Indian Institute of Technology Bombay
Department of Electrical Engineering

Handout 3 EE 764 Wireless and Mobile Communication
Lecture Notes 1 Jan 30, 2014

Gaussian Random Variables

A real Gaussian random variable is defined by the probability density function

$$f_X(x) = \frac{1}{2\pi\sigma^2}\exp(-\frac{(x-\mu)^2}{2\sigma^2}) \tag{1}$$

where μ is the mean and σ^2 the variance of X.

Notice that a real Gaussian RV is symmetric around μ. Complex circularly symmetric Gaussian RV, denoted as $C.C.S.G$, is an extension of the Gaussian RV to the complex domain. Figure below explains its symmetry around a mean value of zero.

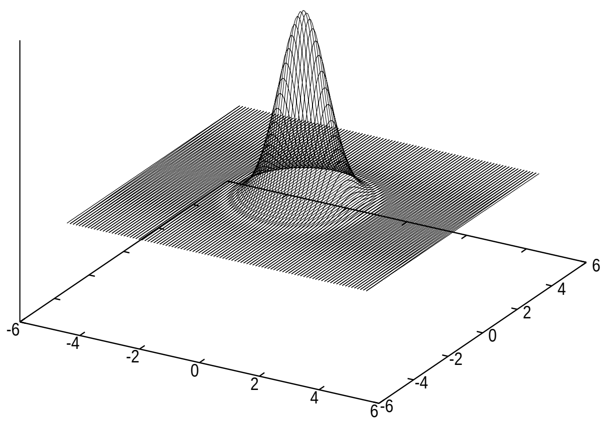


Figure 1: pdf of a zero mean $C.C.S.G$ in complex plane

A $c.c.s.g$ random variable can also be visualized as a complex random variable, with the real and imaginary parts independently distributed as identical Gaussian RVs. So every $c.c.s.g$ Z with zero mean is of the form $U + iV$, where $U \sim \mathcal{N}(0,\sigma^2)$ and $V \sim \mathcal{N}(0,\sigma^2)$, for some $\sigma^2 > 0$. In particular, with $z = u + iv$,

$$\begin{aligned} f_Z(z) &= f_U(u)f_V(v) && (2)\\ &= \frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{u^2}{2\sigma^2})\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{v^2}{2\sigma^2}) && (3)\\ &= \frac{1}{2\pi\sigma^2}\exp(-\frac{u^2+v^2}{2\sigma^2}) && (4)\end{aligned}$$

Notice that $u^2 + v^2 = |z|^2$ and $E|\mathbf{Z}|^2 = E\mathbf{U}^2 + E\mathbf{V}^2 = 2\sigma^2$. Let us call $E|\mathbf{Z}|^2$ as σ_c^2. So the pdf of a zero mean $c.c.s.g$ $\mathbf{Z}$ is

$$f_Z(z) = \frac{1}{\pi\sigma_c^2}\exp(-\frac{|z|^2}{\sigma_c^2}) \tag{5}$$

It is clear that the pdf depends on z only through its magnitude. Thus the value of pdf at $z = r + j0$ will be same as that at all z such that $|z| = r$.

Rayleigh Distribution of |Z|

Consider the random variable $Y = |Z|$. Let us find the distribution of Y. In the last section, we saw that $f_Z(z)$ is the same for all points with $|z| = r$. Furthermore, we know from geometry that all points with $|z| = r$ will lie on a circle with circumference $2\pi r$ and centered at the origin. Thus

$$f_{|Z|}(r) = 2\pi r f_Z(r + j0) \tag{6}$$

Notice that the argument r in the left side is a real number, while the argument to f_Z is a complex number.

