Indian Institute of Technology Bombay

Department of Electrical Engineering

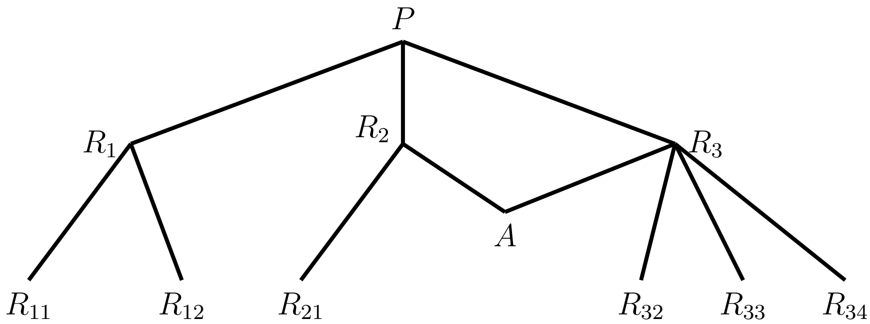
Handout 2 EE 325 Probability and Random Processes

Tutorial 1 January 18, 2019

Question 1) How many 14−letter words are there with the letters $EETHREETWOFIVE$?
Solution:

$$\frac{14!}{2!5!}$$

Question 2) (Rohatgi2001) Consider a bicyclist who leaves a point P (see Figure), choosing one of the roads PR1 , PR2 , PR3 at random. At each subsequent crossroad (or junction), she again chooses an available road at random.



(a) What is the probability that she will arrive at point A?
Solution: Let p be probability of reaching A, starting from P. We can write,

$$p = \frac{1}{3}\frac{1}{5} + \frac{1}{3}\frac{1}{5}p + \frac{1}{3}\frac{1}{3} + \frac{1}{3}\frac{1}{3}p + \frac{1}{3}\frac{1}{3}p$$

which gives $p = \frac{1}{4}$.

(b) What is the conditional probability that she arrived at A via road PR3 ?
Solution: Let us find the probability that the cyclist took the path PR_3A to reach A. A similar recursion as above shows this probability to be 3/32. Since the probability of getting to A is $\frac{1}{4}$, the probability in question is 3/8.

Question 3) There were M boys M girls in the senior secondary graduating batch in a school. Suppose the school pairs each boy with a girl to do lab experiments. They graduated, many boys went to IITs, girls to Medical Schools etc. After 10years they re-assembled for the alumni meet at the school. M tables numbered $1, \cdots, M$, each with a pair of chairs, were arranged for the function. The ladies went first and each lady occupied a table. If the gents now walk in at a random order and occupy the first vacant seat, what is the probability that at least one of the table has an actual pair from the school days.
Solution: Assume that the ladies at the tables are labelled $\{1, \cdots, M\}$ Let $A_i, 1 \le i \le M$ be the event that there is a match at table i. The quantity of interest is $P(\bigcup A_i)$.

Suppose the first table got an actual pair, the rest of the men can now sit in $(M-1)!$ ways.

$$P(A_1) = \frac{(M-1)!}{M!} = \frac{1}{M}.$$

We also know that

$$
\begin{aligned}
P(A_1 \bigcup A_2) &= P(A_1) + P(A_2) - P(A_1, A_2) \\
&= \frac{1}{M} + \frac{1}{M} - \frac{(M-2)!}{M!} \\
&= \frac{1}{M} + \frac{1}{M} - \frac{1}{M(M-1)}.
\end{aligned}
$$

We can generalize this idea.

$$
\begin{aligned}
P(\bigcup A_i) &= \frac{1}{M!}\left(M(M-1)! - \binom{M}{2}(M-2)! + \binom{M}{3}(M-3)! - \cdots\right) && (1) \\
&= \sum_{j=1}^{N} (-1)^{j+1} \binom{M}{j} \frac{(M-j)!}{M!} && (2) \\
&= \sum_{j=1}^{N} (-1)^{j+1} \frac{1}{j!} \ . && (3)
\end{aligned}
$$

Notice that the right hand side quickly tends to $1 - e^{-1} \approx 0.62$ with M.

