Indian Institute of Technology Bombay
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Handout 10 EE 325 Probability and Random Processes
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1 Conditional Probability

Events of an experiment can be related in various ways. For example, mutually exclusive events have $A_i \bigcap A_j = \emptyset$. In other words, for such events, given that the outcome is in A_j, it will not be in the set A_i. We can say that $P(A_i, \text{ given } A_j \text{ happened}) = 0$, which will be represented concisely as

$$P(A_i|A_j) = 0.$$

So conditioning can change the probability of an event. Let us look at another extreme,

$$P(A|\Omega) = P(A).$$

Thus conditioning by Ω does not change the probability of an event. This is intuitive, as we already know that some outcome or the other has to happen. There is nothing new in this conditioning than what we already knew. Let us generalize this notion to any A, B in the sigma-field $\mathcal{F}_1$ associated with Ω. We denote by $P(A|B)$, the conditional probability of A given that the event B has happened.

Definition 1 *For events $A, B \in \mathcal{F}_1$*

$$P(A|B) = \frac{P(A \bigcap B)}{P(B)} \mathbb{1}_{\{P(B)>0\}}.$$

The indicator is to avoid a divide by zero exception. When the events are clear from the context, we will drop the brackets and write

$$P(A|B) = \frac{P_{AB}}{P_B}.$$

We have to verify that $P(A|B)$ is a true probability measure, i.e. it satifies the three probability axioms. Before we show this, it is kind of intriguing to figure out the probability space under consideration. You can think about it in two different, but equivalent ways. One way is to consider the original probability space $(\Omega, \mathcal{F}_1)$, however the events which do not overlap with B (the conditioning event) will be assigned a zero probability by conditioning. Thus we modify the probabiliy measure on $\mathcal{F}_1$. Another alternative is to modify the space itself, i.e. since we know B has happened, we can take $\Omega_2 = B$. Now, we should choose an appropriate sigma-field to associate $P(A|B)$. A possible choice is

$$\mathcal{F}_2 = \{A_i \bigcap B : A_i \in \mathcal{F}_1\}.$$

Exercise 1 *Show that $\mathcal{F}_2$ is a sigma-field defined on B.*

The new sigma-field F_2 is also known as the restriction of $\mathcal{F}_1$ to B. Now $P(A|B)$ can be thought of as a probability measure defined on B if it satisfies all the axioms for events in $\mathcal{F}_2$. Clearly for axiom I,

$$P(A|B) \geq 0.$$

Furthermore

$$P(B|B) = \frac{P(B)}{P_B} = \frac{P_B}{P_B} = 1,$$

As for the other axiom, for pairwise disjoint sets $A_i \in \mathcal{F}_1, i \geq 1$

$$\begin{aligned} P(\bigcup_{i\geq 1} A_i|B) &= \frac{P(\bigcup_{i\geq 1} A_i \bigcap B)}{P_B} \\ &= \frac{P\left(\bigcup_{i\geq 1}(A_i \bigcap B)\right)}{P_B} \\ &= \frac{\sum_{i\geq 1} P(A_i \bigcap B)}{P_B} \\ &= \sum_{i\geq 1} P(A_i|B_i). \end{aligned}$$

We have shown the property for any pairwise disjoint collection in $\mathcal{F}_1$, which ensures the property to be true also for $\mathcal{F}_2$, since $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

1.1 Bayes' Rule

The two pillars of conditional probability are the so called Bayes' rules. We provide the statements below, the proofs are simple applications of the definition of conditional probability.

