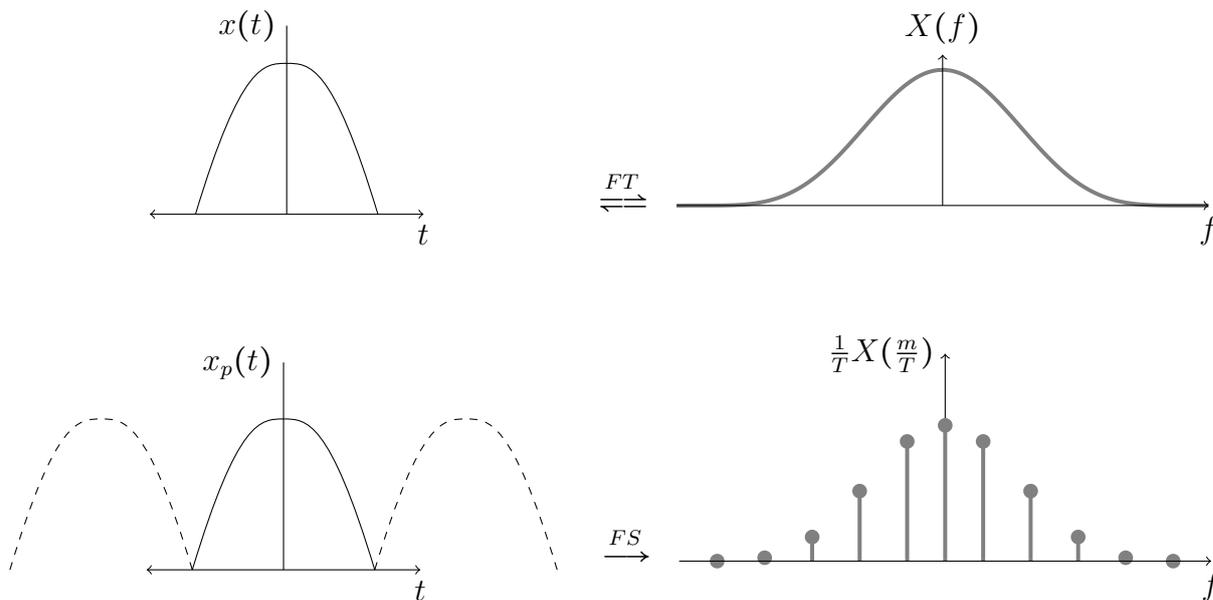
Since both $h(t)$ as well as the impulse train have their respective Fourier Transforms, convolution-multiplication formula will imply.

$$Y(f) = H(f) \sum_m \frac{1}{T} \delta(f - \frac{m}{T}) \tag{12}$$

$$= \sum_m \frac{1}{T} H(\frac{m}{T}) \delta(f - \frac{m}{T}) \tag{13}$$

Thus T -periodic repetitions of $h(t)$ will correspond to samples at $\frac{m}{T}, m \in \mathbb{Z}$ of $\frac{1}{T}H(f)$.

Example 2. Consider a signal $x(t)$ with Fourier Transform as shown in the figure.



We repeat that Fourier Series is nothing but scaled samples of the Fourier transform at intervals of $\frac{1}{T}$, with the understanding that the corresponding time signal is a T -periodic repetition, i.e.

$$x_p(t) = \sum_n x(t - nT) \tag{14}$$

Let us bravely replace the samples of Fourier Transform by impulses and see if everything is intact. See Figure 3 for an illustration.

