Chapter 6

Solutions

6.1 Chapter 1

6.1.1 Subsection 1.1.3

Exercise 6.1.1. De Morgan's rules.
Consider an arbitrary sequence $\{A_n\}_{n\geq 1}$ of subsets of Ω. Prove *De Morgan's identities*:

$$\overline{\left(\bigcap_{n=1}^{\infty} A_n\right)} = \bigcup_{n=1}^{\infty} \overline{A}_n \text{ and } \overline{\left(\bigcup_{n=1}^{\infty} A_n\right)} = \bigcap_{n=1}^{\infty} \overline{A}_n.$$

Prove that if $\mathcal{F}$ is a sigma-field on Ω, and if $A_1, A_2, \ldots$ belong to $\mathcal{F}$, then so does their intersection $\cap_{k=1}^{\infty} A_k$.

Solution (Exercise 6.1.1).
In order to prove a set identity $A = B$, we must show that $\omega \in A \iff \omega \in B$. We do this for the first identity:

$$\begin{aligned}
\omega \in \overline{(\cap_{n=1}^{\infty} A_n)} &\iff \omega \notin \cap_{n=1}^{\infty} A_n \\
&\iff \exists n \geq 1 \text{ such that } \omega \notin A_n \\
&\iff \exists n \geq 1 \text{ such that } \omega \in \overline{A}_n \\
&\iff \omega \in \cup_{n=1}^{\infty} \overline{A}_n.
\end{aligned}$$

For the second identity, replace the A_n's by their complements to obtain

$$\overline{(\cap_{n=1}^{\infty} \overline{A}_n)} = \cup_{n=1}^{\infty} A_n,$$

and then take complements in the last displayed identity.

By successively applying (2) and (3) of Definition 1.1.1, we have that the $\overline{A_k}$'s are in $\mathcal{F}$ and so is their union $\cup_{k=1}^{\infty}\overline{A_k}$. The intersection $\cap_{k=1}^{\infty}A_k$ is, by the corresponding De Morgan's identity, equal to the complement of the union $\cup_{k=1}^{\infty}\overline{A_k}$. We conclude by applying (2) of Definition 1.1.1 once more.

Exercise 6.1.2. Finitely often, infinitely often.
Consider an arbitrary sequence $\{A_n\}_{n\geq 1}$ of subsets of Ω. Show that $\omega \in B := \bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\overline{A_k}$ if and only if there exists at most a *finite* number (depending on ω) of indices k such that $\omega \in A_k$. Show that $\omega \in D := \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}A_k$ if and only if there exist an *infinite* number (depending on ω) of indices k such that $\omega \in A_k$.

Solution (Exercise 6.1.2).

$$\omega \in B \Longleftrightarrow \omega \in \cup_{n=1}^{\infty}C_n$$

where $C_n = \bigcap_{k=n}^{\infty}\overline{A_k}$. Therefore

$$\begin{aligned}
\omega \in B &\Longleftrightarrow \exists n \geq 1 \text{ such that } \omega \in C_n \\
&\Longleftrightarrow \exists n \geq 1 \text{ such that } \omega \in \overline{A_k} \text{ for all } k \geq n \\
&\Longleftrightarrow \exists n \geq 1 \text{ such that } \omega \notin A_k \text{ for all } k \geq n \\
&\Longleftrightarrow \exists \text{ at most a finite number of } k \text{ such that } \omega \in A_k\,.
\end{aligned}$$

$$\begin{aligned}
\omega \in D &\Longleftrightarrow \forall n \geq 1,\, \omega \in \bigcup_{k=n}^{\infty}A_k \\
&\Longleftrightarrow \forall n \geq 1,\, \omega \in A_k \text{ for at least one } k \geq n \\
&\Longleftrightarrow \exists \text{ an infinite number of } k \text{ such that } \omega \in A_k.
\end{aligned}$$

Exercise 6.1.3. Indicator functions.
Prove the following identities for all subsets A, B of a given set Ω, and all sequences $\{A_n\}_{n\geq 1}$ forming a partition of Ω:

$$1_{A\cap B} = 1_A \times 1_B\,, \qquad 1_{\overline{A}} = 1 - 1_A\,, \qquad 1 = \sum_{n\geq 1} 1_{A_n}\,.$$

Solution (Exercise 6.1.3).

$$\begin{aligned}
1_{A\cap B}(\omega) = 1 &\Longleftrightarrow \omega \in A \cap B \\
&\Longleftrightarrow \omega \in A \text{ and } \omega \in B \\
&\Longleftrightarrow 1_A(\omega) = 1 \text{ and } 1_B(\omega) = 1 \\
&\Longleftrightarrow 1_A(\omega)1_B(\omega) = 1
\end{aligned}$$

$$\begin{aligned}
1_{\bar{A}} = 1 &\Longleftrightarrow \omega \in \bar{A} \\
&\Longleftrightarrow \omega \notin A \\
&\Longleftrightarrow 1_A(\omega) = 0 \\
&\Longleftrightarrow 1 - 1_A(\omega) = 1.
\end{aligned}$$

