$x+z$. Now $y+z=1-c$. Therefore, the stationary distribution is

$$p=c^2, \quad q=(1-c)^2, \quad 2r=2c(1-c).$$

Exercise 6.1.29. DOCTORS ARE NOT THAT BAD.
In Example 1.3.3 compute the detection failure probability $P(M\,|\,-)$. If it is not small enough, how to improve things?

Solution (Exercise 6.1.29).

$$P(M\,|\,-)=\frac{P(M,-)}{P(-)}=\frac{P(-\,|\,M)P(M)}{P(-\,|\,M)P(M)+P(-\,|\,\overline{M})P(\overline{M})}\,,$$

that is

$$P(M\,|\,-)=\frac{0.01\times 0.001}{0.01\times 0.001+0.08\times 0.009}\,,$$

which is approximately $1/80$. This is still too large for certain illnesses, and you must in that case have a better $P(+\,|\,M)$. With $P(+\,|\,M)=0.999$, you have

$$P(M\,|\,-)=\frac{0.001\times 0.001}{0.001\times 0.001+0.08\times 0.009}\,,$$

that is, approximately, $1/800$.

6.2 Chapter 2

6.2.1 Subsection 2.1.4

Exercise 6.2.1. POINCARÉ.
Let $A_1,\dots,A_n$ be events and let $X_1,\dots,X_n$ be their indicator functions. From the developped expression of $E\left[\Pi_{i=1}^n(1-X_i)\right]$, deduce *Poincaré's formula*:

$$\begin{aligned}P(\cup_{i=1}^n A_i)=\sum_i P(A_i) \quad &- \quad \sum_{i<j}P(A_i\cap A_j)\\ &+ \sum_{i<j<k}P(A_i\cap A_j\cap A_k)-\cdots+(-1)^{n+1}P(A_1\cap A_2\cap\cdots\cap A_n)\end{aligned}$$

Solution (Exercise 6.2.1).

$$P\left(\cup_{i=1}^n A_i\right)=1-P\left(\overline{\cup_{i=1}^n A_i}\right)=1-P\left(\cap_{i=1}^n\overline{A}_i\right).$$

$$\begin{aligned}P\left(\cap_{i=1}^n\overline{A}_i\right)&=E\left[1_{\cap_{i=1}^n\overline{A}_i}\right]=E\left[\prod_{i=1}^n 1_{\overline{A}_i}\right]\\ &=E\left[\prod_{i=1}^n(1-1_{A_i})\right]=E\left[\prod_{i=1}^n(1-X_i)\right].\end{aligned}$$

$$\prod_{i=1}^{n}(1-X_i) = 1 - \left\{\sum_i X_i - \sum_{i<j} X_iX_j + \sum_{i<j<k} X_iX_jX_k + \dots\right\}.$$

Therefore

$$\begin{aligned} P\left(\cup_{i=1}^{n} A_i\right) &= E\left[\sum_i X_i - \sum_{i<j} X_iX_j + \sum_{i<j<k} X_iX_jX_k + \dots\right] \\ &= E\left[\sum_i X_i\right] - E\left[\sum_{i<j} X_iX_j\right] + E\left[\sum_{i<j<k} X_iX_jX_k\right] + \dots \end{aligned}$$

Observe that

$$E\left[X_{i_1}X_{i_2}\dots X_{i_k}\right] = E\left[1_{\cap_{l=1}^{k} A_{i_l}}\right] = P\left(\cap_{l=1}^{k} A_{i_l}\right).$$

Exercise 6.2.2. NON ESSENTIAL SET.
Let X be a discrete random variable taking its values in E, with probability distribution $p(x)$, $x \in E$. Define $A = \{\omega;\, p(X(\omega)) = 0\}$. Show that $P(A) = 0$.

Solution (Exercise 6.2.2).

$$P(A) = E\left[1_A\right] = E\left[1_{\{p(X)=0\}}\right] = \sum_{x\in E} 1_{\{p(x)=0\}}p(x) = 0\,.$$

Exercise 6.2.3. THE MEAN IS THE CENTER OF INERTIA.
Let X be a real random variable with mean μ and finite variance σ^2. Show that, for all $a \in \mathbb{R}$, $a \neq \mu$,

$$E[(X-a)^2] > E[(X-\mu)^2] = \sigma^2\,.$$

Solution (Exercise 6.2.3).

$$\begin{aligned} E\left[(X-a)^2\right] &= E\left[((X-\mu)+(\mu-a))^2\right] \\ &= E\left[(X-\mu)^2\right] + (\mu-a)^2 + 2(\mu-a)E\left[(X-\mu)\right] \\ &= E\left[(X-\mu)^2\right] + (\mu-a)^2 > E\left[(X-\mu)^2\right] \end{aligned}$$

whenever $a \neq \mu$.