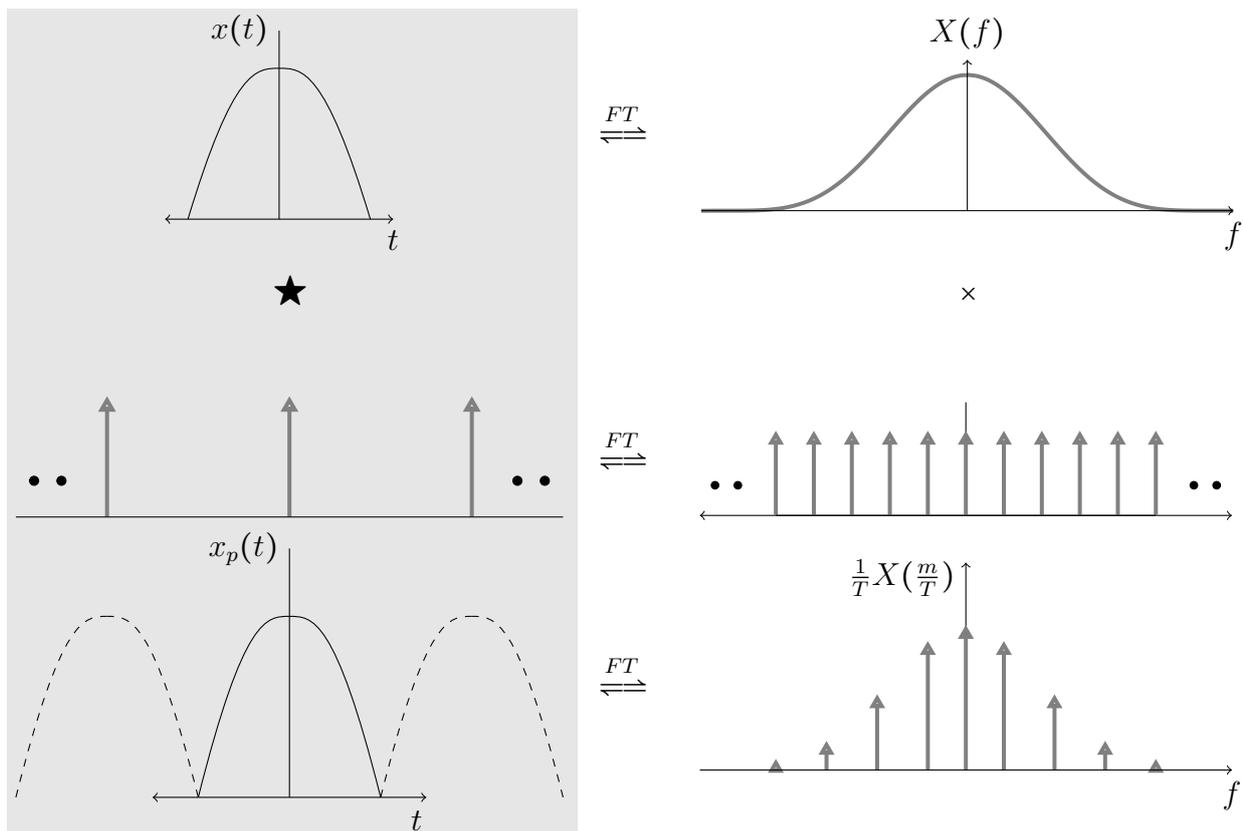


Figure 3: Connecting Fourier Series and Fourier Transform

Simply put, replace the **Fourier Series** coefficients by impulses of corresponding height and call it the **Fourier Transform** in a generalized sense. In other words, for each FS coefficient we are attributing a frequency value. The resulting sampled frequency representation is then termed as Fourier Transform. What results from this transformation is sheer magic, allowing one to seamlessly move between continuous and discrete times as well as frequencies.

The reader should at this point be willing to surrender all his terminology relating to Fourier Series and replace it with Fourier Transform.

5 Sampling Theorem

We promised in the beginning that understanding Fourier Transform will yield us the famous Sampling Theorem (also known as *Shannon-Nyquist Sampling Theorem*) as a by-product. The sampling theorem permits us to store just the samples of a continuous time waveform and yet lose little information, i.e., we can *almost* recreate the continuous waveform from the samples. Furthermore, any meaningful operation can equivalently be done on just the samples. This is a huge advantage, and is a major driving point behind the surge of digital technologies.

Example 3. Let $x(t)$ is a cardinal sine signal (our discussion is applicable to many other signals too). Sampling at intervals of T_s is same as multiplying by an impulse train, see

Figure 4. Thus the sampled waveform is $\sum_n x(t)\delta(t - nT)$.

