Indian Institute of Technology Bombay

Department of Electrical Engineering

Handout 3 EE 325 Probability and Random Processes

Lecture Notes 1 July 21, 2014

1 Introduction

Some basics of probability theory are typically taught in senior high-school, and many students are accustomed to the notion of probability. In fact, many of you might have attempted to correctly predict the results of cricket/football matches, only to loose half your mustache (if you have one), or perhaps some money. Weather forecast is another place where we deal with probability theory on a regular basis. Here there is no money lost, if not obvious, we just need to invert the predictions from our meteorology centers. This is all good, but how come some octopus in Germany is predicting the outcomes of football matches? divine intervention?

"God does not play dice[1]"

This is a popular quote from the formative years of quantum physics. It turned out that quantum theory emerged to be more about probability than anything else. In fact, the world around, in all scales, from the weather clouds to the fundamental atomic particles, are *continuously*[2] tossing their coins or dice, whose outcomes collectively determining the physical world as we see.

2 Reality Bites

You may agree that probability is *really real*, what better place to start than a reality show[3]. In several such shows one out of a given number of marked boxes are to be picked, and a treasure-hunt has to be performed to get your reward. Let us have a miniature model with three boxes as shown below. To make matters important, we have reached the studio

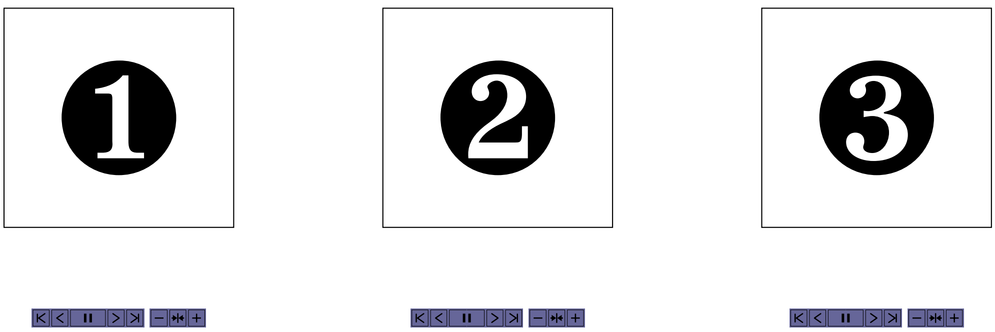

Figure 1: Reality Show (a variant from Starbird and Burger, The heart of mathematics. You can click and play in certain pdf readers.)

[1]Einstein vs Neils Bohr

[2]in the literary sense

[3]see Starbird and Burger, The heart of mathematics

on request from ALICE, who complained over phone that she was trapped in one of the boxes. Our job is to rescue our beloved friend ALICE. The VJ tells you that the other two boxes contain Leopards, so try playing safe in the rescue mission. The game is fair, that each box has the same chance of containing our ALICE, which is $\frac{1}{3}$. You can escape with the prize on figuring out ALICE in one of the boxes. The procedure is as follows.

The suitor chooses one of the boxes and bets his money on it. The box remains closed. Then, our beautiful VJ will spring to action with her seductive voice. She will announce that all is not lost if Alice is not in the box that you have chosen. No! it is not about living with a LEO, rather our VJ will lay bare another box for you, and we all know the outcome there to be ferocious. Now the real game starts.

After seeing the LEO-inhabited box, you are given an option to switch from your initial choice of the box. The question is "Will you switch or not?". No telephone lines or 'ask the audience". In fact, many of the audience shout 'SWITCH', while several others say 'STICK', the VJ unequivocally blabbers garbage, an exercise of building lip-muscles.

It turns out that if you switch your choice, you have more chances of winning the prize. This is slightly surprising, or perhaps confusing. Let us look at it in the following way. Since each box is equally favored to have the prize, we can keep our bet on the first box to start with. Our chances do not change if any other box is picked first. So let us stick to this scheme, i.e. whenever this game is played, we will start with Box I. Now, if ALICE is in any of the other two boxes, you WIN by SWITCHing. Conversely, whenever ALICE is not in the other two boxes you will loose by SWITCHing. It happens that most often, in fact roughly a fraction $\frac{2}{3}$ of the times, we expect the prize to be in the last two boxes, as each of the box is equally favored.