Exercise 6.2.4. NULL VARIANCE.
Prove that a null variance implies that the random variable is almost-surely constant. (Do the proof for an integer-valued random variable.)

Solution (Exercise 6.2.4).

$$\sum_{n=0}^{+\infty}(n-\mu)^2 P(X=n) = 0$$

implies that $n - \mu = 0$ whenever $P(X = n) > 0$. Therefore

$$P(X = \mu) = \sum_{n;n=\mu} P(X = n) = \sum_{n=0}^{+\infty} P(X = n) = 1.$$

Exercise 6.2.5. CHECKING CONDITIONAL INDEPENDENCE.
Let X, Y, and Z be three discrete random variables with values in E, F, and G, respectively. Prove the following: If for some function $g : E \times F \to [0,1]$, $P(X = x \mid Y = y, Z = z) = g(x,y)$ for all x, y, z, then $P(X = x \mid Y = y) = g(x,y)$ for all x, y, and X and Z are conditionally independent given Y.

Solution (Exercise 6.2.5).
We have

$$\begin{aligned} P(X = x, Y = y) &= \sum_z P(X = x, Y = y, Z = z) \\ &= \sum_z P(X = x \mid Y = y, Z = z) P(Y = y, Z = z) \\ &= g(x,y) \sum_z P(Y = y, Z = z) = g(x,y) P(Y = y). \end{aligned}$$

Therefore,

$$P(X = x \mid Y = y) = g(x,y) = P(X = x \mid Y = y, Z = z).$$

Exercise 6.2.6. GIBBS'S INEQUALITY.
Let $(p(x), x \in \mathcal{X})$ and $(q(x), x \in \mathcal{X})$ be two probability distributions on the finite space $\mathcal{X}$. Prove the *Gibbs inequality*

$$- \sum_{x \in \mathcal{X}} p(x) \log p(x) \le - \sum_{x \in \mathcal{X}} p(x) \log q(x)\,, \tag{6.1}$$

with equality if and only if $p(x) = q(x)$ for all $x \in \mathcal{X}$.

Solution (Exercise 6.2.6).
We may suppose that $q(x) > 0$ for all x such that $p(x) > 0$ (otherwise the inequality is trivial, the right-hand side being infinite). One can therefore restrict oneself to the case where $\mathcal{X}$ contains only elements x such that $p(x) > 0$, and where q is a subprobability ($\sum_{x \in \mathcal{X}} q(x) \le 1$) such that $q(x) > 0$ for all x. Using the fact that $z > 0 \Rightarrow \log z \le z - 1$ with equality if and only if $z = 1$, we have that :

$$\begin{aligned} \sum_{x \in \mathcal{X}} p(x) \log \frac{q(x)}{p(x)} &\le \sum_{x \in \mathcal{X}} p(x) \left(\frac{q(x)}{p(x)} - 1 \right) \\ &= \sum_{x \in \mathcal{X}} q(x) - \sum_{x \in \mathcal{X}} p(x) \le 0\,, \end{aligned}$$

with equality if and only if $\frac{q(x)}{p(x)} = 1$ for all $x \in \mathcal{X}$.

6.2.2 Subsection 2.2.5

Exercise 6.2.7. Geometric is memoryless.
Show that a geometric random variable T with parameter $p \in (0,1)$ is *memoriless* in the sense that for all integers $k, k_0 \geq 1$, $P(T = k + k_0 \mid T > k_0) = P(T = k)$.

Solution (Exercise 6.2.7).

$$\begin{aligned} P(T > k_0) &= \sum_{k=k_0+1}^{\infty} (1-p)^{k-1} p \\ &= p(1-p)^{k_0} \sum_{n=0}^{\infty} (1-p)^n = \frac{p(1-p)^{k_0}}{1-(1-p)} = (1-p)^{k_0} . \end{aligned}$$

$$\begin{aligned} P(T = k_0 + k | T > k_0) &= \frac{P(T = k_0 + k, T > k_0)}{P(T > k_0)} = \frac{P(T = k_0 + k)}{P(T > k_0)} \\ &= \frac{p(1-p)^{k+k_0-1}}{(1-p)^{k_0}} = p(1-p)^k = P(T = k) . \end{aligned}$$

Exercise 6.2.8.
Let T_1 and T_2 be two independent geometric random variables with the same parameter $p \in (0,1)$. Give the probability distribution of the sum $X = T_1 + T_2$.