Bayes' Rule I:

$$P(A|B) = \frac{P_A\, P(B|A)}{P_B}.$$

Bayes' rule of exclusive and exhaustive causes: Let $B_i, i \in T$ be disjoint countable events such that $\bigcup_{i\in T} B_i = \Omega$ (sample-space). Then for any event A,

$$P(A) = \sum_{i\in T} P(B_i)P(A|B_i)$$

Let us prove the latter rule using our axioms. We have,

$$\begin{aligned} P(A) &= P(A \bigcap \Omega) \\ &= P(A \bigcap \bigcup_{i\in T} B_i) \\ &= P\left(\bigcup_{i\in T}(A \bigcap B_i)\right) \\ &= \sum_{i\in T} P\left((A \bigcap B_i)\right) \\ &= \sum_{i\in T} P(B_i)P(A|B_i) \end{aligned}$$

where we used the convention that anything multiplied by zero will result in zero.

2 Independent Events

We will start with the notion of two events being independent.

Definition 2 *Events A and B are said to be independent if*

$$P_{AB} = P_A P_B.$$

Exercise 2 *Show that the events A and B with positive probability are independent iff*

$$P(A|B) = P(A).$$

How can we generalize to more than two events. Consider three events A, B, C. Let us look at a weaker notion first, called pairwise independence. As the name implies, we expect any pair of the given events to obey Definition 2.

Example 1 *Let $\Omega = \{a, b, c, d\}$ with $\mathcal{F} = \mathcal{P}(\Omega)$ and the uniform probability measure. Let $A = \{a, b\}, B = \{b, c\}, C = \{a, c\}$. Show that the events are pairwise independent.*

Solution: Clearly,

$$P(A) = P(B) = P(C) = \frac{1}{2},$$

and

$$P(A, B) = P(B, C) = P(A, C) = \frac{1}{4}.$$

■ However, it is natural to expect $P(A, B, C) = P(A)P(B)P(C)$ if these events are independent. In the above example $P(A, B, C) = 0$ or $P(A|B, C) = P(B|A, C) = P(C|A, B) = 0$. Thus two events tell us a lot about the third, and we cannot consider them independent. One may think that ensuring the condition $P(A, B, C) = P(A)P(B)P(C)$ suffices for independence. However, this will also turn out to be a weaker notion, as the example below illustrates.

Example 2 *Let $\Omega = \{a, b, c, d, e, f\}$, with uniform measure and let $A = \{a, b, c\}$, $B = \{c, d, e\}$ and $C = \{a, c, d, f\}$. It is easy to check that*

$$P(A, B, C) = P(\{c\}) = \frac{1}{6} = P(A)P(B)P(C) = \frac{1}{2}\frac{1}{2}\frac{4}{6}.$$

However $P(A, B) = P(A)P(B)$ suggesting that A and B are not independent.

The example gives us the idea about how to claim independence. Make an all inclusive demand like in a patent-application.

Definition 3 *A sequence of events $A_i, i \in T$ are called independent if for any $S \subseteq T$ of finite cardinality, we have*

$$P(\bigcap_{i \in S} A_i) = \prod_{i \in S} P(A_i).$$

A few other examples to clarify these concepts are given as exercises below.

Exercise 3 *Consider $\Omega = \Big\{1, 2, \cdots, 12\Big\}$, and a uniform probability law. Consider the following sets*

$$A = \{1, 2, \cdots, 8\}; \;\; B = \{5, 6, \cdots, 12\} \text{ and } C = \{2, 3, \cdots, 10\}.$$

Show that $P(A\,B\,C) = P(A)P(B)P(C)$ but $P(AB) \neq P(A)P(B)$.

Exercise 4 *Let* $\Omega = \{1, \cdots, 64\}$ *and consider the subsets*

$$\begin{aligned} A_1 &= \{1, \cdots, 15\} \\ A_2 &= \{16, \cdots, 30\} \\ A_3 &= \{31, \cdots, 45\} \\ A_4 &= \{46, \cdots, 63\} \\ A_5 &= \{64\}. \end{aligned}$$

Let us take events $B_1 = A_1 \cup A_5$, $B_2 = A_2 \cup A_5$ *and* $B_3 = A_3 \cup A_5$. *Compute* $P(B_1, B_2, B_3)$, $P(B_1, B_2)$ *and* $P(B_1)P(B_2)$.