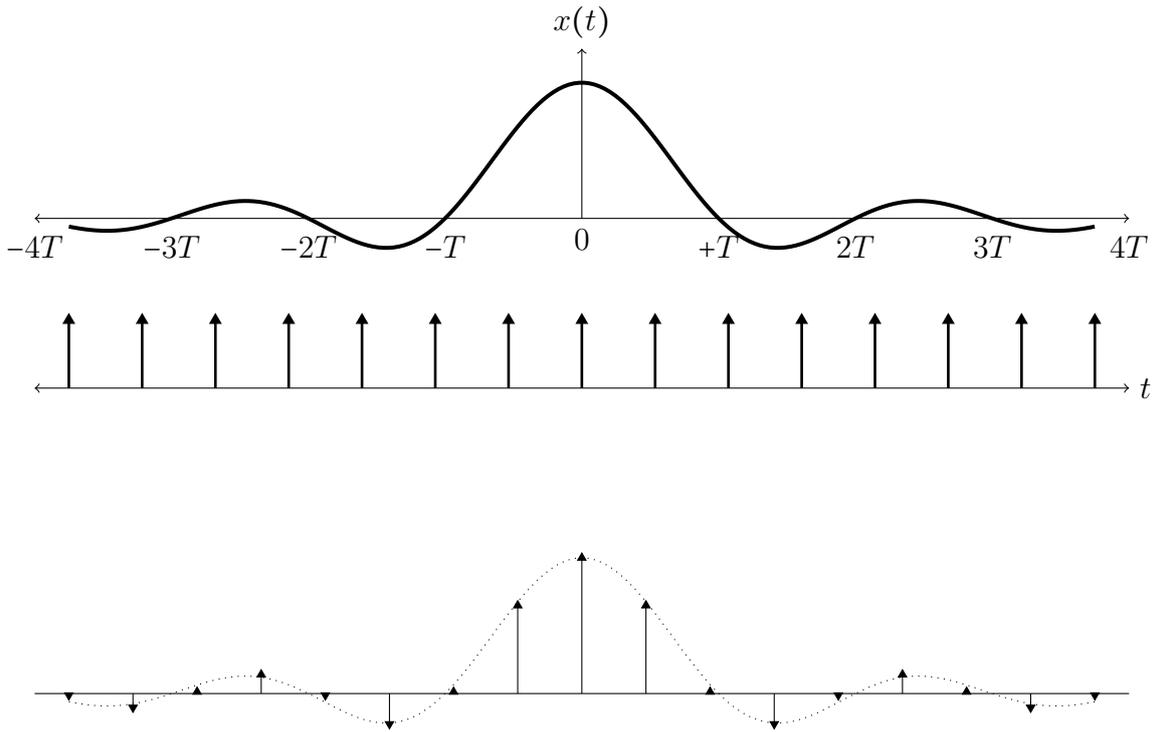


Figure 4: Sampling sinc waveform

It is very evident that sampling makes huge savings in space and computations possible. Ideally, we wish to deal with as few samples as possible. On the other hand, far and few samples make it hard to recreate all the information contained in the original signal. Our question is

What is the optimal rate of sampling?

Let us answer this question using the relation that we developed between FS and FT. To this end, let us enlist the steps that we followed in the last section to connect FT and FS.

1. Consider a signal $x(t)$ defined on a bounded interval in time and,

$$x(t) \xrightarrow{FT} X(f). \quad (15)$$

Here, $X(f)$ is defined over all $f \in \mathbb{R}$.

2. We repeated $x(t)$ to generate a periodic waveform, and as a result, the FT $X(f)$ got sampled. These steps used the convolution-multiplication theorem, where on one side we convolved by a Dirac train $\Delta_T(t)$ (see equation (7)). Correspondingly, we used multiplication by a Dirac train $\Delta_{\frac{1}{T}}(f)$ in frequency.

Equivalently, if we sample at intervals of $\frac{1}{T}$ in the frequency, then periodic repetitions with interval T occurs in the time domain. If the support of $x(t)$ is less than T , each period will contain the undistorted $x(t)$.