Solution (Exercise 6.2.8).
For $n \geq 2$:

$$\begin{aligned} P(T_1 + T_2 = n) &= \sum_{k=1}^{n-1} P(T_1 + k = n, T_2 = k) \\ &= \sum_{k=1}^{n-1} P(T_1 = n - k) P(T_2 = k) \\ &= \sum_{k=1}^{n-1} (1-p)^{n-k-1} p (1-p)^{k-1} p \\ &= \sum_{k=1}^{n-1} (1-p)^{n-2} p^2 = (n-1) \left(\frac{p}{1-p} \right)^2 (1-p)^n \end{aligned}$$

Exercise 6.2.9. The return of the coupon collector.
In the coupon's collector problem of Example 2.2.4, compute the variance σ_X^2 of X (the number of chocolate tablets needed to complete the collection of the n different coupons) and show that $\frac{\sigma_X^2}{n^2}$ has a limit (to be identified) as n grows indefinitely.

Exercise 6.2.10. The ambitious coupon collector.

In the coupon's collector problem of Example 2.2.4, prove that for all $c > 0$,

$$P\left(X > \lceil n \ln n + cn \lceil\right) \leq e^{-c}.$$

Hint: you might find useful to define A_i to be the event that Type i coupon has not shown up during in first $\lceil n \ln n + cn \lceil$ tablets.

Solution (Exercise 6.2.10).
Then

$$\begin{aligned} P\left(X > \lceil n \ln n + cn \rceil\right) &= P\left(\cup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P\left(A_i\right) \\ &= \sum_{i=1}^{n} \left(1 - \frac{1}{n}\right)^{\lceil n \ln n + cn \rceil} \leq n \exp\left(n \ln n + cn\right) = e^{-c}. \end{aligned}$$

Exercise 6.2.11. FACTORIAL OF POISSON.
1. Let X be a Poisson random variable with mean $\theta > 0$. Compute the mean of the random variable $X!$ (factorial, not exclamation mark!).

2. Compute $E\left[\theta^X\right]$.

Solution (Exercise 6.2.11).

1.

$$E\left[X!\right] = \sum_{n=0}^{\infty} n! e^{-\theta} \frac{\theta^n}{n!} = e^{\theta} \sum_{n=0}^{\infty} \theta^n.$$

If $\theta \geq 1$, $E\left[X!\right] = \infty$, if $\theta < 1$, $E\left[X!\right] = e^{-\theta} \frac{1}{1-\theta}$.

2.

$$\begin{aligned} E\left[\theta^X\right] &= \sum_{n=0}^{\infty} \theta^n P(X = n) = \sum_{n=0}^{\infty} \theta^n e^{-\theta} \frac{\theta^n}{n!} \\ &= e^{-\theta} \sum_{n=0}^{\infty} \frac{\theta^{2n}}{n!} = e^{-\theta} e^{\theta^2} = e^{-\theta(1-\theta)}. \end{aligned}$$

Exercise 6.2.12. EVEN AND ODD POISSON.
Let X be a Poisson random variable with mean $\theta > 0$. What is the probability that X is odd?

Solution (Exercise 6.2.12).

$$P\left(X \textbf{ is odd}\right) = \sum_{k \textbf{ odd}} \frac{\theta^k}{k!}$$

Now

$$e^{\theta} = \sum_{k \text{ even}} \frac{\theta^k}{k!} + \sum_{k \text{ odd}} \frac{\theta^k}{k!},$$

and

$$e^{-\theta} = \sum_{k \text{ even}} \frac{\theta^k}{k!} - \sum_{k \text{ odd}} \frac{\theta^k}{k!}.$$

Therefore

$$\sum_{k \text{ odd}} \frac{\theta^k}{k!} = \frac{e^{\theta} - e^{-\theta}}{2},$$

and

$$P(X \text{ is odd}) = e^{-\theta} \times \frac{e^{\theta} - e^{-\theta}}{2} = \frac{1 - e^{-2\theta}}{2}.$$

Exercise 6.2.13. ENTROPY.

Let X be a discrete random variable taking its values in a finite set F. Let $p(x) := P(X = x)$, $x \in F$, be its distribution.

i) Compute $-E[\log p(X)]$. (This quantity is called the *entropy* of X and is denoted by $H[X]$.)

ii) Let now $X = (X_1, \ldots, X_n)$ where the X_i's are IID random variables taking their values in a finite set E with the common distribution π. Express $H[X]$ in terms of $H[X_1]$.

iii) What is the entropy of a binomial random variable of size n and parameter $p \in (0,1)$?