Suppose that there is an ω such that $\sum_{n\geq 1} 1_{A_n}(\omega) = 0$. For such an ω, $1_{A_n}(\omega) = 0$ for all $n \geq 1$, and therefore $\omega \notin \cup_{n=1}^{\infty} A_n$, which contradicts exhaustivity ($\cup_{n=1}^{\infty} A_n = \Omega$). Suppose that there is an ω such that $\sum_{n\geq 1} 1_{A_n}(\omega) \geq 2$. For such an ω, there exist at least 2 distinct indices n_1 and n_2 (say $n_1 = 1$ and $n_2 = 2$) such that $1_{A_1}(\omega) = 1$ and $1_{A_2}(\omega) = 1$. Therefore $\omega \in A_1 \cap A_2$, which contradicts mutual exclusion ($A_i \cap A_j = \varnothing$, $\forall i \neq j$).

Exercise 6.1.4. Small sigma-fields.
Is there a sigma-field on Ω with 6 elements (including of course Ω and $\varnothing$)?

Solution (Exercise 6.1.4).
$\mathcal{F}$ must contain $\varnothing$ and Ω and some other element, A, and therefore $\overline{A}$. This makes 4 elements. Suppose there is a fifth, B. Then $\overline{B}$ is a sixth element, distinct from the 5 previous ones. But $A \cap B$ and $\overline{A} \cap B$ are in $\mathcal{F}$ and at least one of them is new. Therefore there cannot be a sigma-field with 6 elements.

Exercise 6.1.5. Operations on measurable sets.
Let $\mathcal{F}$ be a sigma-field on some set Ω. Show that if A_1, A_2 are in $\mathcal{F}$, then so is their *symmetric difference* $A_1 \triangle A_2 := A_1 \cup A_2 - A_1 \cap A_2$.

Solution (Exercise 6.1.5).
$C := A_1 \cup A_2$ and $D := A_1 \cap A_2$ are in $\mathcal{F}$ and therefore $A_1 \triangle A_2 = C - D := C \cap \overline{D} \in \mathcal{F}$.

Exercise 6.1.6. Sigma-field generated by a collection of sets.
(1) Let $\{\mathcal{F}_i\}_{i\in I}$ be an arbitrary non-empty family of sigma-fields on some set Ω (the non-empty index set I is arbitrary). Show that the family $\mathcal{F} = \cap_{i\in I}\mathcal{F}_i$ ($A \in \mathcal{F}$ if and only if $A \in \mathcal{F}_i$ for all $i \in I$) is a sigma-field.

(2) Let $\mathcal{C}$ be an arbitrary family of subsets of some set Ω. Show the existence of a smallest sigma-field $\mathcal{F}$ containing $\mathcal{C}$. (This means, by definition, that $\mathcal{F}$ is a sigma-field on Ω containing $\mathcal{C}$, such that if $\mathcal{F}'$ is a sigma-field on Ω containing $\mathcal{C}$, then $\mathcal{F} \subseteq \mathcal{F}'$.)

Solution (Exercise 6.1.6).
Obvious.

Exercise 6.1.7. Union of sigma-fields.
Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be two sigma-fields on the set Ω. Give a counterexample contradicting the assertion that $\mathcal{F}_1 \cup \mathcal{F}_2$ is a sigma-field.

Solution (Exercise 6.1.7).
$\Omega := \{1, 2, 3, 4\}$, $\mathcal{F}_1 = \{\Omega, \varnothing, \{1, 2\}, \{3, 4\}\}$, $\mathcal{F}_2 = \{\Omega, \varnothing, \{1, 3\}, \{2, 4\}\}$. If $\mathcal{F}_1 \cup \mathcal{F}_2$ were a sigma-field, it would contain the intersection of any pair of its elements, for instance, $\{1, 2\} \cap \{1, 3\} = \{1\}$. But $\{1\} \notin \mathcal{F}_1 \cup \mathcal{F}_2$.

Exercise 6.1.8. Atoms.
Let the non-empty subsets $A_1, \ldots, A_k$ of a set Ω form a partition of the latter. How many elements are there in the sigma-field $\mathcal{F}$ they generate on Ω? (The sets $A_1, \ldots, A_k$

are called the *atoms* of $\mathcal{F}$.) Show that, conversely, if a sigma-field $\mathcal{F}$ on Ω contains a finite number of elements, it is generated by a finite number of sets that form a partition of Ω.

Solution (Exercise 6.1.8).
2^k (Hint: how many subsets in the set $\{1, 2, \dots k\}$ including the whole set and the empty subset?). Hint: The atoms are the sets in $\mathcal{F}$ other that Ω and $\varnothing$ that do not intersect with any other set in $\mathcal{F}$ other that Ω and $\varnothing$.