Exponential Distribution of |Z|²

Consider the random variable $W = |X|^2$. To find the pdf of any random variable, a robust way is to start from the cumulative distribution function (cdf). In particular, for a positive random variable W with a pdf,

$$P(W \le w) = \int_0^w f_W(y)dy. \tag{7}$$

The quantity $f_W(\cdot)$ on the right hand side is the pdf that we are looking for. Let us see whether we can obtain an expression similar to this.

$$\begin{aligned}
P(W \le w) &= P(|X|^2 \le w) && (8)\\
&= P(|X| \le \sqrt{w})\ ;\ (\text{true since}|X| \ge 0) && (9)\\
&= \int_0^{\sqrt{w}} f_{|X|}(u)du && (10)\\
&= \int_0^w f_{|X|}(\sqrt{v})\frac{dv}{2\sqrt{v}}\ ;\ (\text{put}v = u^2) && (11)\\
&= \int_0^w 2\pi\sqrt{v} f_X(\sqrt{v} + j0)\frac{dv}{2\sqrt{v}}\ ;\ (\text{from equation (6)}) && (12)\\
&= \int_0^w \pi f_X(\sqrt{v})dv && (13)
\end{aligned}$$

By equating equation (7) and (13), we find that

$$f_W(w) = \pi f_X(\sqrt{w}) = \frac{1}{\sigma_c^2}\exp(-\frac{w}{\sigma_c^2}) \tag{14}$$

Thus W is distributed as an exponential random variable with mean parameter σ_c^2.

Comment: The step from equation (8) to (9) depended crucially on the fact that $|X|$ is a positive valued RV. If X was a real random variable with non-zero probability of taking negative as well as positive values, then $P(X^2 \le w) = P(-\sqrt{w} \le X \le \sqrt{w})$. Students many a times forget this fact. Note that this derivation is a bit different from what was done in *class. Here we obtained the density of W in terms of the pdf of X, whereas we derived it in terms of pdf of $|X|$ in the class. Both should give the same answer.*

Exercise 1.1: If X and Y are independent RVs, with pdfs $f_X(\cdot)$ and $f_Y(\cdot)$, what is the distribution of $Z = X + Y$.

Exercise 1.2: If W_0 and W_1 are two exponential RVs with parameters λ_0 and λ_1 respectively, show that

$$P(W_0 \leq W_1) = \frac{\lambda_0}{\lambda_0 + \lambda_1} \tag{15}$$

Gaussian Random Vector $X \in \mathbb{R}^n$

See last exercise in work-sheet 3 distributed in class.

C.C.S.G **Random Vector** $X \in \mathbb{C}^n$

See last exercise in work-sheet 3 distributed in class.

Some Fundamentals of Probability

This is not an exhaustive discussion. We expect the students to know the fundamentals. There are many good books, for example S. Ross, "A first course in probability".

We denote events by letters A, B, C, random variables by U, V, W, X, Y, Z. The probability space is not explicitly stated each time, but clear from the context. Further, statements are made for either continuous or discrete variables, but clearly applicable to both by appropriate modifications, for example - changing integrals to summations.

1. **Baye's Rule**

$$P(A, B) = P(A)P(B|A) \tag{16}$$
$$P(A, B|C) = P(A|C)P(B|A, C) \tag{17}$$

2. **Independence**
   For independent events A and B,

$$P(B|A) = P(B) \tag{18}$$
$$P(A, B) = P(A)P(B) \tag{19}$$

   A sequence of RV $\{X_i\}_{i=1}^n$ are independent if

$$P(X_1, X_2, \cdots, X_n) = \prod_{i=1}^{n} P(X_i) \tag{20}$$

   A sequence of RVs are *i.i.d* if, (20) is true and each X_i has the same probability distribution.

3. **Conditional Independence**
   The RVs X and Y are conditionally independent given Z iff

$$P(X|Y, Z) = P(X|Z) \tag{21}$$

   Notice that this doesn't imply that X and Y are independent.
   Exercise 1.3: Find an example of X, Y and Z where the above property is true.

4. **Expectation**

$$E[g(\mathbf{X})] = \int g(x) f_X(x) dx \tag{22}$$

   For independent RVs X and Y,

$$E[f(\mathbf{X})g(\mathbf{Y})] = E[f(\mathbf{X})]E[g(\mathbf{Y})] \tag{23}$$

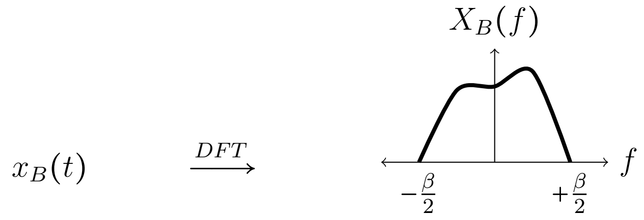
   Expectation is a **linear** operation.