Solution (Exercise 6.2.13).

i) $-E[\log p(X)] = -\sum_{x \in F} P(X = x) \log p(x) = -\sum_{x \in F} p(x) \log p(x)$.

ii) $p(x) = P(X = x) = P((X_1 = x_1, \ldots, X_n = x_n)) = \prod_{i=1}^n P(X_i = x_i) = \prod_{i=1}^n \pi(x_i)$.

$-E[\log p(X)] = -E[\log \prod_{i=1}^n \pi(X_i)] = -\sum_{i=1}^n E[\log \pi(X_i)] = nH[X_1]$.

(iii) $-n(p \log p + (1-p)\log(1-p))$.

Exercise 6.2.14. POISSON–BERNOULLI SUM.

Let $\{X_n\}_{n \geq 1}$ be independent random variables taking the values 0 and 1 with probability $q = 1 - p$ and p, respectively, where $p \in (0,1)$. Let T be a Poisson random variable with mean $\theta > 0$, independent of $\{X_n\}_{n \geq 1}$. Let $S := X_1 + \cdots + X_T$. Show that S is a Poisson random variable with mean $p\theta$.

Solution (Exercise 6.2.14).

$$\begin{aligned}
P(S=k) &= P(X_1+\cdots+X_T=k)\\
&= P\left(\sum_{n=k}^{\infty}\{X_1+\cdots+X_n=k, T=n\}\right)\\
&= \cup_{n=k}^{\infty} P(X_1+\cdots+X_n=k, T=n)\\
&= \sum_{n=k}^{\infty} P(X_1+\cdots+X_n=k)P(T=n),
\end{aligned}$$

that is

$$\begin{aligned}
P(S=k) &= \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k q^{n-k} e^{-\theta} \frac{\theta^n}{n!}\\
&= e^{-\theta}\frac{(p\theta)^k}{k!}\sum_{n=k}^{\infty}\frac{(q\theta)^{n-k}}{(n-k)!}\\
&= e^{-\theta}\frac{(p\theta)^k}{k!}\sum_{i=0}^{\infty}\frac{(q\theta)^i}{i!}\\
&= e^{-\theta}\frac{(p\theta)^k}{k!}e^{q\theta} = e^{p\theta}\frac{(p\theta)^k}{k!}.
\end{aligned}$$

Thus, if one "thins out" with thinning probability $1-p$ a population sample of Poissonian size, the remaining sample has also a Poissonian size, with the obvious mean that is p times that of the original sample.

Exercise 6.2.15. Bernoulli sum of products.
Let $X_1, \dots, X_{2n}$ be independent random variables taking the values 0 or 1, and such that for all i, $1 \le i \le 2n$, $P(X_i = 1) = p \in [0, 1]$. Define $Z = \sum_{i=1}^{n} X_i X_{n+i}$. Compute $P(Z = k)$, $1 \le k \le n$.

Solution (Exercise 6.2.15).
Define $Z_i = X_i X_{n-i}$. The sequence $\{Z_i\}_{1\le i\le n}$ is a Bernoulli sequence with

$$P(Z_i=1) = P(X_i=1)\,P(X_{n-i}=1) = p^2.$$

Therefore $Z = \sum_{i=1}^{n} Z_i$ is a binomial random variable of size n and parameter p^2. Therefore, for $0 \le k \le n$,

$$P(Z=k) = \binom{n}{k} p^{2k}\left(1-p^2\right)^{n-k}.$$

Exercise 6.2.16. Hazard rate.
The hazard rate function $\lambda : \mathbb{N} \to [0, 1]$ of an integer-valued function X is defined by $\lambda(n) = P(X = n | X \ge n)$.
(i) Compute $P(X \ge n)$ in terms of $\lambda(0), \cdots, \lambda(n)$.

(ii) Let $\{U_n\}_{n\geq 0}$ be a sequence of IID random variables uniformly distributed on $[0,1]$. Show that the random variable $Z := \min\{n \geq 0 : U_n \leq \lambda(n)\}$ has the same distribution as X.

