Indian Institute of Technology Bombay
Department of Electrical Engineering

Handout 5 EE 325 Probability and Random Processes
Lecture Notes 3 July 28, 2014

1 Axiomatic Probability

We have learned some paradoxes associated with traditional probability theory, in particular the so called Bertrand's paradox. The problem there was an inaccurate or incomplete specification of what the term *random* means. In fact, formulating a random experiment needs to be more *constructive*, clearly outlining the way probabilities of various events are generated. It is here that the **axiomatic framework** or modern framework[1] takes over. The modern notion defines a **probability space**, sometimes referred to as *probability trinity*. Much like the Indian trinity of *srushti(brahma), sthithi(vishnu), samhara(maheshwara)* or western *father, son, holy soul*, the probability space is about how outcomes, events and probabilities are constructed unambiguously. Indeed a 'probabilistic universe' in action. However, do no read further into this mythological connection, our trinity has more creation, maintenance and lending out, its Mumbai after all.

We will define a few axioms on events and their probability assignment, and further use this to construct suitable probability spaces.

2 Probability Space $(\Omega, \mathcal{F}, P)$

The probability space has three constituents, Ω, $\mathcal{F}$ and P.

1. The first entity Ω (borrowed from classical probability) denotes the *sample-space*, where each outcome of the experiment is a sample-point.

2. The second entity $\mathcal{F}$, also known as *event-space*, is a class of subsets of Ω, having certain extra qualifications than its classical counterpart. In particular, the event-space $\mathcal{F}$ is a $\sigma-$**field** (pronounced as sigma field).

3. The third entity is the *probability measure*, which associates a value in $[0,1]$ for any event $A \in \mathcal{F}$.

In fact, the three entities appear pretty much the way they were introduced in classical probability. However, we will refine our definitions and list a set of axioms that governs the construction of the last two entities. We already had several examples about the first entity. So let us start with the second one, the so called event-space $\mathcal{F}$.

3 Event-Space $\mathcal{F}$

Given several events, event-management is a necessity (imagine Olympics or Techfest). Event-management allows us to do operations on events in $\mathcal{F}$. Since events themselves are subsets of Ω, the permitted operations are the well-known set operations. In order to do event-management, the space $\mathcal{F}$ should be **stable** with respect to set operations. The word

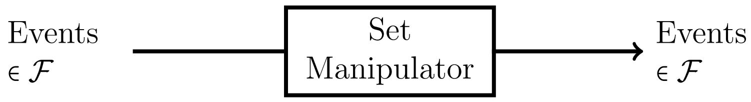

Figure 1: A familiar linear systems view of stability

stable here is very important. It emphasizes that if you perform permissible operations on event(s) in $\mathcal{F}$, what results is an event (possibly different) in $\mathcal{F}$, see Figure 1.

Now this resembles very much of a say in cricket, "what happens in a cricket-field stays there". In fact a cricket-field provides a level-playing ground for various actions and events to occur, and ideally no event boils out of the field, that is the ground, stadium and participants[2]. The word 'field' applies to a more general context than a cricket field, feel free to think about a cricket field and the above proverb.

Those who are used to set-theory may guess immediately that the property *stable* is similar in spirit to 'a class of subsets being *closed* with respect to *usual* set-operations'. However, there are subtle differences between the terms *closed* and *stable*, at least in the conventional use of the first. Conventional set operations on event(s) are

1. Complement, denoted as A^c or $\bar{A}$.
2. Union: $A_1 \bigcup A_2$.
3. Intersection: $A_1 \bigcap A_2$.
4. Set difference: $A_1 \setminus A_2$, which is $A_1 \bigcap A_2^c$.
5. Symmetric difference: $A_1 \triangle A_2$ which is $(A_1 \setminus A_2) \bigcup (A_2 \setminus A_1)$.

3.1 Stability and Complements

Events of interests in an experiment are usually associated with a YES/NO question. For example, has A happened?, or is it HEAD?. An event A has not happened will mean that the event A^c has happened. Thus, 'has A not happened' can be asked equivalently in terms of A^c happening. As a result, we want both parties to be members of $\mathcal{F}$ for it to be stable.

