Indian Institute of Technology Bombay
Dept of Electrical Engineering

Handout 14 EE 603 Digital Signal Processing and Applications
Tutorial 5 October 29, 2016

Question 1) *Half-band filters:* Let us consider an L-interpolator where the output at periodic sampling instants which are apart by L gives the input itself. Show that when $L = 2$, the interpolating filter's frequency response exhibits a symmetry around the half-band to yield $H(z) + H(-z) = 2$ (also called half-band filter).
Solution: Notice that an L-interpolator is nothing but an upsampler followed by an appropriate low pass filter. In particular, this is a multi-rate system with the input and output having different sampling rates. Thus, we have to be a bit careful in using the term frequency response. Like interpolators we learned in the digital to analog conversion, one can visualize each sample being replaced by L samples, however, due to edge effects, more values may become necessary at the endpoints.

For $L = 2$, the filter $h[n]$ can be seen as operating on $x[n]$ up sampled by 2. For the input to be same as the output at $n = 0$, it is compulsory that $h[2n] = 0, n \in \mathbb{Z}, n \neq 0$. Let us prove the above claim by contradiction. Assume $h[2m] = \epsilon \neq 0$. Imagine the output for an input $x[n] = \delta[n]$. Clearly $y[2m] = \epsilon \neq x[2m] = 0$. Thus the filter $h[n]$ has a unit value at $n = 0$, and other possible non-zero values only at odd integers. Putting these together

$$H(z) = 1 + \sum_{n \text{ odd}} h[n] z^{-n},$$

which implies

$$H(z) + H(-z) = 2. \tag{1}$$

For real filters, we know that $H(z)$ exhibits a symmetry in the unit circle, i.e. $H(z = e^{j\omega}) = H^*(z = e^{-j\omega})$. This, together with (1), gives

$$H(z = e^{j(\frac{\pi}{2}+\omega)}) + H(z = e^{j(\frac{\pi}{2}-\omega)}) = 2,$$

called half-band symmetry. For $\omega \in [0, \pi]$, the values on either side of $\frac{\pi}{2}$ determines the complete response, this is called half-band symmetry.

Question 2) Let us consider perfect reconstruction filter-banks for sub-band coding. Take $H_i(z)\tilde{H}_i(z) := P_i(z)$, $i = 0, 1$.

(a) Show that under aliasing-free conditions we described in class,

$$P_1(z) = -P_0(-z).$$

Thus the distortion-free constraint becomes $P_0(z) - P_0(-z) = 2z^{-l}$ for some $l \geq 0$.
Solution: We have already seen the alias free conditions

$$\tilde{H}_0(z) = H_1(-z) \text{ and } H_0(-z) = -\tilde{H}_1(z).$$

This gives

$$P_1(z) = H_1(z)\tilde{H}_1(z) = \tilde{H}_0(-z)\,(-H_0(z)) = -P_0(-z).$$

The distortion free constraint now reads

$$2z^{-l} = P_0(z) + P_1(z) = P_0(z) - P_0(-z).$$

(b) Show that l can only be an odd integer in above.
Solution: Assume l is not odd, then $P_0(z) - P_0(-z)$ will have an even degree term remaining, which is impossible.

(c) Let us take $P(z) = z^l P_0(z)$, an we want $P_0(z)$ to be of linear phase. Show that $P(z)$ is zero phase half-band LPF, (i.e. there is symmetry around half-band). Further $P(z)$ has the form

$$P(z) = 1 + \sum_{i \geq 1} a_i (z^i + z^{-i}).$$

Solution:

$$\begin{aligned} P(z) + P(-z) &= z^l P_0(z) + (-z)^l P_0(-z) \\ &= z^l P_0(z) - z^l P_0(-z) \\ &= z^l (P_0(z) - P_0(-z)) \\ &= z^l 2z^{-l} \\ &= 2. \end{aligned}$$

Since we are interested in a $P(z)$ with real coefficients, the half-band symmetry now follows from Question 1. Since $P(z)$ is linear phase, the symmetry of the coefficients will explain the structure of $P(z)$ above. Notice that the coefficient $a_0 = 1$, ruling out any antisymmetry. Furthermore, clearly we can take all $a_{2n}, n \in \mathbb{Z}, n \neq 0$ to be zero in the above equation for $P(z)$.

