Indian Institute of Technology Bombay

Dept of Electrical Engineering

Handout 10 EE 210 Signals and Systems

Tutorial 4 Sep 4, 2015

Question 1) A string is tied from $(0,0)$ to $(\frac{L}{2},0)$. Now, a point at distance p from the origin is given a vertical displacement of h upward. Let $f(t), t \in [0, \frac{L}{2}]$ be the initial position of the string.

(a) Find the coefficients A_m such that

$$f(t) = \sum_{m\geq 1} A_m \sin(\frac{2\pi}{L}mt).$$

Solution: Please see solutions to Homework 2.

(b) Find the coefficients B_m such that

$$f(t) = \sum_{m\geq 0} B_m \cos(\frac{2\pi}{L}mt).$$

Solution: Please see solutions to Homework 2.

(c) Find the coefficients C_m such that

$$f(t) = \sum_{m\in\mathbb{Z}} C_m \exp(j\frac{4\pi}{L}mt).$$

Solution: Please see solutions to Homework 2. **(d)** We have learnt that FS coefficients can uniquely identify a continuous function. However, there seems to be three expansions given in parts $(a), (b)$ and (c). How do you reconcile these different expansions.
Solution: Please see solutions to Homework 2.

(e) Which of the expansions above is useful in identifying the harmonics of a vibrating string. Write the first harmonic frequency and suggest a position p for which all the even harmonics are missing.
Hint: 'Fourier Transform+Poisson sum-formula+Convolution-multiplication" can handle Fourier Series
Solution: Please see solutions to Homework 2.

Question 2) Consider a signal $x(t) = \text{sinc}^2(\frac{t}{T})$. It is sampled at the rate of B samples per second to obtain a discrete-time signal $x[n]$. What will be the Fourier representation $X_s(f)$ of $x[n]$. Let us call $\hat{X}(f) := X_s(Bf)$, the latter is also known as the DTFT (Discrete-Time Fourier Transform) of $x[n]$.
Solution: Notice that the function after sampling, $x_s(t)$ can be written as

$$x_s(t) = \sum_{n\in\mathbb{Z}} x(\frac{n}{B})\delta(t - \frac{n}{B}).$$

The Fourier representation of $x[n]$ is nothing but the Fourier Transform of $x_s(t)$. Thus

$$
\begin{aligned}
X_s(f) &= \int x_s(t)\exp(-j2\pi ft)dt \\
&= \int \sum_{n\in\mathbb{Z}} x(\frac{n}{B})\delta(t-\frac{n}{B})\exp(-j2\pi ft)dt \\
&= \sum_{n\in\mathbb{Z}} x(\frac{n}{B}) \int \delta(t-\frac{n}{B})\exp(-j2\pi ft)dt \\
&= \sum_{n\in\mathbb{Z}} x(\frac{n}{B})\exp(-j2\pi f\frac{n}{B}) \\
&= \sum_{n\in\mathbb{Z}} x[n]\exp(-j2\pi \frac{f}{B}n),
\end{aligned}
$$

as we represent $x(\frac{n}{B})$ by $x[n]$. Notice that

$$X_s(Bf) = \sum_{n\in\mathbb{Z}} x[n]\exp(-j2\pi fn),$$

where $X_s(Bf)$ is a periodic function of unit period.
(a) Show that $\hat{X}(u)$ is a periodic waveform, and find its period.
Solution: From the previous part,

$$\hat{X}(u) = \sum_{n\in\mathbb{Z}} x[n]\exp(-j2\pi - \frac{u}{B}n). \tag{1}$$

Clearly this has unit period.

(b) We know that locally integrable periodic functions admit Fourier series expansions. What are the FS coefficients of $\hat{X}(u)$.
Solution: Note that $\hat{X}(u)$ is a periodic waveform in the variable u. It does not matter whether u denotes the time-variable, frequency variable, space variable etc. We can evaluate the FS coefficients of any periodic waveform. From (1), it is evident that the RHS is a FS reconstruction formula, albeit with a $-f$. Thus

$$a_{-m} = x[m], \forall m$$

This is a very pleasing result, if you take the DTFT, then time domain samples are nothing but the FS coefficients of DTFT. This is not surprising though, as all we learn in this class are connected together by a single thread, and we will completely unravel it.

(c) If $B = \frac{2}{\sqrt{2}T}$, can you get back $x(t)$ from the samples $x[n]$ (Draw $\hat{X}(f)$ and see).
Solution: Observe that the Fourier Transform of the given waveform is a triangle. Time-sampling the waveform will create a repetition in frequency, as shown in the Figure below. The spectrum $X_s(f)$ is given by the superposition of the shifted triangles. Clearly, we cannot get back $X(f)$ by extracting a segment of $X_s(f)$. So it is impossible to get back $x(t)$ from $x[n]$ using a LTI filter.