A *stable* class needs to be closed under complementing, i.e.

$$A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}.$$

3.2 Stability and Unions

A class $\mathcal{F}$ is closed under set-unions if the union of any two elements in the class will give rise to another element in the class. By repetition, we can say that if $A_i \in \mathcal{F}, 1 \le i \le n$, then

$$\bigcup_{i=1}^{n} A_i \in \mathcal{F}.$$

In other words, $\mathcal{F}$ is closed under finite unions.

Definition 1 *A collection of subsets which includes Ω and closed with respect to (i)complements and (ii) finite unions is known as a field or an algebra.*

[1]attributed to A. N. Kolmogorov, the Russian mathematician

[2]let us forget the idiot box and sidduisms

On the other hand, if $\mathcal{F}$ is **stable**, then it needs to be closed under *countable* unions, i.e.

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}.$$

Thus countable unions cannot result in anything outside the event-space, pointing to the stability or robustness of our space. The rest of the set-operations can be written in terms of unions and complements, and thus the definition of stability can be clearly stated in the first two operations.

Notice that *stability* with respect to countably infinite unions is an extremely important property, which is necessary in many circumstances. We provide a simple example to differentiate between finite and infinite (countable) unions.

Example 1 *Let $\Omega = [0,1)$ and consider intervals in $\mathbb{R}$ of the form $A_i = [0 + \frac{1}{i+2}, 1), i \in \mathbb{N}$. For any finite n, we know that $\bigcup_{i\leq n} A_i = [\frac{1}{n+2}, 1)$, which is closed on the left. However, $\bigcup_{i\geq 2} A_i = (0,1)$, which is open at the left, and thus not included in the set of subintervals of the form $[a,b)$. For our purposes, we demand that the field be stable with respect to countably infinite unions too.*

So far we have used the terminology *stable* to convey a more intuitive notion familiar from our electrical engineering background. We will now give a proper definition to the event space that we are interested and drop the word 'stable' from further usage.

Definition 2 *A collection of subsets which includes Ω and closed with respect to (i)complements and (ii) countable unions is known as a* ***sigma- field*** *or a* sigma-algebra.

From now onward, the meaning of '$\mathcal{F}$ is a σ-field' should be clear, and we may, at times, will simply say $\mathcal{F}$, that it is a $\sigma-$field will be clear from the context.

3.3 Event-Axioms

We will summarize the natural desired properties of the event-space as event-axioms. While this is rephrasing some of the statements in the previous sub-section, their importance warrants a repetition.

1. **Axiom-I:** $\Omega \in \mathcal{F}$.

2. **Axiom-II:** $A \in F \Rightarrow A^c \in \mathcal{F}$.

3. **Axiom-III:** For any arbitrary countable index set T,

$$A_i \in \mathcal{F}, \forall i \in T \Rightarrow \bigcup_{i \in T} A_i \in \mathcal{F}.$$

In literature you may find a different set of axioms defining the event-space. This should not be causing confusion, as the whole space can be constructed based on either of the axioms. For example, many text-books replace axiom-I by $\emptyset \in \mathcal{F}$. This is certainly true in the light of axioms I and II above, as $\Omega \in \mathcal{F}$ and $\emptyset = \Omega^c \in \mathcal{F}$ ($\emptyset$ will be called the *null-set* or *emptyset*). To emphasize again, the set T contains arbitrary indices, but countable. Thus T can be the set of all even numbers, or the collection of primes (is this countably infinite?) or finite sets as in $T = \{i \in \mathbb{N} : 10 \leq i \leq 10000\}$, so on and so forth.

3.4 De Morgan's Laws

Let us start with a simple exercise.

Exercise 1 *Simplify the expressions below.*
(i)$A \bigcup \varnothing$, *(ii)*$A \bigcap \varnothing$, *(iii)* $A \bigcup A$, *(iv)* $A \bigcap A$, *(v)* $A \bigcup (B \setminus A)$, *(vi)* $A \bigcap (B \setminus A)$.

Familiarity with most of these operations will be assumed in this course[3]. A set of laws, though familiar, we will devote some time are the De Morgan's laws.