(d) Many sub-band coding techniques can be derived by factorizing $P(z)$ into a set of analysis and synthesis filters, i.e. $P(z) = z^l H_0(z) \tilde{H}_0(z)$. A suitable form of $P(z)$ with wide applicability is

$$P(z) = (1 + z^{-1})^m (1 + z)^m R(z),$$

where $R(z)$ is a polynomial of the form $r_0 + \sum_{i=1}^{m-1} r_i (z^{-i} + z^i)$. For $m = 1$, can you identify the analysis and synthesis filter.
Solution: For $m = 1$, we have

$$\begin{aligned} P(z) &= r_0 (1 + z^{-1})(1 + z) \\ &= r_0 z^{-1} (1 + z^{-1})(1 + z^{-1}) \\ &= z^{-1} \left(\frac{1 + z^{-1}}{\sqrt{r_0}} \right)^2 . \end{aligned}$$

To find r_0, notice that

$$P(z) + P(-z) = r_0 (2 + z + z^{-1} + 2 - z - z^{-1}) = 2,$$

yielding $r_0 = 0.5$. Thus, the only possible choices for the analysis and synthesis filter are

$$H_0(z) = \frac{1 + z^{-1}}{\sqrt{2}} \text{ and } \tilde{H}_0(z) = \frac{1 + z^{-1}}{\sqrt{2}}.$$

(e) Consider $R(z) = az + b + az^{-1}$. Show that

$$P(z) = \frac{1}{16} z^3 \left(1 + 2z^{-1} + z^{-2}\right)^2 (-1 + 4z^{-1} - z^{-2}).$$

Solution: Comparing with formula for $R(z)$, we have taken $m = 2$. Thus

$$P(z) = (1 + z^{-1})^2(1 + z)^2(az + b + az^{-1}).$$

We have to find the unknowns a and b. Expanding

$$\begin{aligned} P(z) &= (1 + z^{-1})^2(1 + z)^2(az + b + az^{-1}) \\ &= [(1 - z^{-1})(1 - z)]^2(b + a(z + z^{-1})) \\ &= (2 + z + z^{-1})^2(b + a(z + z^{-1})) \\ &= 4b + 4b(z + z^{-1}) + b(z + z^{-1})^2 + 4a(z + z^{-1}) + 4a(z + z^{-1})^2 + a(z + z^{-1})^3 \end{aligned}$$

Since $P(z) + P(-z) = 2$,

$$8b + (2b + 8a)(z + z^{-1})^2 = 2.$$

Clearly, $2b + 8a = 0$ and $8b = 2$ for this formula to work, yielding

$$a = -\frac{1}{16} \text{ and } b = \frac{1}{4}.$$

Equivalently, we have

$$P(z) = \frac{1}{16}z^3\left(1 + 2z^{-1} + z^{-2}\right)^2(-1 + 4z^{-1} - z^{-2}).$$

(f) With $H_0(z) = \frac{1}{2}(1 + 2z^{-1} + z^{-2})$ we get the LeGall 3/5 tap filter. What are the other three filters for constructing a two sub-band coding scheme. Explain the name 3/5-tap filter.
Solution: By identification $P(z) = z^3H_0(z)\tilde{H}_0(z)$. Thus

$$\begin{aligned} \tilde{H}_0(z) &= \frac{1}{8}(1 + 2z^{-1} + z^{-2})(-1 + 4z^{-1} - z^{-2}) \\ &= \frac{1}{8}(-1 + 2z^{-1} + 6z^{-2} + 2z^{-3} - z^{-4}). \end{aligned}$$

Also

$$\begin{aligned} H_1(z) &= \tilde{H}_0(-z) = \frac{1}{8}(-1 - 2z^{-1} + 6z^{-2} - 2z^{-3} - z^{-4}) \\ \tilde{H}_1(z) &= -H_0(-z) = \frac{1}{2}(-1 + 2z^{-1} - z^{-2}) \end{aligned}$$

(g) With $H_0(z) = \frac{1}{8}\left(1 + 3z^{-1} + 3z^{-2} + z^{-3}\right)$ we get another decomposition scheme. What are the remaining filters to generate a two sub-band coding.
Solution:

$$\begin{aligned} H_0(z) &= \frac{1}{8}(1 + 3z^{-1} + 3z^{-2} + z^{-3}) \\ \tilde{H}_0(z) &= \frac{1}{2}(-1 + 3z^{-1} + 3z^{-2} - z^{-3}) \\ H_1(z) &= \frac{1}{2}(-1 - 3z^{-1} + 3z^{-2} + z^{-3}) \\ \tilde{H}_1(z) &= \frac{1}{8}(-1 + 3z^{-1} - 3z^{-2} + z^{-3}) \end{aligned}$$

(h) In the last two questions, notice that the obtained filters $H_i(z), \tilde{H}_i(z)$ are of linear phase. A Daubechies-4/4 filter is obtained by splitting the second polynomial of $P(z)$ such that one root is there in $H_0(z)$ and the other in $\tilde{H}_0(z)$. Given that it is 4/4 tap filter, give the analysis and synthesis filters.