(d) If $B = \frac{2\sqrt{2}}{T}$, give two 'different' reconstruction formulae to obtain $x(t)$ from $x[n]$.
Solution: With the new sampling frequency, the spectral representation becomes as in Figure 2. One can get back $x(t)$ from any one period of the waveform $X_s(f)$. In particular, the filter shown in dashed lines will give us a scaled version of $x(t)$, whereas the dotted filter targets a scaled and shifted version of $X(f)$, from which we can downconvert to get $x(t)$.

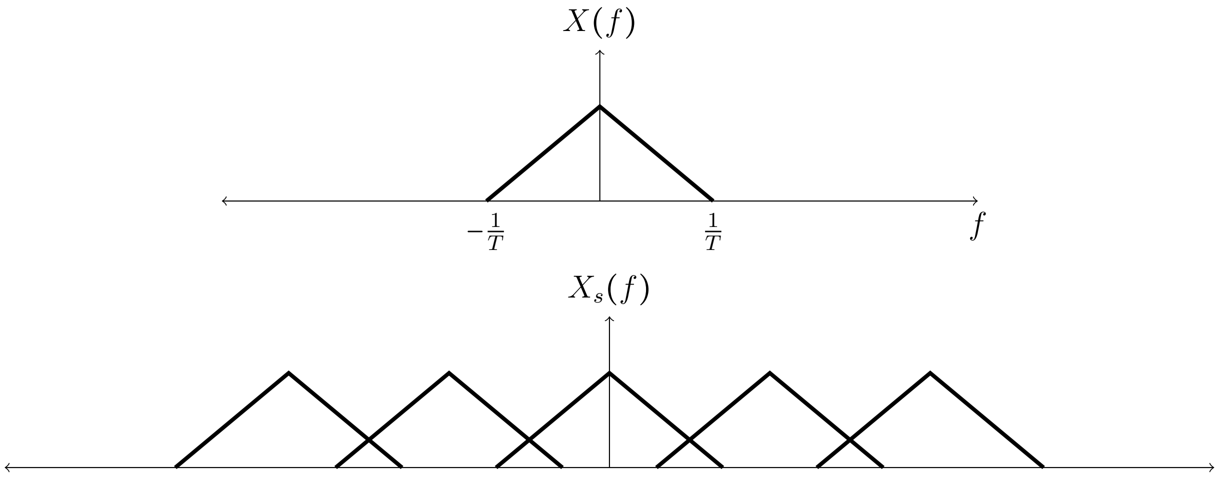

Figure 1: Under Sampling leads to Overlap in Frequency

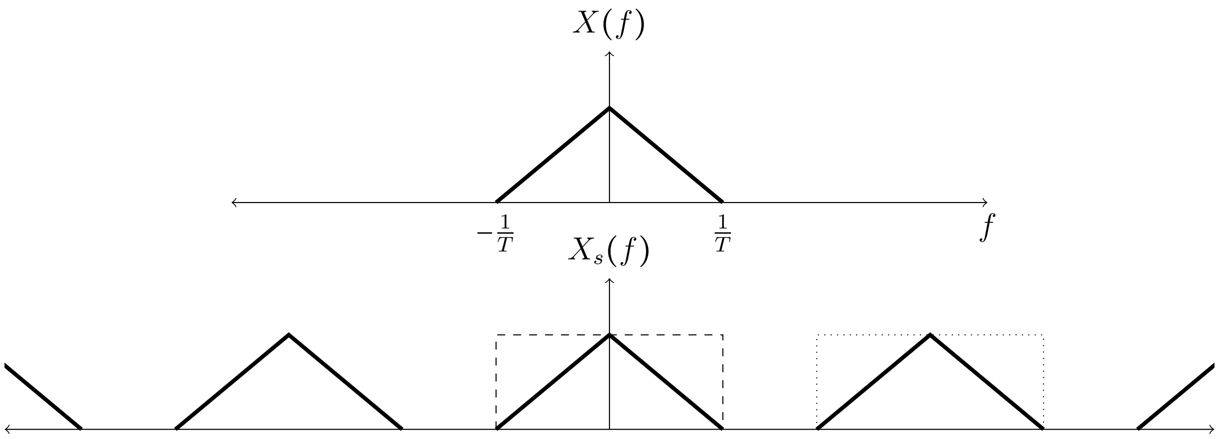

Figure 2: Over Sampling allows perfect reconstruction

Question 3) Notice that periodic functions have a Fourier-series expansion. Suppose a $T-$periodic function $x(t)$ is sampled to obtain $x_p[n] = x(n\frac{T}{k})$, where k is some integer greater than two. Since $x_p[n]$ is also periodic, we should attempt to find the FS coefficients of $x_p[n]$. Feel free to use the formalisms (Dirac or Impulse formalisms), to find the FS coefficients of $x_p[n]$.

Solution: Since $x_p[n]$ is obtained as a periodic repetition of $x[n]$, we can sample the DTFT of $x[n]$ to obtain the Series coefficients for $x_p[n]$.