Law I:

$$(A \bigcup B)^c = A^c \bigcap B^c$$

Law II:

$$(A \bigcap B)^c = A^c \bigcup B^c$$

We are in fact interested in the extension of these laws to operations over arbitrary countable set of indexes.
Extension Laws:

$$\left(\bigcup_{i \in T} A_i \right)^c = \bigcap_{i \in T} A_i^c \tag{1}$$

$$\left(\bigcap_{i \in T} A_i \right)^c = \bigcup_{i \in T} A_i^c \tag{2}$$

Proof:

$$\begin{aligned} x \in \left(\bigcup_{i \in T} A_i \right)^c &\Leftrightarrow x \notin A_i, \forall i \in T \\ &\Leftrightarrow x \in A_i^c, \forall i \in T \\ &\Leftrightarrow x \in \bigcap_{i \in T} A_i^c. \end{aligned}$$

■

Exercise 2 *Prove the second extension stated above.*

3.5 Examples of σ−fields

1. Trivial σ−field : $\{\Omega, \varnothing\}$.

2. Gross σ−field : $\mathcal{P}(\Omega)$, the power-set of Ω. This is the most useful σ-field when Ω is finite or countable. However, keeping track of the events in $\mathcal{P}(\Omega)$ for real-valued Ω can turn to be a mundane, if not impossible, task(this is even true when $\Omega = \mathbb{N}$).

3. A non-trivial example: $\{\Omega, A, A^c, \varnothing\}$

4. Borel σ-field : $\mathcal{B}$, this is the single most important σ−field when Ω is a real interval or rectangle (higher dimensional) in Cartesian coordinates.

Exercise 3 *Verify that the third example is indeed a σ−field.*

[3]otherwise it is time that you grab the set-theory books

3.6 Recipe for Finite Sigma Fields

There are indeed many sigma fields containing a class of events, let us say, E. The clarity of understanding can be helped by constructing a recipe by which one can construct sigma-fields. More precisely, what we construct are more aptly fields, finite sets which are closed with respect to complements and unions. For example, we have already constructed a field of size 4 above. Let us construct a field of size 8 now.

Example 2 *Let A and B are disjoint subsets of Ω and consider $C = (A \bigcup B)^c$. Let us enumerate a 3-bit binary numbering system, and choose the subset in $\mathcal{F}$ using these bits as shown below.*

0	*0*	*0*	-	$\varnothing$	*1*	*0*	*0*	-	C
0	*0*	*1*	-	A	*1*	*0*	*1*	-	$A \bigcup C$
0	*1*	*0*	-	B	*1*	*1*	*0*	-	$B \bigcup C$
0	*1*	*1*	-	$A \bigcup B$	*1*	*1*	*1*	-	Ω

It is easy to verify that this is a sigma-field and one is free to choose the disjoint sets A and B at convenience.

Exercise 4 *Construct a sigma-field which has* 16 *events in it.*

3.7 Mutually Exclusive Events

The events $A_1, \cdots, A_n$ are **mutually exclusive** (or pairwise disjoint [4]) if at most one of them is true. i.e. no two share a common outcome of the experiment.

$$A_i \text{ happened } \Rightarrow A_j, j \neq i \text{ did NOT happen.}$$

4 Probability Axioms

In addition to the three event axioms that we had described, we will also outline a set of axioms on the third entity of the probability space, namely the probability measure $P(\cdot)$.

Axiom-I:

For any event A, $P(A) \geq 0$.

Axiom-II:

For any countable index set T, and a collection $A_i, i \in T$ of mutually exclusive events,

$$P(\bigcup_{i \in T} A_i) = \sum_{i \in T} P(A_i).$$

Axiom-III:

The sample-space $\Omega \in \mathcal{F}$ is *certain*, i.e. $P(\Omega) = 1$.

We will include some discussion on countable sets and reals in the coming lecture notes. To be frank, there is not much need to know anything more than the distinction between countable and uncountable, throughout the course, once we are proficient enough to avoid any pit-falls. So feel free to concentrate on the rest of the aspects if this countable discussion is not appealing to you

[4] pairwise disjoint is a stronger notion than disjoint. The latter means the intersection of the given sets is $\varnothing$, whereas the former says that the intersection of any two sets is $\varnothing$.