Exercise 6.1.9. Set inverse function.
Let $f : U \to E$, where U and E are arbitrary sets. For any subet $A \subseteq E$, define

$$f^{-1}(A) = \{u \in U\,;\, f(u) \in A\}\,.$$

(i) Show that for all $u \in U$, $1_A(f(u)) = 1_{f^{-1}(A)}(u)$.
(ii) Prove that if $\mathcal{E}$ is a sigma-field on E, then the collection of subsets of U

$$f^{-1}(\mathcal{E}) := \left\{f^{-1}(A)\,;\, A \in \mathcal{E}\right\}$$

is a sigma-field on U.

Solution (Exercise 6.1.9).
(i)

$$\begin{aligned} 1_A(f(u)) = 1 &\iff f(u) \in A \\ &\iff u \in f^{-1}(A) \\ &\iff 1_{f^{-1}(A)}(u) = 1. \end{aligned}$$

(ii) This is a direct consequence of the definition of a sigma-field and of the following set identities. For all subsets $A, A_1, A_2, \dots$ of E,

$$f^{-1}(\bar{A}) = \overline{f^{-1}(A)},$$

$$f^{-1}\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} f^{-1}(A_n),$$

$$f^{-1}\left(\bigcup_{n=1}^{\infty} A_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(A_n).$$

6.1.2 Subsection 1.2.3

Exercise 6.1.10. Why just sigma-additive?
Consider the probability model of Example 1.2.3 (random point on the square). Observing that any singleton $\{a\}$ ($a \in [0,1]^2$) is in the Borel sigma-field and has a null surface in the usual sense, show that there exists no totally additive probability P on the Borel sigma-field on the square $[0,1]^2$ that assigns to rectangles therein their surface. (By "totally additive", it is meant that the probability of the union

of an arbitrary —not necessarily countable— collection of mutually disjoint sets in the Borel sigma-field is the sum of the individual probabilities.)

Solution (Exercise 6.1.10).
If such P existed,

$$1 = P(\Omega) = P(\cup_{a\in\Omega}\{a\}) = \sum_{a\in\Omega} P(\{a\}) = \sum_{a\in\Omega} 0 = 0\,,$$

a contradiction.

Exercise 6.1.11. Identities.
Prove the set identities

$$P(A \cup B) = 1 - P(\overline{A} \cap \overline{B}), \qquad P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Solution (Exercise 6.1.11).
Apply de Morgan's rule to obtain

$$A \cup B = \overline{\overline{A} \cap \overline{B}}$$

and then use property (1.1). For the second identity, observe that $A \cup B = A + (B - A \cap B)$ and therefore

$$\begin{aligned} P(A \cup B) &= P(A) + P(B - A \cap B) \\ &= P(A) + P(B) - P(A \cap B)\,. \end{aligned}$$

Exercise 6.1.12. Sub-sigma-additivity.
Let $(\Omega, \mathcal{F}, P)$ be a probability space. Prove the *sub-sigma-additivity* property: for any sequence $\{A_n\}_{n\geq 1}$ of events,

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n).$$

Solution (Exercise 6.1.12).
Let $A'_1 := A_1$ and for $n \geq 2$, $A'_n := A_n \cap \overline{\left(\cup_{k=1}^{n-1} A_k\right)}$. We have that the A'_n's are mutually disjoint and that $A'_n \subseteq A_n$ (and in particular $P(A'_n) \leq P(A_n)$). Also $\bigcup_{k=1}^{\infty} A'_n \equiv \bigcup_{k=1}^{\infty} A_n$ and therefore $P\left(\bigcup_{n=1}^{\infty} A_n\right) = P\left(\bigcup_{n=1}^{\infty} A'_n\right) = \sum_{k=1}^{\infty} P(A'_n) \leq \sum_{k=1}^{\infty} P(A_n)$.

Exercise 6.1.13. Sequential continuity, the decreasing case.
Prove Corollary 1.2.1.

Solution (Exercise 6.1.13).
To obtain (1.8), use De Morgan's identity (Exercise 6.1.1) and apply (1.5) with

$A_n = \overline{B}_n$:

$$\begin{aligned}P(\cap_{n=1}^{\infty} B_n) &= 1 - P\left(\overline{\cap_{n=1}^{\infty} B_n}\right)\\ &= 1 - P(\cup_{n=1}^{\infty} \overline{B}_n)\\ &= 1 - \lim_{n\uparrow\infty} P(\overline{B}_n)\\ &= \lim_{n\uparrow\infty}(1 - P(\overline{B}_n)) = \lim_{n\uparrow\infty} P(B_n)\,.\end{aligned}$$

Exercise 6.1.14. Roll it!
You roll fairly and simultaneously three unbiased dice.
(i) What is the probability that one die shows 4, another 2, and another 1?
(ii) What is the probability that *some* die shows 2, given that the sum of the 3 values equals 5?