1 Digital Communication Model for Wireless

We will assume that a transmission band of $[f_c - \frac{\beta}{2}, f_c + \frac{\beta}{2}]$ is available to communicate between a transmitter-receiver pair over a wireless channel. From our first level communication courses, we can generate an information bearing baseband signal in $[-\frac{\beta}{2}, +\frac{\beta}{2}]$ and then modulate it to the appropriate carrier frequency using amplitude modulation. Imagine that the baseband input waveform $x_B(t)$ is given to us. Let $X_B(f)$ denote the frequency domain representation, also known as the Fourier Transform of $x_B(t)$. Remember that our assumption of bandlimited will imply that $x_B(t)$ is unlimited in time. For real input signals, $X_B(f)$ is complex symmetric around the origin. However, there is no reason that our baseband signal should be real, and we will consider complex signals of the form,

$$x_B(t) = x_{BR}(t) + jx_{BI}(t).$$

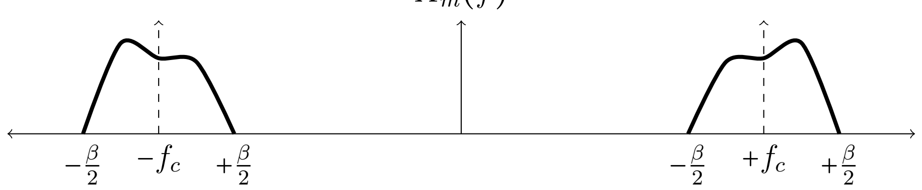
For illustration, we will consider a $x_B(t)$ waveform with Fourier Transform $X_B(f)$ as depicted below.



We will repeatedly use the above pictorial representation. We should keep couple of things in mind while using these pictures. First, the inverse Fourier transform (IFT) of such an $X_B(f)$ may correspond to a non-causal signal. Thus our descriptions are not for the absolute time, and we do not really worry about causality. This is the reason why you will not find additional path delays appearing in this subsection. However, we will incorporate the delays while considering the physical propagation models. Secondly, we do not emphasize the phase information in these pictures. This is partly due to the fact that we expect the phase response to be linear in the passband frequencies, i.e. all frequencies are delayed by an equal amount. So the pictures do not depict the phase component at all, for convenience.

We can represent the modulated output radiated from an ideal transmit antenna as

$$x_m(t) = \text{Real}\left(x_B(t)\exp(j2\pi f_c t)\right) = x_{BR}(t)\cos 2\pi f_c t - x_{BI}(t)\sin 2\pi f_c t.$$



A natural question here is about the effect of the transmit antenna, and also contributions from the RF circuitry, amplifiers and so on. It is reasonable to assume that the antenna has good response over the transmission bandwidth. In reality, the antenna response depends on various factors like frequency, the angle between the transmitter and receiver, distance etc. Similar effects are contributed by the receive antenna too. In between the antennas, the wireless medium pitches in with its own effects on physical propagation, like reflection, refraction, interference etc.

