Indian Institute of Technology Bombay

Department of Electrical Engineering

Handout 9 EE 325 Probability and Random Processes

Tutorial 3 Feb 15, 2019

Question 1) The Generating Function of a random variable X is given to be

$$g_X(z) = \frac{(1+z)\,c}{2z^2 - 7z + 6}, \; c > 0.$$

(a) Find $P(X = 3)$.
Solution: Notice that $g(1) = 1$ implies $c = 0.5$. Now

$$g_X(z) = \frac{1+z}{2}\left(\frac{1}{z-2} - \frac{1}{z-1.5}\right).$$

Expanding the terms $P(X = 3) = 0.1531$

(b) Find $\mathbb{E}[X]$.
Solution: $\mathbb{E}X = g'(1) = \frac{7}{2}$.

Question 2) Let T be a random variable taking values in the positive integer. Consider an IID sequence $X_i, i \geq 1$ independent of the random variable T. Consider $Y = \sum_{j=1}^{T} X_j$.

(a) Find the generating function $g_Y(z)$ of Y in terms of $g_X(z)$ and $g_T(z)$.
Solution: $g_Y(z) = g_T(g_X(z))$.
(b) Find $\mathbb{E}[Y]$.
Solution: $\mathbb{E}[Y] = \mathbb{E}[X]\mathbb{E}T$.

Question 3) Amit purchases a 6 digit lottery ticket. Find the probability that the sum of the first three digits is greater than the sum of the last three digits. (Note that each digit is chosen uniformly from $\{0, \cdots, 9\}$).
Solution: Let X_i represent digit i. Let the event A represent $X_1 + X_2 + X_3 = X_4 + X_5 + X_6$, and B representst $X_1 + X_2 + X_3 \geq X_4 + X_5 + X_6$. By symmetry,

$$P(B) = \frac{1}{2}\left(1 - P(A)\right)$$

Consider the random variable

$$Y = 27 + X_1 + X_2 + X_3 - X_4 - X_5 - X_6.$$

Notice that $P(A) = P(Y = 27)$. Let us compute the generating function of Y.

$$g_Y(z) = \frac{1}{10^6}\frac{(1 - z^10)^6}{(1-z)^6}.$$

Using Tayloy series, we can show that

$$P(Y = 27) = \frac{1}{10^6}\left(\binom{32}{5} - \binom{6}{1}\binom{22}{5} + \binom{6}{2}\binom{12}{5}\right).$$

Question 4) Consider a voting model where there are two candidates A and B. Of the n registered votes, candidate A polled l votes, while candidate B polled m votes, with $l > m$

and $l+m=n$. The ballots were well shuffled before the counting started. Given that A was leading B after counting the first 3 votes, what is the probability that A did not surrender the lead for the rest of the counting process.
Solution: By denoting $F3L$ as the event of A leading after 3 votes and W as the event of A leading from vote number 3 to n, we can write,

$$\begin{aligned} P(W|F3L) &= P(AAA|F3L)P(W|AAA) + P(AAB|F3L)P(W|AAB) \\ &\qquad + P(ABA|F3L)P(W|ABA) + P(BAA|F3L)P(W|ABA) \\ &= P(AAA|F3L)P(W|AAA) + 3P(AAB|F3L)P(W|AAB). \end{aligned}$$

Now

$$P_{AAA} = \frac{\binom{n-3}{m}}{\binom{n}{m}} \text{ and } P_{AAB} = \frac{\binom{n-3}{m-1}}{\binom{n}{m}}.$$

So

$$P(AAA|F3L) = \frac{P_{AAA}}{P_{AAA} + 3P_{AAB}}. \text{ and } P(AAB|F3L) = \frac{P_{AAB}}{P_{AAA} + 3P_{AAB}}.$$

Let us first write the answer for A strictly leading B.

$$P(W|AAA) = 1 - \frac{\binom{n-3}{l}}{\binom{n-3}{m}}$$

$$P(W|AAB) = 1 - \frac{\binom{n-3}{l-1}}{\binom{n-3}{m-1}}$$

If you replace l in the last two equations by $l+1$, you will get the answer where ties are also allowed (you find this out).