We should be very careful NOT to state that one will win a fraction $\frac{2}{3}$ of the times by switching your choice. In fact, what we have is a rough estimate, or an approximate guess of what the expected winnings are. As we go further into this course, we will rigorously state the quantitative and qualitative properties of 'the probability of winning'. To convince you, Figure 2 shows the results of computer simulations where the *fraction-of-wins-by-switching* vs the *number-of-trials* is shown. The two curves correspond to different runs of the experiment. Each time we restart the experiment, we get a possibly new curve, but it can be visualized that the success percentage *tends to* $\frac{2}{3}$ with the number of trials in a run. We will make this statement mathematically rigorous later in the course.

3 Birth Day Problems

This is another interesting example on probability computation. Keep in mind that we have not introduced the notion of probability. In fact, the following discussion relies on your past familiarity on probability theory to convey ideas. Our question is

> Of N people sitting in a room, what is the chance that there is a pair who shares the same birthday?

To demystify, we are looking at people who celebrate their birthday on the same day every year. For simplicity, let us ignore the leap years, and assume there are 365 possible birthdays up for grabs. In order that the above problem makes sense, we need to also state that each day in any year is favored equally as a birthday[4]. In lieu of our *fairness*

[4] I apologize if there is confusion about this, the term *not favored* is to account for fairness, or lack of partiality, which is vital for our answers. In many western countries, a large number of kids are actually born around the new-year, and our results will not apply then

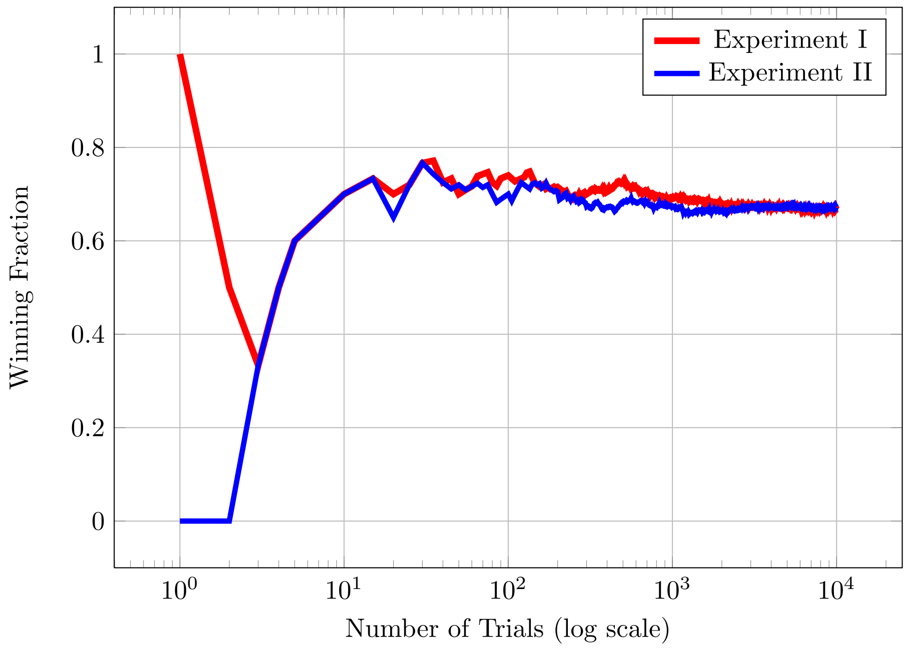

Figure 2: Computer Simulation showing Winning Fraction for 2 different runs

assumption, on picking a person, the chance that his birthday happens to be on January 1 is $\frac{1}{365}$, the same being true for any other day.

Example 1 *How many birthday combinations are possible for N people?*

Solution: Consider two people, the first person can be born in any of the 365 days, and so be the second person. Thus a total of $365.365 = 365^2$ birthday pairs are possible. For N people, we can have any of 365^N choices.

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Example 2 *If 100 people are in a room, what is the chance that nobody has(had) his/her birthday today.*

Solution: If we replace 'birthday' by 'breakfast' the answer is obvious. As for birthday, take the first person, the chance that he/she was born some other day is $\frac{364}{365}$, which is same as $1 - \frac{1}{365}$. In particular, there are 364 other possible days, as opposed to today. Now take the second person, his/her chance of being born some other day is also $\frac{364}{365}$. Now, put them together and assume that none of them have their birthday today. So each birthday can take any one of the 364 values, and the total possible pairs is 364×364. Using the previous example, the chance of two persons not having their birthday can be evaluated as $\frac{364 \times 364}{365 \times 365} = \frac{364^2}{365^2}$. The argument extends to N users, thus the chance that none of N persons having their birthday today is $\frac{364^N}{365^N}$. For 100 people, this evaluates to *approximately* 0.76. Conversely, the chance of at least one person having his/her birthday today is about 0.24.