Solution (Exercise 6.2.16).
(i) Letting $A_n = P(X \geq n)$, we have $\lambda(n) = \frac{A_n - A_{n+1}}{A_n} = 1 - \frac{A_{n+1}}{A_n}$, from which we obtain, observing that $A_0 = 1$,

$$P(X \geq n) = \prod_{i=0}^{n} (1-\lambda(i)) .$$

(ii)

$$\begin{aligned} P(Z \geq n) &= P(U_0 > \lambda(0), \dots, U_n > \lambda(n)) \\ &= P(U_0 > \lambda(0)) \cdots P(U_n > \lambda(n)) = \prod_{i=0}^{n} (1-\lambda(i)) = P(X \geq n) . \end{aligned}$$

Exercise 6.2.17. STOCHASTICALLY LARGER.
Let X and Y be two integer-valued random variables. Then X is said to be *stochastically larger* than Y if for all $n \geq 0$, $P(X \geq n) \geq P(Y \geq n)$. Show that in this case $E[u(X)] \geq E[u(Y)]$ whenever $u : \mathbb{N} \to \mathbb{R}$ is a non-negative and non-decreasing function.

Solution (Exercise 6.2.17).
The proof immediately follows from the observation that

$$\begin{aligned} E[u(X)] &= u(0)P(X=0) + u(1)P(X=1) + \cdots + u(n)P(X=n) + \cdots \\ &= u(0)(P(X \geq 0) - P(X \geq 1)) + u(1)(P(X \geq 1) - P(X \geq 2)) \\ &\qquad + \cdots + u(n)(P(X \geq n) - P(X \geq n+1)) + \cdots \\ &= u(0)P(X \geq 0) + (u(1)-u(0))P(X \geq 1) + \cdots (u(n+1)-u(n))P(X \geq n+1) + \cdots \end{aligned}$$

Exercise 6.2.18. THE MATCHBOX.
A smoker has one matchbox with N matches in each pocket. He reaches at random for one box or the other. What is the probability that, having eventually found an empty matchbox, there will be k matches left in the other box?

Solution (Exercise 6.2.18).
Let $X_n = 1$ if the match is taken from the box in the left pocket, $= 0$ otherwise. The sequence $\{X_n\}_{n\geq 1}$ is a Bernoulli sequence. The event that the box in the right pocket has k matches left when the box in the left pocket is empty for the first time is

$$P(X_1+\cdots+X_{N+k-1} = N-1, X_{N+k} = 1) = P(X_1+\cdots+X_{N+k-1} = N-1)P(X_{N+k} = 1) = \binom{N+k-1}{N-1} \times \frac{1}{2} .$$

Inverting the role of the pockets, we find the same result, by symmetry. Therefore the answer is $\binom{N+k-1}{N-1}$.

Exercise 6.2.19. THE BLUE PINKO.

The blue pinko, an extravagant australian bird, lays T eggs, each egg blue or pink, with probability p that a given egg is blue. The colors of the successive eggs are independent, and independent of the number of eggs laid. Exercise 6.2.14 shows that if the number of eggs is Poisson with mean θ, then the number of blue eggs is Poisson with mean θp and the number of pink eggs is Poisson with mean θq. Show that the number of blue eggs and the number of pink eggs are independent random variables.

Solution (Exercise 6.2.19).

If S is the number of blue eggs, $T - S$ is the number of pink eggs. One must show that for any integers $k \geq 0, \ell \geq 0$,

$$\begin{aligned} P(S = k, T - S = \ell) &= P(S = k)P(T - S = \ell) \\ &= e^{-\theta p}\frac{(\theta p)^k}{k!}e^{-\theta q}\frac{(\theta q)^\ell}{\ell!}. \end{aligned}$$

But

$$\begin{aligned} P(S = k, T - S = \ell) &= P(S = k, T = k + \ell) \\ &= P(X_1 + \cdots + X_{k+\ell} = k, T = k + \ell) \\ &= P(X_1 + \cdots + X_{k+\ell} = k)P(T = k + \ell) \\ &= \frac{(k+\ell)!}{k!\ell!}p^k q^\ell e^{-\theta}\frac{\theta^{k+\ell}}{(k+\ell)!} = e^{-p\theta}\frac{(p\theta)^k}{k!}e^{-q\theta}\frac{(q\theta)^\ell}{\ell!}. \end{aligned}$$

Exercise 6.2.20. THE ENTOMOLOGIST.

Each individual of a specific breed of insects has, independently of the others, the probability θ of being a male. An entomologist seeks to collect exactly $M > 1$ males, and therefore stops hunting as soon as M males are captured. What is the distribution of X, the number of insects that must be caught in order to collect *exactly* M males?

Solution (Exercise 6.2.20).