Solution **(Exercise 6.1.14).**
Call X, Y, Z the results of the three dices A, B, C respectively. We want to compute

$$\begin{aligned}x = P(X=4, Y=2, Z=1) &+ P(X=4, Y=1, Z=2)\\ &+ P(X=2, Y=4, Z=1)\\ &+ P(X=2, Y=1, Z=4)\\ &+ P(X=1, Y=4, Z=2)\\ &+ P(X=1, Y=2, Z=4).\end{aligned}$$

All these probabilities are equal to $\frac{1}{6^3}$ and therefore $x = 6 \times \frac{1}{6^3} = \frac{1}{36}$.

Exercise 6.1.15. One is the sum of the two others.
You perform three independent tosses of an unbiased die. What is the probability that one of these tosses results in a number that is the sum of the two other numbers?

Solution **(Exercise 6.1.15).**
By accounting. Enumerate all possibilities, and beware that the dice have to be ordered. For instance $(1,2,3)$ is different from $(3,1,2)$. The result is $\frac{5}{24}$. See also Exercise 6.2.26 where a less fastidious method is given.

6.1.3 Subsection 1.3.4

Exercise 6.1.16. Urns.

1. An urn contains 17 red balls and 19 white balls. Balls are drawn in succession at random and without replacement. What is the probability that the first 2 balls are red?

2. An urn contains N balls numbered from 1 to N. Someone draws n balls $(1 \le n \le N)$ simultaneously from the urn. What is the probability that the lowest number drawn is k?

Solution (Exercise 6.1.16).

1. Let R_1 and R_2 be the events that the first and second ball respectively that are drawn out from the urn are red. $P(R_1 \cap R_2) = P(R_1)P(R_2|R_1)$. But $P(R_1) = \frac{17}{17+19} = \frac{17}{36}$. If the first ball is red, there remain 16 red balls and 19 white balls. Therefore $P(R_2|R_1) = \frac{16}{16+19} = \frac{16}{35}$, so that $P(R_1 \cap R_2) = \frac{17\times16}{36\times35} = \frac{68}{315}$.

2. There are $\binom{N}{n}$ subsets of n balls among N balls. If ball k is in the subset that is drawn, and if it is the ball with the lowest number, the remaining $n-1$ balls must be chosen among $N-k$ balls (that is $k+1,\dots,N$). This leaves $\binom{N-k}{n-1}$ choices. The probability to be found is therefore $\binom{N-k}{n-1}/\binom{N}{n}$.

Exercise 6.1.17. Heads and tails as usual.
A person, A, tossing an *unbiased* coin N times obtains T_A tails. Another person, B, tossing her own unbiased coin $N+1$ times has T_B tails. What is the probability that $T_A \geq T_B$? *Hint:* Introduce H_A and H_B the number of heads obtained by A and B respectively, and use a symmetry argument.

Solution (Exercise 6.1.17).
We have

$$N+1 = H_B + T_B$$
$$N = H_A + T_A$$

Therefore

$$T_A - T_B = H_B - H_A - 1$$

and

$$P\left(T_A - T_B \geq 0\right) = P\left(H_B - H_A > 0\right).$$

By symmetry,

$$P\left(H_B - H_A > 0\right) = P\left(T_B - T_A > 0\right) = 1 - P\left(T_A - T_B \geq 0\right).$$

Therefore

$$P\left(T_A - T_B \geq 0\right) = 1 - P\left(T_A - T_B \geq 0\right),$$

which gives $P\left(T_A - T_B \geq 0\right) = 1/2$.

Exercise 6.1.18. The switches.
Two nodes A and B in a communications network are connected by three different routes and each route contains a number of links that may fail. These are represented symbolically in Fig. 6.1 by switches that are in the lifted position if the link is in a failure state. In this figure, the number associated with a switch is the probability that the corresponding link is out of order. The links fail independently. What is the probability that A and B are connected?

Solution (Exercise 6.1.18).
Let U_1 be the event "no switch lifted in the upper path". Defining similarly U_2 and U_3, we see that the probability to be computed is that of $U_1 \cup U_2 \cup U_3$, or by De

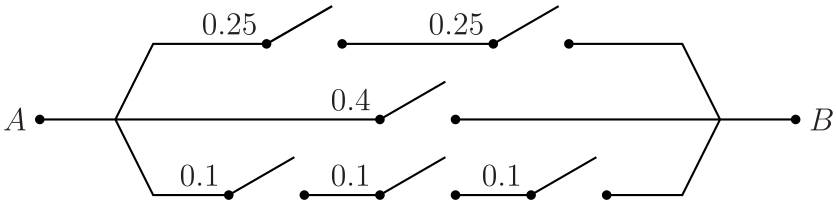

Figure 6.1: All switches up.