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Example 3 *If two persons who do not share their birthdays are chosen, how many birthday combinations are possible.*

Solution: We can arrange all possible birthday combinations on a two dimensional matrix of size 365×365. Any element in the diagonal corresponds to identical birthdays, and should not be counted. So the total different birthday combinations are $365^2 - 365$, i.e. total entries minus the number of diagonal entries, which is same as 365.364. ■

Example 4 *How many ways N birthdays can be chosen, if no birthdays are shared?*

Solution: The answer is an extension of the previous problem. The first person's birthday can be any one of the 365 values. Given the first birthday, the second one can be any of the remaining 364. Now the third can have any from 363 possibilities, and so on. Thus the number of ways are $365 \times 364 \times 363 \times, \cdots, \times (365 - N + 1)$. The notation $, \cdots,$ will be used frequently to concisely write a string of characters. ■

We are now in a position to answer our birthday problem. In fact, it is easy to find the the complement, i.e. the chance of N different birthdays is

$$\frac{365 \times 364 \times 363 \times, \cdots, \times (365 - N + 1)}{365^N}.$$

From what we learn later, subtracting the above value from one will give the answer that we seek.

4 Numbers and Counting

Most of what you learned as probability(in earlier classes) are actually related to a branch of mathematics called ENUMERATIVE COMBINATORICS, on which several beautiful books are available. A minimal revision is done here, we need to move beyond this primary concepts as early as possible. Consider the decimal number system, where the elements are $0, 1, \cdots, 9$. We can form 10 numbers of 1 digit. How about 2 digits? $00, 01, \cdots, 99$ will give us 100, or 10^2 numbers. The principle: the first digit can take each of the 10 values, and for each first digit the second one can take 10 values, leading to 100 numbers. This property extends beyond the decimal system. Imagine a number system with elements $0, 1, 2, 3$. With 2 digits we can form 16 numbers and with 10 digits we can form $4^{10} = 1048576$ numbers.

Example 5 *How many numbers are possible if the first digit is decimal and the second one is from $0, 1, 2, 3, 4$?*

Certainly our philosophy of counting should work. Consider a 5×10 matrix, where the entry in position (i, j) of the matrix is $10.i + j$, where $0 \le i \le 9$ and $0 \le j \le 4$. This will include all the permitted numbers and hence $10.5 = 50$ numbers are possible. ■

Exercise 1 *Suppose you have 5 different number systems, let us call it S_1, S_2, S_2, S_4 and S_5. Let n_i denote the number of elements in the system S_i. You are supposed to form 5 'digit' numbers, with position i employing elements from system S_i. How many different numbers are possible.*

Exercise 2 *Consider the decimal system with elements $0, \cdots, 9$, and the hexa-decimal system with elements $0, \cdots, 9, A, \cdots F$. Furthermore, imagine a 'chinese' numbering system with 20 pictures representing various digits, we will call it $c_a, c_b, \cdots, c_t$. You are required to form 2 digit numbers. The first digit can use any one of the number systems. For example, you can decide it to be the chinese system. The second digit can be taken from the remaining* *two systems, i.e. you can pick elements from both the remaining systems for the second digit. Notice that* $c_a 1$, $c_a F$ *etc are valid numbers here. Under these constraints, maximize the total number of possible 2 digit numbers.*

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We need to add one more trick to our bag to do some counting-magic. Let us first generalize Example 4, and say that N elements can be arranged in $N!$ ways if no element repeats. What if we are only interested in the first $M < N$ elements? The first M elements can now be placed in $\frac{N!}{(N-M)!}$ ways, i.e. any arrangement of the last $N - M$ does not count, once we take the first M digits. Suppose, in addition, that we are interested only in the values that we picked and not their positions, i.e. we will not count again if the same elements are permuted or re-arranged. Thus, for each choice of the first M values we will have $M!$ re-arrangements which will NOT get counted. Since the first M digits can be taken in $\frac{N!}{(N-M)!}$ ways, the total number of ways different elements can be picked up (sans permutation) is

$$\frac{N!}{M!(N-M)!} = \binom{N}{M},$$

where the RHS is a short hand notation, which expands to N choose M, i.e. ways of choosing M elements from N, where the position does not matter. For brevity, we will crop out the statement 'position-does-not-matter' and simply say choose M elements from N.