Question 4) For events $A_i, 1 \leq i \leq n$ show that

$$
P(\bigcup_{i=1}^{n} A_i) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{1 \leq i_1 < \cdots < i_j \leq n} P(A_{i_1}, \cdots, A_{i_j}).
$$

Solution: See Lecture Notes 4.

Question 5) Ballot Problem: In an election, MAMA and RAHU polled a and b votes respectively, with $a > b$. Given this, what is the probability that MAMA always lead RAHU in the counting process?
Solution: Let us first write the solution for MAMA being ahead all the time, i.e. if they become equal, then MAMA has not lead RAHU. This can be found by applying the principle of reflection. In particular, if the first vote went to NARI, the number of ways in which $a+b$ votes are reached is $\binom{a+b-1}{a}$. For every such sequence, there is a another sequence such that KEJU got the first vote, and his lead was not maintained at some juncture. Thus the number of ways KEJU leads all the time is $\binom{a+b}{a} - 2\binom{a+b-1}{a}$. The required probability is then,

$$
\frac{\binom{a+b}{a} - 2\binom{a+b-1}{a}}{\binom{a+b}{a}},
$$

which will give $\frac{a-b}{a+b}$.

There are two alternate techiques. Suppose we pick a vote at random, with probability $\frac{a}{a+b}$ it will be won by KEJU. So NARI wins the first vote with probability $\frac{b}{b+a}$. This is a bad event. The reflection principle says twice this much is the probability that KEJU surrenders the lead. So the probability in question is

$$
1 - 2\frac{b}{a+b} = \frac{a-b}{a+b}
$$

Another way is to start counting from behind. Suppose the last vote be denoted as X_n, where $n = a + b$. We can write

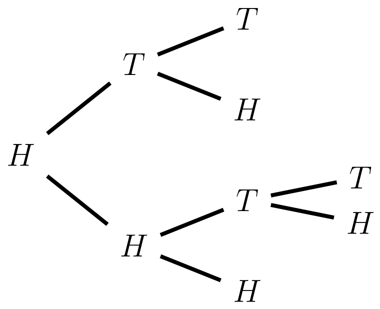
$$
P_n(a, b) = \frac{a}{n} P_{n-1}(a-1, b) + \frac{b}{n} P_{n-1}(a, b-1),
$$

where $P_l(x,y)$ is the probability that KEJU lead all the time if x out of the total l votes went to him. The boundary conditions are $P_{2b}(x,y) = 0$. See $\max\{\frac{x-y}{x+y}, 0\}$ solves these set of equations.

Question 6) If a coin is tossed repeatedly (non-stop), what is the sample space Ω.

(a) Find the probability that the sequence HTH occurs before the sequence HHH.

Solution: Consider the following tree



What is the probability P_0 of you winning starting from the root node. Enumerate the leaves, starting from the topmost to bottom.

1. Leaf 1: If you reach leaf 1, you are back to the 'square you started'. Your probability of winning from now is P_0.
2. Leaf 2: reaching here means you have won, so the probability is 1.
3. Leaf 3: This is like leaf 1, the probability of win now is P_0.
4. Leaf 4: Hey, you have won.
5. Leaf 5: Please head back to pavilion, you have lost the game.

The above happens if you start with a HEAD. If you start with a TAIL, well, we can cut all the outcomes before the first HEAD appears, and so we need to consider only the tree above. Writing what we said 'compactly'

$$P_0 = \frac{1}{4}P_0 + \frac{1}{4}.1 + \frac{1}{8}P_0 + \frac{1}{8}.1$$

and $P_0 = \frac{3}{5}$.