Sampling Theorem is a reverse statement of the above idea, which relates samples in time to repetition in frequency. To start with, consider a band-limited waveform $s(t)$.

i.e., the Fourier Transform $S(f)$ of the signal has bounded support (length of the non-zero interval) in frequency¹. Sampling a signal $s(t)$ in time is same as multiplying $s(t)$ by $\Delta_T(t)$.

$$s(nT) = s(t) \times \Delta_T(t). \quad (16)$$

By Fourier Transform relation,

$$s(nT) \xLeftrightarrow{FT} S(f) * \frac{1}{T} \Delta_{\frac{1}{T}}(f) \quad (17)$$

Notice that the RHS is nothing but the periodic repetitions of the waveform $\frac{1}{T}S(f)$. Thus, our sampling question can be rephrased as.

How close should we sample $s(t)$ in time so that the frequency spectrum $S(f)$ is non-overlapping under periodic repetitions?

To have a sensible answer to this question, the spectrum $S(f)$ should be defined on a finite interval in frequency. If not, no single period of the repetition can contain $S(f)$. Suppose $S(f)$ is defined for $f \in (-\frac{\beta}{2}, +\frac{\beta}{2})$, and zero elsewhere. If $S(f)$ is convolved with a Dirac train $\Delta_{F_s}(f)$, $F_s \geq \beta$, then each period of $S(f) * \Delta_{F_s}(f)$ will contain $S(f)$ undistorted. But convolving with $\Delta_{F_s}(f)$ in frequency is same as multiplying $s(t)$ by $\Delta_{\frac{1}{F_s}}(t)$ in the time domain. Thus if the time domain samples are separated by $T_s = \frac{1}{F_s} \leq \frac{1}{\beta}$, then we can recover $S(f)$ by just taking any one period of the pattern in frequency. Once $S(f)$ is recovered, taking an inverse Fourier Transform will yield $s(t)$. Thus storing the samples $s(nT_s), n \in \mathbb{Z}$ is enough. We now state the sampling theorem.

Sampling Theorem:

Let $x(t)$ be a signal, bandlimited to the frequency interval $[-\frac{\beta}{2}, \frac{\beta}{2}]$. Then

$$x(t) = \sum_n x(nT_s) \text{sinc}\left(\frac{t - nT_s}{T_s}\right) \quad (18)$$

where

$$T_s \leq \frac{1}{\beta}. \quad (19)$$

In other words, sampling theorem gives the recovery formula to get $x(t)$ back from the samples taken at a rate greater than β samples per second. The proof of this statement is straightforward. Since the FT $X(f)$ is defined in $[-\frac{\beta}{2}, \frac{\beta}{2}]$, (19) ensures that a scaled version of $X(f)$ is contained in each period of $X(f) * \frac{1}{T_s} \Delta_{\frac{1}{T_s}}(f)$. Thus,

$$X(f) = T_s \left(X(f) * \frac{1}{T_s} \Delta_{\frac{1}{T_s}}(f) \right) \text{rect}_{\frac{1}{T_s}}(f). \quad (20)$$

In the time domain, by convolution-multiplication theorem,

$$x(t) = T_s \left(x(t) \sum_n \delta(t - nT_s) \right) * \frac{1}{T_s} \text{sinc}\left(\frac{1}{T_s}t\right) \quad (21)$$

$$= \sum_n x(nT_s) \text{sinc}\left(\frac{t - nT_s}{T_s}\right). \quad (22)$$

¹In reality, many signals are not strictly bandlimited, and some part of their spectrum lingers all the way to infinity. In practice, we will consider a large enough frequency band, beyond which the energy of the signal is negligible, as the support of the signal spectrum.

Our assumption that the signal is band-limited is quite strong, and rarely met in practise. For example, no time-limited waveform can be band-limited (why?). Nevertheless, many signals are *almost* band-limited. That is, most of the signal energy is contained in a finite band. In this generalized sense, sampling theory is applied ubiquitously, whether strictly bandlimited or not, where it is understood that samples preserve most of the information contained in the signal.