Consider the (independent) random variables $Z_1, Z_2, \ldots$ where $Z_i = 1$ if and only if the i-th captured insect is a male, $Z_i = 0$ otherwise. If $k < M$, $P(X = k) = 0$. If $k \geq M$,

$$\begin{aligned} P(X = k) &= P(Z_1 + \cdots + Z_{k-1} = M - 1, Z_k = 1) \\ &= P(Z_1 + \cdots + Z_{k-1} = M - 1) \times P(Z_k = 1) \\ &= \frac{(k-1)!}{(k-M)!(M-1)!}\theta^{M-1}(1-\theta)^{k-M} \times \theta \\ &= \frac{(k-1)!}{(k-M)!(M-1)!}\theta^{M}(1-\theta)^{k-M}. \end{aligned}$$

where we have used the fact that $Z_1 + \cdots + Z_{k-1}$ is a binomial random variable.

Exercise 6.2.21. THE RETURN OF THE ENTOMOLOGIST.

The situation is as in Exercise 6.2.20. What is the distribution of X, the smallest number of insects that the entomologist must catch to collect *at least* M males and N females?

Solution (Exercise 6.2.21).
$P(X=k)=0$ **if** $k<N+M$**. If** $k\geq N+M$**,**

$$P(X=k)=P\left(Z_k=1,\sum_{i=1}^{k-1}Z_i=M-1\right)+P\left(Z_k=0,\sum_{i=1}^{k-1}(1-Z_i)=N-1\right)$$

(the first term on the right is the probability of obtaining at time k exactly M males and at therefore at least N females since $k\geq N+M$, the second term is the probability of obtaining at time k exactly N females and therefore at least M males). Therefore

$$P(X=k)=1_{k\geq N+M}\left(\binom{k-1}{M-1}p^k(1-p)^{M-k}+\binom{k-1}{N-1}(1-p)^k p^{N-k}\right).$$

6.2.3 Subsection 2.3.4

Exercise 6.2.22. Mean and variance via generating functions.

Compute the mean and variance of the binomial random variable B of size n and parameter p from its generating function. Do the same for the Poisson random variable P of mean θ.

Solution (Exercise 6.2.22).

By (2.24) and (2.25), when it is defined, the variance σ_X^2 of any integer-valued random variable X of generating function $g_X(z)$ is given by the formula

$$\sigma_X^2=g_X''(1)+g_X'(1)-g_X'(1)^2.$$

Applying this formula to B and P of respective generating function $g_B(z)=(1-p+pz)^n$ and $g_P(z)=e^{\theta(z-1)}$, we obtain $\sigma_B^2=np(1-p)$ and $\sigma_P^2=\theta$.

Exercise 6.2.23. Variance of the geometric function.

What is the generating function g_T of the geometric random variable T with parameter $p\in(0,1)$ (recall $P(T=n)=(1-p)^{n-1}p$, $n\geq 1$). Compute its first two derivatives and deduce from the result the variance of T.

Solution (Exercise 6.2.23).
We have

$$\begin{aligned} g_T(z) &= \sum_{n=1}^{\infty} (1-p)^{n-1} p z^n \\ &= pz \sum_{n=1}^{\infty} [(1-p)z]^{n-1} \\ &= pz \sum_{j=1}^{\infty} [(1-p)z]^j = \frac{pz}{1-(1-p)z} = \frac{pz}{1-qz}, \end{aligned}$$

where $q = 1-p$ **(the radius of convergence of the power series is** $1/1-p$**, and therefore the domain of definition of** g_T **is** $\{z; |z| \le 1/1-p\}$**). In the interior of the disk of absolute convergence**

$$g_T'(z) = \frac{p}{(1-qz)^2}, \quad g_T''(z) = \frac{2pq}{(1-qz)^3}$$

Therefore

$$g_T'(1) = \frac{p}{(1-q)^2} = \frac{1}{p}, \quad g_T''(1) = \frac{2pq}{(1-q)^3} = \frac{2q}{p^2}$$

and

$$E[T] = g_T'(1) = \frac{1}{p}.$$

$$\begin{aligned} \text{Var }(T) &= g_T''(1) + E[T] - E[T]^2 \\ &= \frac{2p}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2q+p-1}{p^2} \\ &= \frac{q+p+q-1}{p^2} = \frac{q+1-1}{p^2} = \frac{q}{p^2}. \end{aligned}$$

Exercise 6.2.24. Factorial moment of Poisson.
What is the n**-th factorial moment (**$E[X(X-1)\cdots(X-n+1)]$**) of a Poisson random variable** X **of mean** $\theta > 0$**?**