Alternate Solution: While the above approach quickly solves the problem, we will present a formal solution technique which can be applied in a wide variety of situations. Consider the first three flips, this is equally likely to be in

$$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Observe that each of these choices is equally likely. Let us name these states respectively as $\{1,2,3,4,5,6,7,8\}$ for convenience. Denoting by $Q(i)$ the probability that HHH occurs before HTH while the initial state is i, we have $Q(1) = 1$. As for $Q(2)$, we can write that

$$Q(2) = \frac{1}{2}Q(3) + \frac{1}{2}Q(4), \tag{4}$$

which is a restatement of the fact that from $Q(2)$ with probability $\frac{1}{2}$ we will go to $Q(3)$, and can go to $Q(4)$ with remaining probability. Thus reaching HHH starting from $Q(2)$

is broken in to two paths, one which goes through $Q(3)$, and other which goes through $Q(4)$. This technique is known as one-step analysis. Similarly,

$$\begin{aligned}
Q(3) &= 0 \\
Q(4) &= \frac{1}{2}Q(7) + \frac{1}{2}Q(8) \\
Q(5) &= \frac{1}{2} + \frac{1}{2}Q(2) \\
Q(6) &= \frac{1}{2}.0 + \frac{1}{2}Q(4) \\
Q(7) &= \frac{1}{2}Q(5) + \frac{1}{2}Q(6) \\
Q(8) &= \frac{1}{2}Q(7) + \frac{1}{2}Q(8)
\end{aligned}$$

We can convert everything in terms of $Q(2)$.

$$Q(4) = 2Q(2); Q(5) = \frac{1}{2} + \frac{1}{2}Q(2); Q(6) = Q(2); Q(7) = \frac{1}{4} + \frac{3}{4}Q(2); Q(8) = \frac{1}{4} + \frac{3}{4}Q(2).$$

Since $Q(4) = Q(8)$, we have

$$2Q(2) = \frac{1}{4} + \frac{3}{4}Q(2),$$

and we get

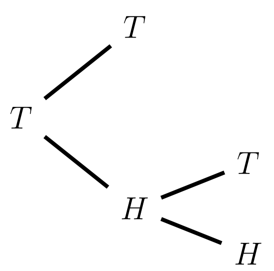
$$Q(2) = \frac{1}{5}$$

Since the initial value i is equally likely,

$$\begin{aligned}
\frac{1}{8}\sum Q(i) &= \frac{1}{8}\left(1 + \frac{1}{2} + \frac{1}{2} + Q(2)(1 + 2 + \frac{1}{2} + 1 + \frac{6}{4})\right) && (5) \\
&= \frac{1}{4} + \frac{6}{40} && (6) \\
&= \frac{2}{5}. && (7)
\end{aligned}$$

(b) What is the probability of the sequence HHT occurring before THT.

Solution: Let us first find the probability P_T of winning by starting from a TAIL.



Thus

$$P_T = \frac{1}{2}P_T + \frac{1}{4}1$$

$$P_T = \frac{1}{2}.$$

Now consider starting with HEADS. Given this, if the second toss is a HEAD you win, 'eventually'. On the other hand, if the second is a TAIL, your probability becomes P_T. Thus

$$\frac{1}{2}P_T + \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{1}{2}P_T$$

is the probability of winning, which is $\frac{5}{8}$.

Question 7) Suppose we have a set $C = \{A, B, C\}$. What do you think is the maximum possible number of elements in the sigma-field generated by C.
Solution: 256 elements are possible.

Question 8) Show that $\Pr\left((\bigcup_{i \in T} A_i) \bigcap B\right) = \Pr\left(\bigcup_{i \in T} \left(A_i \bigcap B\right)\right)$
Solution: Consider the set in the LHS for which probability has to be computed. Each element x in this set is present in B, and also in at least one of the A_i. Thus, an element x in the LHS is present in some $A_i \bigcap B$. Thus the corresponding sets in the LHS and RHS are identical, implying the same probability assignment.