Morgan's law, that of the complement of $\overline{U}_1 \cap \overline{U}_2 \cap \overline{U}_3$:

$$1 - P(\overline{U}_1 \cap \overline{U}_2 \cap \overline{U}_3) = 1 - P(\overline{U}_1)P(\overline{U}_2)(P\overline{U}_3),$$

where the last equality follows from the independence assumption for the states of the link. Letting now $U_1^1 =$ "switch 1 (first from left) in the upper path is not lifted" and $U_1^2 =$"switch 2 in the upper path is not lifted", we have $U_1 = U_1^1 \cap U_1^2$, therefore, in view of the independence of the bridges,

$$P(\overline{U}_1) = 1 - P(U_1) = 1 - P(U_1^1)P(U_1^2).$$

We must now use the data $P(U_1^1) = 1 - 0.25, P(U_1^2) = 1 - 0.25$ to obtain $P(\overline{U}_1) = 1 - (0.75)^2$. Similarly $P(\overline{U}_2) = 1 - 0.6$ and $P(\overline{U}_3) = 1 - (0.9)^3$. The final result is $1 - (0.4375)(0.4)(0.271) = 0.952575$.

Exercise 6.1.19. PAIRWISE INDEPENDENCE DOES NOT SUFFICE.
1. Give a simple example of a probability space $(\Omega, \mathcal{F}, P)$ with three events A_1, A_2, A_3 that are pairwise independent, but *not* globally independent (that is, the family $\{A_1, A_2, A_3\}$ is not independent).

2. If $\{A_i\}_{i\in I}$ is an independent family of events, is it true that $\{\tilde{A}_i\}_{i\in I}$ is also an independent family of events, where for each $i \in I$, $\tilde{A}_i = A_i$ or $\overline{A}_i$ (your choice, for instance, with $I = \mathbb{N}$, $\tilde{A}_0 = A_0, \tilde{A}_1 = \overline{A}_1, \tilde{A}_3 = A_3, \ldots$)?

Solution (Exercise 6.1.19).

1. For all $i \in \Omega = \{1, 2, 3, 4\}$, take $P(\{i\}) = \frac{1}{4}$. Define $A_1 = \{1, 2\}$, $A_2 = \{2, 3\}$, $A_3 = \{3, 1\}$. We have

$$P(A_1 \cap A_2 \cap A_3) = P(\varnothing) = 0\,.$$

On the other hand,

$$P(A_1)P(A_2)P(A_3) = \frac{1}{8}.$$

Therefore

$$P(A_1 \cap A_2 \cap A_3) \neq P(A_1)P(A_2)P(A_3),$$

which contradicts independence of the family $\{A_1, A_2, A_3\}$. However, A_1, A_2, A_3 are *pairwise* independent. For instance,

$$P(A_1 \cap A_2) = P(\{2\}) = \frac{1}{4} = P(A_1)P(A_2).$$

2. Yes. By recurrence, it suffice to prove the result when just one A_i is changed in $\overline{A}_i$. Say $i = 0$. The result follows from the following type of computations:

$$\begin{aligned}
P\left(\overline{A}_0 \cap A_1 \cap A_2\right) &= P\left(A_1 \cap A_2\right) - P\left(A_0 \cap A_1 \cap A_2\right) \\
&= P\left(A_1\right) P\left(A_2\right) - P\left(A_0\right) P\left(A_1\right) P\left(A_2\right) \\
&= \left(1 - P\left(A_0\right)\right) P\left(A_1\right) P\left(A_2\right) \\
&= P\left(\overline{A}_0\right) P\left(A_1\right) P\left(A_2\right).
\end{aligned}$$

Exercise 6.1.20. Conditional independence and Markov property.
1. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Define for a fixed event C of positive probability, $P_C(A) := P(A \mid C)$. Show that P_C is a probability on $(\Omega, \mathcal{F})$. (And note that A and B are independent with respect to this probability if and only if they are conditionally independent given C.)

2. Let A_1, A_2, A_3 be three events of positive probability. Show that events A_1 and A_3 are conditionally independent given A_2 if and only if the "Markov property" holds, that is, $P(A_3 \mid A_1 \cap A_2) = P(A_3 \mid A_2)$.