Solution (Exercise 6.2.24).
The generating function of a Poisson variable X **of mean** $\theta > 0$ **is**

$$g_X(z) = e^{\theta(z-1)},$$

and

$$g_X^{(n)}(z) = \theta^n e^{\theta(z-1)}.$$

But

$$\begin{aligned} \frac{d^n}{dz^n} E\left[z^X\right] &= E\left[\frac{d^n}{dz^n} z^X\right] \\ &= E\left[X(X-1)\dots(X-n+1)z^X\right]. \end{aligned}$$

Therefore

$$E\left[X\left(X-1\right)\ldots\left(X-n+1\right)\right]=g_X^{(n)}\left(0\right)=\theta^n\,.$$

Exercise 6.2.25. FROM THE GENERATING FUNCTION TO THE DISTRIBUTION.
What is the probability distribution of the integer-valued random variable X with generating function $g(z)=\frac{1}{(2-z)^2}$?

Solution (Exercise 6.2.25).

$$\begin{aligned}g(z)&=\frac{1}{(2-z)^2}=\frac{1}{2}\frac{d}{dz}\frac{1}{1-\frac{z}{2}}\\&=\frac{1}{2}\frac{d}{dz}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots+\frac{z^n}{2^n}+\cdots\right)\\&=\frac{1}{4}\left(1+\frac{2z}{2}+\frac{3z^2}{4}+\cdots+\frac{nz^{n-1}}{2^{n-1}}+\cdots\right).\end{aligned}$$

Therefore, for all $n\ge 0$,

$$P(X=n)=\frac{1}{4}\frac{n+1}{2^n}\,.$$

Exercise 6.2.26. THROW A DIE.
You perform three independent tosses of an unbiased die. What is the probability that one of these tosses results in a number that is the sum of the two other numbers? (You are required to find a solution using generating functions.)

Solution (Exercise 6.2.26).
We seek to compute

$$P\left(X_1=X_2+X_3\right)+P\left(X_2=X_1+X_3\right)+P\left(X_3=X_1+X_2\right)\,,$$

which is equal, by symmetry, to $3P\left(Y=6\right)$, where $Y=6+X_2+X_3-X_1$. The generating function of any X_1 is

$$\frac{1}{6}\left(z+\ldots+z^6\right)=\frac{1}{6}z\frac{1-z^6}{1-z}\,,$$

and therefore

$$\begin{aligned}g_Y\left(z\right)&=E\left[z^Y\right]\\&=z^6E\left[z^{X_2}\right]E\left[z^{X_3}\right]E\left[\left(z^{-1}\right)^{X_1}\right]\\&=z^6\frac{1}{6^3}z^2\frac{\left(1-z^6\right)^2}{\left(1-z\right)^2}z^{-1}\frac{1-z^{-6}}{1-z^{-1}}=\frac{1}{216}z^2\frac{\left(1-z^6\right)^3}{\left(1-z\right)^3},\end{aligned}$$

that is

$$\frac{1}{216}z^2\left(1-\binom{3}{1}z^6+\cdots\right)\left(1+\binom{3}{2}z+\binom{4}{2}z^2+\binom{5}{2}z^3+\binom{6}{2}z^4+\cdots\right)$$

$P(Y=6)$ **is the term in** z^6**, that is** $\frac{1}{216}\binom{6}{2}$**. The result is therefore**

$$3P(Y=6) = \frac{3}{216}\frac{5\times 6}{2} = \frac{5}{24}.$$

Exercise 6.2.27. Residual time.
Let X be a random variable with values in $\mathbb{N}$ an with finite mean m. Show that $p_n = \frac{1}{m}P(X > n)$, $n \in \mathbb{N}$, defines a probability distribution on $\mathbb{N}$. Compute its generating function G in terms of the generating function g and the mean m of X.

Solution (Exercise 6.2.27).
By the telescope formula,

$$m := E[X] = \sum_{n\geq 0} P(X > n)\,.$$

Now,

$$\begin{aligned}
mG(z) &= \sum_{n\geq 0} P(X>n)z^n \\
&= P(X=1) + P(X=2) + P(X=3) + P(X=4) + \cdots \\
&\quad + P(X=2)z + P(X=3)z + P(X=4)z + \cdots \\
&\quad + P(X=3)z^2 + P(X=4)z^2 + \cdots \\
&\quad + P(X=4)z^3 + \cdots \\
&= P(X=1) + P(X=2)(1+z) + P(X=3)(1+z+z^2) + P(X=4)(1+z+z^2+z^3) + \cdots \\
&= \frac{1}{1-z}\left(P(X=1)(1-z) + P(X=2)(1-z^2) + P(X=3)(1-z^3) + P(X=4)(1-z^4) + \cdots\right) \\
&= \frac{1}{1-z}(1 - P(X=0)) - \frac{1}{1-z}(g(z) - P(X=0))\,.
\end{aligned}$$

