Indian Institute of Technology Bombay
Department of Electrical Engineering

Handout 6 EE 708 Information Theory and Coding
Homework 3 Mar 27, 2018

Question 1) Consider a continuous real-valued random variable X. If it is known that $\mathbb{E}|X| \le \mu$, what is the maximum possible differential entropy for X and find the maximizing distribution.

Question 2) Let us show that the capacity of a point-to-point channel with IID state is given by $\max_{p(x|s)} I(X;Y|S)$, when the receiver and transmitter know the IID state causally before the transmission instant.

(a) Assume a reliable communication scheme, and justify $(a)-(e)$.

$$\begin{aligned} nR &\overset{(a)}{=} H(W) \\ &\overset{(b)}{=} H(W|S^n) \\ &\overset{(c)}{\approx} I(W;Y^n|S^n) \\ &\overset{(d)}{\le} \sum_{i=1}^{n} H(Y_i|S_i) - H(Y_i|S^n, W, Y^{i-1}) \\ &\overset{(e)}{=} \sum_{i=1}^{n} I(X_i : Y_i|S_i). \end{aligned}$$

(b) Consider a scalar narrowband AWGN model with IID fading, given by

$$Y = GX + Z,$$

where the gain G takes IID complex values according to the distribution $f_G(\cdot)$, and $Z \sim \mathcal{N}_c(0, \sigma^2)$ is independent of the transmissions X and state G. Find the capacity of this model under an average power constraint $\mathbb{E}|X|^2 \le P$.

(c) How does the above capacity compare with that of the model $Y = |G|X + Z$.

(d) What is the capacity in part (b), if the output Y is causally fed back to the transmitter through a noiseless link.

Question 3) Consider a complex AWGN channel given by $Y = X + Z$ (standard notation and assumptions). We have two codewords u and v, each of length n. What is the error probability when each codeword is chosen with probability $\frac{1}{2}$ for transmission.

Question 4) Consider a multiple input multiple output channel

$$y = HX + Z,$$

where $H = U\Delta$, where U is a $N_r \times N_r$ unitary matrix, and Δ is a $N_r \times N_t$ diagonal matrix. Let $Z \sim \mathcal{N}_c(0, I_{N_r})$. For a fixed matrix H, find the capacity of this channel under an average power constraint $\mathbb{E}||X||^2 \le P$.

Question 5) Recall that the discrete memoryless channel (DMC) is defined by a collection of probability laws $\{p(y|x)\}$, one for each value of the input x, often written as the triple $(\mathcal{X}, p(y|x), \mathcal{Y})$. Consider X and Y chosen from discrete and finite alphabets. Instead of drawing the picture of a DMC, we can write the channel laws in a matrix form. In this matrix W, each row corresponds to an input value, while columns denote output values. In particular, the entry at (i, j), denoted as W_{ij}, will contain the value

$$W_{ij} = p(Y = j|X = i)$$

Thus W completely specifies the channel law. The matrix W is also known as a stochastic matrix.

(a) Consider a 3×3 stochastic matrix W_1. Let the last row of W_1 be a positive linear combination of the first two rows. Consider a distribution q on the inputs (x_1, x_2, x_3). Show that there exists a distribution p with $p(x_3) = 0$ and

$$H_p(Y) = H_q(Y),$$

where $H_r(Y)$ is for the distribution of the output Y, under the input law r and channel W_1.

(b) Continuing from part (a), show that the capacity of the channel W_1 is same as that of W, where W contains only the first two rows of W_1.

(c) Generalize this result to show that for any channel law W_1 (with finite number of input values), the capacity is same as that of W, where W is the maximal submatrix with linearly independent rows.

Question 6) Consider a DMC defined by the collection of transition probabilities given by

$$\{p(y|x), \forall x \in \mathcal{X}, \forall y \in \mathcal{Y}\}.$$

Let the input alphabet $\mathcal{X}$ and the output alphabet $\mathcal{Y}$ have the same cardinality, say M. The transition probabilities $p(y|x)$ can be arranged to a $M \times M$ matrix, with x as the rows and y as the columns. Call this matrix W. Suppose the rows of W are permutations of each other. Similarly the columns of W are also permutations of each other. Given that the values $\rho_1, \rho_2, \cdots, \rho_{M-1}$ are present in the second row of W, compute the capacity of this channel in terms of $\rho_i, 1 \le i \le M - 1$.

Question 7) For a DMC described by the triplet $\{\mathcal{X}, p(y|x), \mathcal{Y}\}$, let C be the capacity. Suppose this channel is serially cascaded with an erasure channel, such that the output at each channel use is erased in an IID manner with probability ρ. Find the capacity of the resulting cascaded channel.

Question 8) Find the capacity of the channel