Solution (Exercise 6.1.20).
1. Recall that $P_C(A) := \frac{P(A \cap C)}{P(C)}$ and therefore clearly $0 \le P_C(A) \le 1$. Also

$$P_C(\Omega) = \frac{P(\Omega \cap C}{P(C)} = \frac{P(C)}{P(C)} = 1,$$

and

$$\begin{aligned}
P_C(\sum_{n=1}^{\infty} A_n) &= \frac{P((\sum_{n=1}^{\infty} A_n) \cap C}{P(C)} = \frac{P(\sum_{n=1}^{\infty} (A_n \cap C)}{P(C)} \\
&= \frac{\sum_{n=1}^{\infty} P(A_n \cap C)}{P(C)} = \sum_{n=1}^{\infty} \frac{P(A_n \cap C)}{P(C)} = \sum_{n=1}^{\infty} P_C(A_n).
\end{aligned}$$

2. Assume conditional independence. Then

$$\begin{aligned}
P(A_3 \mid A_1 \cap A_2) &= \frac{P(A_1 \cap A_2 \cap A_3)}{P(A_1 \cap A_2)} = \frac{P(A_1 \cap A_3 \mid A_2) P(A_2)}{P(A_1 \cap A_2)} \\
&= \frac{P(A_1 \mid A_2) P(A_3 \mid A_2) P(A_2)}{P(A_1 \mid A_2) P(A_2)} = P(A_3 \mid A_2).
\end{aligned}$$

Similar computations yield the converse implication.

Exercise 6.1.21. Roll it once more!
You roll fairly and simultaneously three unbiased dice. What is the probability that *some* die shows 2, given that the sum of the 3 values equals 5?

Solution (Exercise 6.1.21).
Of course, 1.

Exercise 6.1.22. APARTHEID UNIVERSITY.
In the renowned Social Apartheid University, students have been separated into three social groups for "pedagogical" purposes. In group A, one finds students who individually have a probability of passing equal to 0.95. In group B this probability is 0.75, and in group C only 0.65. The three groups are of equal size. What is the probability that a student passing the course comes from group A? B? C?

Solution (Exercise 6.1.22).
By the Bayes retrodiction formula,

$$P(A|\text{pass}) = \frac{P(\text{pass}|A)\,P(A)}{P(\text{pass})}\,.$$

By the Bayes rule of exclusive and exhaustive causes,

$$\begin{aligned} P(\text{pass}) &= P(\text{pass}|A)\,P(A) + P(\text{pass}|B)\,P(B) + P(\text{pass}|C)\,P(C) \\ &= 0.95 \times \frac{1}{3} + 0.75 \times \frac{1}{3} + 0.65 \times \frac{1}{3} = \frac{2.35}{3}\,. \end{aligned}$$

Therefore

$$P(A|\text{pass}) = \frac{0.95/3}{2.35/3} = \frac{95}{235}\,.$$

Similar computations give $P(B|\text{pass}) = \frac{75}{235}$ and $P(C|\text{pass}) = \frac{65}{235}$.

Exercise 6.1.23. WISE BET.
There are 3 cards. The first one has both faces red, the second one has both faces white, and the third one is white on one face, red on the other. A card is drawn at random, and the color of a randomly selected face of this card is shown to you (the other remains hidden). What is the winning strategy if you must bet on the color of the hidden face?

Solution (Exercise 6.1.23).
In the following computations, C stands for "card", HF for "hidden face", FS for "face shown".

$$\begin{aligned} P(HF = R|FS = R) &= P(HF = R, FS = R)\,/P(FS = R) \\ &= P(C = RR)\,/P(FS = R)\,. \end{aligned}$$

$$\begin{aligned} P(C = RR) &= \frac{1}{3} \\ P(FS = R) &= P(FS = R|C = RR)\,P(C = RR) + P(FS = R|C = RW)\,P(C = RW) \\ &= 1 \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{3}{2} \times \frac{1}{3}. \end{aligned}$$

Therefore

$$P(HF = R|FS = R) = \frac{2}{3}\,.$$

The best strategy is therefore: bet that the hidden face has the same color as the face exposed.

Exercise 6.1.24. A SEQUENCE OF LIARS.
Consider a sequence of n ''liars'' $L_1, \dots, L_n$. The first liar L_1 receives information about the occurrence of some event in the form ''yes or no'', and transmits it to L_2, who transmits it to L_3, etc... Each liar transmits what he hears with probability $p \in (0,1)$, and the contrary with probability $q = 1-p$. The decision of lying or not is made independently by each liar. What is the probability x_n to obtain the correct information from L_n? What is the limit of x_n as n increases to infinity?

Solution (Exercise 6.1.24).
We have the recurrence equation

$$x_{n+1} = px_n + q(1-x_n),$$

where $q = 1-p$ and $x_1 = p$. Rewrite it as

$$x_{n+1} = q + (p-q)x_n.$$

Thus, if $\lim x_n = x$ exists, then necessarily

$$x = q + (p-q)x,$$

that is $x = 1/2$. Subtracting the last displayed equality with $x = 1/2$ from the previous one yields

$$\left(x_{n+1} - \frac{1}{2}\right) = (p-q)\left(x_n - \frac{1}{2}\right).$$

and therefore

$$\begin{aligned}\left(x_n - \frac{1}{2}\right) &= (p-q)^{n-1}\left(x_1 - \frac{1}{2}\right)\\ &= (p-q)^{n-1}\left(p - \frac{1}{2}\right).\end{aligned}$$

That is

$$x_n = \frac{1}{2} + (p-q)^{n-1}\left(p - \frac{1}{2}\right).$$

Exercise 6.1.25. THE CAMPUS LIBRARY COMPLAINT.
You are looking for a book in the campus libraries. Each library has it with probability 0.60 but the book of each given library may have been stolen with probability 0.25. If there are 3 libraries, what are your chances of obtaining the book?

