Indian Institute of Technology Bombay
Dept of Electrical Engineering

Quiz I — EE 325 Probability and Random Processes

25 marks — Feb 9, 2019

Question 1) Consider an election process between candidates A and B, where a total of n people cast their votes. The ballots were collected, shuffled well, and then counted. When the votes were announced, candidates A and B polled m and l votes respectively with $m > l$, and $l + m = n$. These numbers are made available to all. Given this information, our interest now is in the sequence of votes.

(a) Suppose it came to be known that the first vote went to A. Given this additional information, find the probability that A was ahead of B throughout the counting process.

[5 marks]

Solution: Once the first vote goes to a, there are $\binom{m+l-1}{l}$ possible voting sequences. Of that $\binom{m+l-1}{m}$ will lead to an outcome where A is not leading throughout.

$$1 - \frac{l}{m}.$$

(b) In the case where there is no prior knowledge on any part of the vote sequence (other than the total votes polled by each candidate), find the probability of A having at least as many votes as that of B throughout the counting process.

[5 marks]

Solution: Imagine a hypothetical voting model where A has $m+1$ votes, l votes for B, and the first vote goes to A. From the second vote onwards, the hypothetical model is similar to the original voting model.

Whenever the votes get equal in the actual system, it is 1 vote ahead in the hypothetical system. If A has fewer votes than B at any instant on the actual system, the hypothetical system allots at least as many votes to B, when compared to A.

Thus, we need to find the probability of A being ahead of B probability that in the hypothetical system. By the previous part this is

$$1 - \frac{l}{m+1}.$$

(c) If it came to be known that the first two votes went to A, find the probability that A had at least as many votes as B throughout the counting process.

[5 marks]

Solution: Put the above ideas together.

Question 2) Consider a random variable X with probability law $\Pr(X = x) = Q(x)$. Let us take for some $\alpha \in (0, 1)$,

$$Q(k) = (1 - \alpha)^{k-1}\alpha, \text{ for } k = 1, 2, 3, \cdots$$

(a) Show that $Y = Q(X)$ is a random variable, and find its distribution.

[3 marks]

Solution: $Q(x)$ is a function and X being a random variable suffice to support that $Y = Q(X)$ is a random variable. In particular for $y_k = (1-\alpha)^{k-1}\alpha$,

$$P(Y = y_k) = y_k.$$

(b) Find the expectation of $Y = Q(X)$.
Solution:

$$\mathbb{E}[Y] = \sum_{k\geq 1} y_k P(Y = y_k) = \sum_{k\geq 1} y_k^2 = \frac{\alpha^2}{1-(1-\alpha)^2}.$$

[3 marks]

Solution:

(c) The entropy of a random variable X with probability distribution $\Pr(X = x) = Q(x)$ is given by $-\mathbb{E}\log Q(X)$. For $Q(x)$ as given above, find the entropy of X.

[4 marks]

$$\begin{aligned}
-\mathbb{E}\log Q(X) &= -\mathbb{E}\log(1-\alpha)^{X-1}\alpha \\
&= -\mathbb{E}\log\alpha - \mathbb{E}(X-1)\log(1-\alpha) \\
&= -\log\alpha + \log(1-\alpha)[\mathbb{E}(X) - 1] \\
&= -\log\alpha + \log(1-\alpha)(\frac{1}{\alpha} - 1) \\
&= \frac{1}{\alpha}\left(\alpha\log\frac{1}{\alpha} + (1-\alpha)\log\frac{1}{1-\alpha}\right).
\end{aligned}$$