Therefore

$$G(z) = \frac{1-g(z)}{m(1-z)}\,.$$

Exercise 6.2.28. Wald's expectation formula.
Let $\{Y_n\}_{n\geq 1}$ be a sequence of integer-valued integrable random variables such that $E[Y_n] = E[Y_1]$ for all $n \geq 1$. Let T be an integer-valued random variable such that for all $n \geq 1$, the event $\{T \geq n\}$ is independent of Y_n. Define $X := \sum_{n=1}^{T} Y_n$. Prove that

$$E[X] = E[Y_1]E[T]\,.$$

Solution (Exercise 6.2.28).
We have

$$E[X] = E[\sum_{n=1}^{\infty} Y_n 1_{\{n\leq T\}}] = \sum_{n=1}^{\infty} E[Y_n 1_{\{n\leq T\}}].$$

But

$$E[Y_n 1_{\{n \le T\}}] = E[Y_n] E[1_{\{n \le T\}}] = E[Y_1] P(n \le T\}).$$

The result then follows from the telescope formula.

Exercise 6.2.29. THE ENTOMOLOGIST STRIKES AGAIN!
Recall the setup of Exercise 6.2.20. Each individual of a specific breed of insects has, independently of the others, the probability θ of being a male. An entomologist seeks to collect exactly $M > 1$ males, and therefore stops hunting as soon as she captures M males. She has to capture an insect in order to determine its gender. What is the expectation of X, the number of insects she must catch to collect *exactly* M males? (In Exercise 6.2.20, we computed the distribution of X, from which we can of course compute the mean. However you can give the solution more quickly, and this is what is required in the present exercise)

Solution (Exercise 6.2.29).
Apply Wald's expectation formula of Exercise 6.2.28 to obtain $M = E[X] \times \theta$. For the result of Exercise6.2.28 to apply, one must be sure that for each $n \ge 0$, the random variable Z_n ($= 1$ if the n-th insect captured is a male, $= 0$ otherwise) and the event $\{T \ge n\}$ are independent. Equivalently (see Exercise 6.1.19) we must show that Z_n and $\{T < n\}$ are independent. This is true since $\{T < n\}$ depends only on $Z_1, \dots, Z_{n-1}$.

Exercise 6.2.30. A RECURRENCE EQUATION.
Recall the notation $a^+ = \max(a, 0)$. Consider the recurrence equation,

$$X_{n+1} = (X_n - 1)^+ + Z_{n+1},\ (n \ge 0),$$

where X_0 is a random variable taking its values in $\mathbb{N}$, and $\{Z_n\}_{n\ge1}$ is a sequence of independent random variables taking their values in $\mathbb{N}$, and independent of X_0. Express the generating function ψ_{n+1} of X_{n+1} in terms of the generating function φ of Z_1.

Solution (Exercise 6.2.30).

$$X_{n+1} = X_n - 1_{X_n>0} + Z_{n+1}$$

Observe that Z_{n+1} is independent of X_n (the latter depends only on $X_0, Z_1, \dots, Z_n$). Therefore

$$\begin{aligned} E\left[z^{X_{n+1}}\right] &= E\left[z^{X_n - 1_{X_n>0}}\right] E\left[z^{Z_{n+1}}\right] \\ &= E\left[z^{X_n - 1_{X_n>0}}\right] \varphi(z). \end{aligned}$$

Now

$$\begin{aligned} z^{X_n - 1_{X_n>0}} &= \left(z^{X_n - 1_{X_n>0}}\right) 1_{X_n>0} + \left(z^{X_n - 1_{X_n>0}}\right) 1_{X_n=0} \\ &= z^{X_n - 1} 1_{X_n>0} + z^{X_n} 1_{X_n=0} \\ &= z^{X_n - 1} - z^{X_n - 1} 1_{X_n=0} + z^{X_n} 1_{X_n=0} \\ &= z^{X_n - 1} + -z^{-1} 1_{X_n=0} + 1_{X_n=0} \\ &= z^{X_n} z^{-1} + \left(1 - z^{-1}\right) 1_{X_n=0}. \end{aligned}$$