Solution (Exercise 6.1.25).
The probability of not having the book from a given library is

$$0.4 + (0.6)(0.25) = 0.55$$

(it does not hold a copy of the book, or it does but the book was stolen). Therefore the probability that the book is not available in *all* three libraries is $(0.55)^3$ and that of finding it in *some* library is therefore $1 - (0.55)^3 = 0.833625$.

Exercise 6.1.26. SAFARI BUTCHERS.
Three tourists participate in a safari in Africa. Here comes an elephant, unaware of the rules of the game. The innocent beast is killed, having received two out of the three bullets simultaneously shot by the tourists. The probability of hit of the tourists are: Tourist A: $\frac{1}{4}$, Tourist B: $\frac{1}{2}$, Tourist C: $\frac{3}{4}$. Give for each of the tourists the probability that he was the one who missed.

Solution (Exercise 6.1.26).

$$P(A \text{ missed} \mid 2 \text{ tourists hit }) = \frac{P(A \text{ missed and } 2 \text{ tourists hit})}{P(\text{ } 2 \text{ tourists hit })}.$$

But "A missed and 2 tourists hit" means "A missed, B hit, C hit". Therefore (using independence for the second equality),

$$\begin{aligned} P(\text{A missed and 2 tourists hit}) &= P(\text{A missed, B hit, C hit}) \\ &= P(\text{A missed})P(\text{B hit})P(\text{C hit}) \\ &= (1 - P(\text{A hit}))P(\text{B hit})P(\text{C hit}) \\ &= \frac{3}{4} \times \frac{1}{2} \times \frac{3}{4} = \frac{9}{32}. \end{aligned}$$

Now, "two tourists hit" means "one and only one missed". Therefore (again using independence for the second equality),

$$\begin{aligned} P(\text{two hits}) &= P(\text{A missed, B hit, C hit}) + P(\text{A hit, B missed, C hit}) + P(\text{A hit, B hit, C missed}) \\ &= P(\text{A missed})P(\text{B hit})P(\text{C hit}) + P(\text{A hit})P(\text{B missed})P(\text{C hit}) + P(\text{A hit})P(\text{B hit})P(\text{C} \\ &= \frac{3}{4} \times \frac{1}{2} \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{2} \times \frac{3}{4} + \frac{1}{4} \times \frac{1}{2} \times \frac{1}{4} = \frac{9+3+1}{32} = \frac{13}{32}. \end{aligned}$$

Therefore

$$P(A \text{ missed} \mid 2 \text{ tourists hit }) = \frac{9}{13}.$$

Similar computations give

$$P(B \text{ missed} \mid 2 \text{ tourists hit }) = \frac{3}{13}.$$

$$P(C \text{ missed} \mid 2 \text{ tourists hit }) = \frac{1}{13}.$$

Exercise 6.1.27. PROFESSOR NEBULOUS.
Professor Nebulous travels from Los Angeles to Paris with stopovers in New York and London. At each stop his luggage is transferred from one plane to another. In each airport, including Los Angeles, chances are that with probability p his luggage is not placed in the right plane. Professor Nebulous finds that his suitcase has not reached Paris. What are the chances that the mishap took place in Los Angeles, New York, and London, respectively ?

Solution (Exercise 6.1.27).
Think of the misplacement procedure as follows: a demoniac probabilist throws three coins independently. This results in three random variables X_1, X_2 and X_3,

with values in $\{0,1\}$ (1 is for heads and 0 for tails), and with $P(X_1 = 1) = P(X_2 = 1) = P(X_3 = 1) = p$. If $X_1 = 1$, the misplacement happened in Los Angeles. If $X_1 = 0$ and $X_2 = 1$, it happenend in New York, and if $X_1 = 0$ and $X_2 = 0$ and $X_3 = 1$, it happened in London. The event $M =$ " the luggage has been misplaced " is the sum of these three disjoint (incompatible) events and its probability is therefore $P(M) = P(X_1 = 1) + P(X_1 = 0, X_2 = 1) + P(X_1 = 0, X_2 = 0, X_3 = 1)$. It is natural to assume that the staff in different airports misbehave independently of one another, so that $P(M) = P(X_1 = 1)+P(X_1 = 0)P(X_2 = 1)+P(X_1 = 0)P(X_2 = 0)P(X_3 = 1) = p+(1-p)p+(1-p)^2p = 1-(1-p)^3$. This result could have been obtained more simply: $P(M) = 1-P(\overline{M}) = 1-P(X_1 = 0, X_2 = 0, X_3 = 0) = 1-P(X_1 = 0)P(X_2 = 0)P(X_3 = 0) = 1-(1-p)^3$. We want to compute the probabilities x, y, and z for the luggage to be stranded in Los Angeles, New York, and London, respectively , knowing that it does not reach Paris: $x = P(X_1 = 1|M), y = P(X_1 = 0, X_2 = 1|M), z = P(X_1 = 0, X_2 = 0, X_3 = 1|M)$. One finds