Question 5) N boys enter a clean room facility to do experiments. They all reached the facility with their lab shoes on, however were asked to leave the shoes outside the lab. (Imagine the shoes are identical and brand new!)

After finishing the experiments, students were requested to hand over the report and leave one by one. The first student who came out took the shoe of someone else (uniformly at random among the choices). Subsequent students took their own shoe whenever it is 'available', or otherwise took a *random* shoe from the remaining (i.e. if you are not the first student and your shoe is available, no option but to take it).

(a) If $N > 12$, find the probability that the last person walks back with his own shoe.
Solution: Let us call the event of the last person finding his shoe as WIN. Let us define

$$P(k) = P(\text{WIN}|\text{Person } k \text{ does not get his shoe}).$$

The quantity of interest is $P(1)$ with the evident boundary condition that $P(N) = 0$. We can now write

$$P(k) = \frac{1}{N-1} \sum_{i=2}^{N-2} P(i).$$

In other words, with probability $\frac{1}{N-1}$, the first fellow picks shoes i $(2 \le i \le N)$, and from there we win with probability $P(i)$. The funniest thing to notice is that $P(i) = \frac{1}{2}$, identically for all $2 \le i \le N-1$. The reason is that a WIN occurs if any of these fellows pick shoes 1, and a failure occurs if someone picks up shoes N. However, for people $2, \cdots, N-1$, these two shoes are identical from the statistical point of view, getting one among these two shoes picked up is exactly $\frac{1}{2}$. Thus

$$P(1) = \frac{1}{N-1}\frac{1}{2}(N-2).$$

You can also compute $P(k)$ recursively, starting from $P(N-1)$ backwards.

(b) Given that persons numbered $2, 5, 10$ and 12 found their own shoes and 8 did not find the correct shoes, what is the probability that the last person coming out finds his/her own shoes (Take $N > 12$).

Solution: We can remove the students from 2 to 7, as we know 8 lost his shoe. Thus student 8 becomes student 2 in the new list, and we have to find the probability $\Pi(2)$ of winning after 2 lost its shoes. Purge the other fellas that we know to have had their shoes back, and let there be $M-1$ remaining users after 2.

$$\Pi(2) = \frac{1}{M} + \frac{1}{M}\sum_{i=3}^{M+1}\Pi(i),$$

where the first term captures the fact that we win if 2 takes the first shoe. Notice $\Pi(M+1) = 0$, and

$$\Pi(i) = \frac{1}{2}, 3 \le i \le M,$$

as both the first and last shoes are equally preferred by any intermediate person. Solving

$$\Pi(2) = \frac{1}{M} + (M-2)\frac{1}{2} = \frac{1}{2}.$$

Question 6) N people are called for a 'Walk-In Walk-Out' interview for a single post.

Suppose each candidate has a rank, say a value $r_i \in \{1, \cdots, N\}$ for applicant i. The rank denotes the suitability of a candidate for the job: lower the r_i value, better the suitability. Furthermore, the rank-vector $(r_1, r_2, \cdots, r_N)$ is chosen uniformly from all possible permutations of $\{1, 2, \cdots, N\}$.

The interview board has **no** direct access to the above rank order. However from the interview process, the board learns whether the current candidate is better than the ones already interviewed, i.e. a binary variable. Observe that the ranks are not revealed to the board. At this point, the board can choose either one of the two options below.

1. They can offer the job to the current candidate and stop the interview process.
2. Decline the current application and move on to the next candidate. A declined candidate will no longer be considered for the same job.

Thus, there is a clear risk if the board goes with a plan to interview all the N candidates anyway, since, in this case the last one will get the job, and the probability that the last one being the best is a dismal $1/N$. Let us do some thing better to maximize the probability of success, i.e. finding the best candidate (one with rank 1).

(a) Show that any optimal selection scheme will NOT offer the job to the k^{th} candidate, if she is not found to be better in the interview than all others who appeared before her.

(b) Suppose after interviewing k candidates, the board decides to choose the next aspirant who performs better than the k aspirants so far. Under this scheme, find the probability of getting the best candidate for a given k.

(c) If the interview board decides on a strategy as in part (b), how can it maximize the probability of success. Find the maximal success probability.

Solution: Please see secretary problem on Wikipedia.