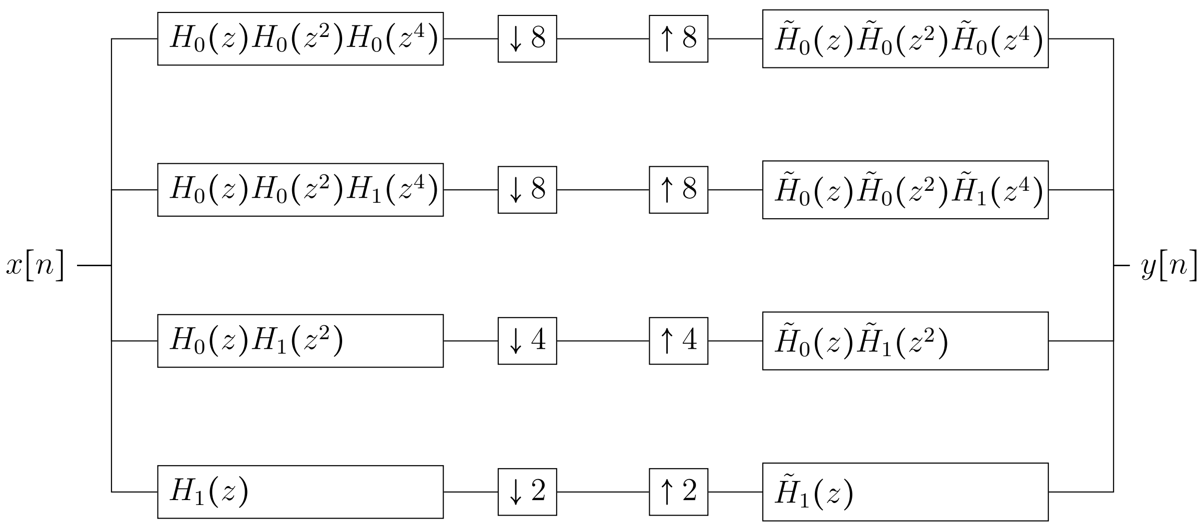
Solution:

$$\begin{aligned}
H_0(z) &= \frac{1}{4(\sqrt{3}-1)}(1+z^{-1})^2(1-(2-\sqrt{3})z^{-1}) \\
&= 0.3415 + 0.5915z^{-1} + 0.1585z^{-2} + 0.0915z^{-3} \\
\tilde{H}_0(z) &= -0.0915 + 0.1585z^{-1} + 0.5915z^{-2} + 0.3415z^{-3} \\
H_1(z) &= -0.0915 - 0.1585z^{-1} + 0.5915z^{-2} - 0.3415z^{-3} \\
\tilde{H}_1(z) &= -0.3415 + 0.5915z^{-1} - 0.1585z^{-2} - 0.0915z^{-3}
\end{aligned}$$

Question 2) Give the complete 4 band analysis and synthesis diagram for Haar MRA. Only 4 down-samplers and 4-up samplers are allowed.

Solution:

The MRA can be obtained by appropriate filtering and downsampling. The general picture of a 4− subband octave decomposition is



Remember that the mother wavelet is nothing but the inverse of $H_1(z)$. Sometimes people call $-H_1(z)$ as the mother wavelet, or some other times a scaled version of $H_1(z)$ as the mother wavelet. We will stick to the first convention. We know the formula for the Haar filter from Question (1.d). Substitute the appropriate filter above to obtain the decomposition.

Question 3) Give the complete 4 band analysis and synthesis diagram for Daub-4/4 MRA. Only 4 down-samplers and 4-up samplers are allowed.

Solution: The answer is obtained similar to the last question, except that the MRA filters are to be taken from Question (1.h).

Question 4) Given the sequence $x[n] = [1, 2, 2, 3, 3, 4, 3, 3, 3, 5, 7, 7, 7, 7, 3, -1]$, find the 4− band Haar decomposition.

Solution: At the lowest sub-band, we get

$$x_h[n] = [-0.70711, -0.70711, -0.70711, 0.00000, -1.41421, 0.00000, 0.00000, 2.82843]$$

It is okay to get the negative of this sequence as the output. On the other hand, if you are getting a scaled version, ensure that an appropriate scale is applied at the synthesis side. In general, though I have occasionally neglected the scalars, you may get the wrong answer in MRA if you ignore the scales. So please ensure the correct scalars.
Notice that the first output is $\frac{1}{\sqrt{2}}(x[0] - x[1]) = -0.707$. If you shift the sequence and take the first output as $\frac{1}{\sqrt{2}}(x[-1] - x[0]) = 0 - 0.707 = -0.707$, then the subsequent outputs will be different from the above sequence. While this is okay, try to match the numbers such that an N length input sequence has N output values in total, and the last(highest) two subbands have only a single output.
For the other subbands we get

$$\begin{aligned} x_{lh}[n] &= [-1.0, 0.5, -3.0, 6.0] \\ x_{llh}[n] &= [-1.76777, 2.12132] \\ x_{lllh}[n] &= [-4.25] \\ x_{llll}[n] &= [14.75] \end{aligned}$$