$$\begin{aligned}
x &= P(X_1 = 1, M)/P(M) = P(X_1 = 1)/P(M) \\
&= \frac{p}{1-(1-p)^3} \\
y &= P(X_1 = 0, X_2 = 1, M)/P(M) = P(X_1 = 0, X_2 = 1)/P(M) \\
&= \frac{p(1-p)}{1-(1-p)^3} \\
z &= P(X_1 = 0, X_2 = 0, X_3 = 1, M)/P(M) = P(X_1 = 0, X_2 = 0, X_3 = 1)/P(M) \\
&= \frac{p(1-p)^2}{1-(1-p)^3}
\end{aligned}$$

Exercise 6.1.28. HARDY–WEINBERG'S LAW.
In Example 1.3.1, show that the genotypic distributions of all generations, starting from the third one, are the same (*Hardy–Weinberg's law*) and that the stationary distribution depends only on the proportion c of alleles of type A in the initial population.

Solution (Exercise 6.1.28).
Define the functions f_1, f_2, and f_3 by

$$\begin{aligned}
f_1(x, y, z) &= (x + z)^2 \\
f_2(x, y, z) &= (y + z)^2 \\
f_3(x, y, z) &= (x + z)(y + z).
\end{aligned}$$

To be proven: for all nonnegative numbers x, y, z such that $x + y + 2z = 1$,

$$f_i(x, y, z) = f_i[f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)], \quad i = 1, 2, 3.$$

The third equality, for instance, is

$$(x + z)(y + z) = [(x + z)^2 + (y + z)(x + z)][y + z)^2 + (y + z)(x + z)].$$

It holds since $(x+z)^2+(y+z)(x+z) = (x+z)(x+2z+y) = x+z$ and $(y+z)^2+(y+z)(x+z) = (y + z)(y + 2z + x) = y + z$. The ratio c of alleles of type A in the initial population is

$x + z$. Now $y + z = 1 - c$. Therefore, the stationary distribution is

$$p = c^2, \quad q = (1-c)^2, \quad 2r = 2c(1-c).$$

Exercise 6.1.29. Doctors are not that bad.
In Example 1.3.3 compute the detection failure probability $P(M \,|\, -)$. If it is not small enough, how to improve things?

Solution (Exercise 6.1.29).

$$P(M \,|\, -) = \frac{P(M, -)}{P(-)} = \frac{P(- \,|\, M)P(M)}{P(- \,|\, M)P(M) + P(- \,|\, \overline{M})P(\overline{M})},$$

that is

$$P(M \,|\, -) = \frac{0.01 \times 0.001}{0.01 \times 0.001 + 0.08 \times 0.009},$$

which is approximately $1/80$. This is still too large for certain illnesses, and you must in that case have a better $P(+ \,|\, M)$. With $P(+ \,|\, M) = 0.999$, you have

$$P(M \,|\, -) = \frac{0.001 \times 0.001}{0.001 \times 0.001 + 0.08 \times 0.009},$$

that is, approximately, $1/800$.

6.2 Chapter 2

6.2.1 Subsection 2.1.4

Exercise 6.2.1. Poincaré.
Let $A_1, \dots, A_n$ be events and let $X_1, \dots, X_n$ be their indicator functions. From the developped expression of $E\left[\Pi_{i=1}^n (1 - X_i)\right]$, deduce *Poincaré's formula*:

$$\begin{aligned} P(\cup_{i=1}^n A_i) = \sum_i P(A_i) \quad & - \sum_{i<j} P(A_i \cap A_j) \\ & + \sum_{i<j<k} P(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n) \end{aligned}$$

Solution (Exercise 6.2.1).

$$P\left(\cup_{i=1}^n A_i\right) = 1 - P\left(\overline{\cup_{i=1}^n A_i}\right) = 1 - P\left(\cap_{i=1}^n \overline{A}_i\right).$$

$$\begin{aligned} P\left(\cap_{i=1}^n \overline{A}_i\right) &= E\left[1_{\cap_{i=1}^n \overline{A}_i}\right] = E\left[\prod_{i=1}^n 1_{\overline{A}_i}\right] \\ &= E\left[\prod_{i=1}^n (1 - 1_{A_i})\right] = E\left[\prod_{i=1}^n (1 - X_i)\right]. \end{aligned}$$