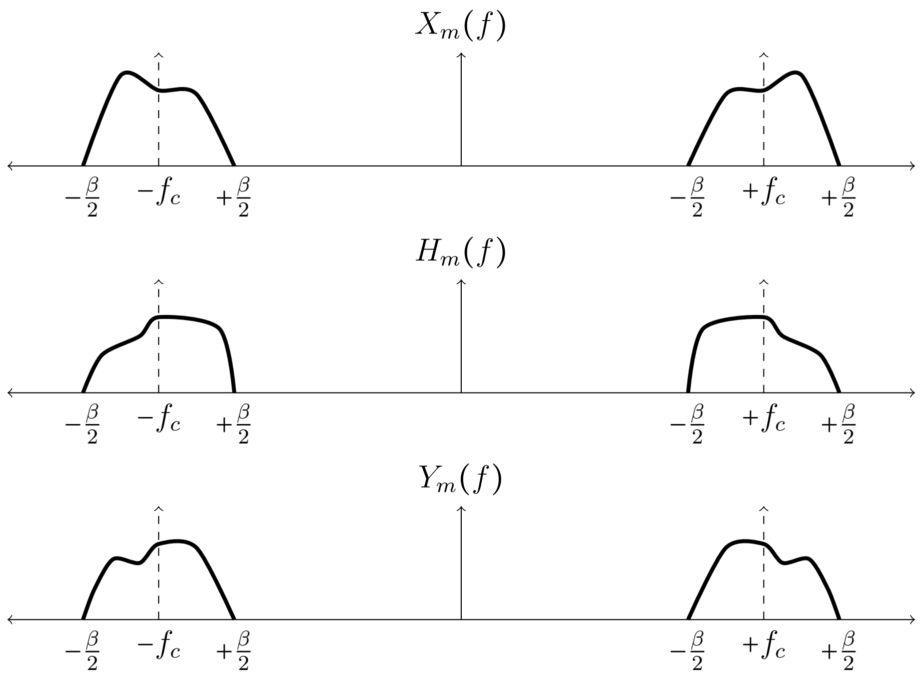
Our strategy is to visualize the contribution of these three entities, i.e. the transmit antenna-wireless channel-receive antenna as a combined system, which is the channel. We will in particular assume a linear system for the channel, this not only enables forming a cohesive theory, but also works very well in practice, that most modern day communication systems are designed for linear models. Keep in mind that linear does not mean LTI, as the systems we consider are typically time-varying. Thus, the actual time-scales become important. We will argue that the medium response stays LTI for reasonable time-intervals, and we can exploit this to approximate the medium as locally LTI. This, in turn, arms us with the popular and powerful tools of Fourier Theory. Thus, our objective now is to obtain a meaningful model for the channel, in the *vicinity* of any chosen time t_0. Since an LTI system is completely specified by its transfer-function, we need to specify the response $H_m(f)$ to each frequency $f \in [f_c - \frac{\beta}{2}, f_c + \frac{\beta}{2}]$.

1.1 Frequency Response

The frequency response of the medium can be measured by sending adequately spaced complex sinusoids and measuring the output attenuation, in terms of magnitude decrements and phase-rotations. From this we can find the response which is continuous over the frequency range of interest. Note that certain frequencies may propagate better for a given transmit-receive pair, whereas some other frequencies may combine destructively at the receiver. Since the antenna is excited by the active power of the signals impinging on it, the received passband signal without including the effect of additive noise can be written as

$$Y_m(f) = X_m(f)H_m(f).$$

The effect of noise was excluded in the above received waveform to clarify some aspects of propagation. Thus, we may continue to call $Y_m(f)$ or its IFT as the received signal. The effect of noise is additive, and it will be added eventually. All the responses above are conjugate-symmetric w.r.t to the origin, guaranteeing that the signals under consideration are real. This can be drawn pictorially as follows.



The diagram above illustrates that the time domain passband signal $y_m(t)$ at the receiver is

$$y_m(t) = x_m(t) \star h_m(t),$$

where $h_m(t)$ is the IFT of $H_m(f)$. We can transfer all these passband operations to equivalent baseband operations to obtain

$$Y_B(f) = X_B(f) H_m(f + f_c) \mathbb{1}_{\{-\frac{\beta}{2} \le f \le \frac{\beta}{2}\}},$$

where we applied ideal lowpass filtering after demodulation. Notice that such a $Y_B(f)$ may no longer be conjugate symmetric.

Exercise 1. *Argue that $y_B(t)$ may correspond to a complex waveform even when the input is real.*

Our equivalent model now becomes,

$$y_B(t) = x_B(t) \star h_B(t),$$

where $h_B(t)$ is the IFT of $H_B(f) = H_m(f + f_c) \mathbb{1}_{\{-\frac{\beta}{2} \le f \le \frac{\beta}{2}\}}$.

Clearly, $h_B(t)$ appears to be a time continuous impulse response function. This will imply that the system possibly introduces a continuum of delayed replicas of the transmitted signal, also known as echos. It turns out that a much more simpler visualization is possible. Notice that

$$Y_B(f) = X_B(f) \tilde{H}_B(f) \mathbb{1}_{\{-\frac{\beta}{2} \le f \le \frac{\beta}{2}\}},$$

for any $\tilde{H}_B(f)$ such that

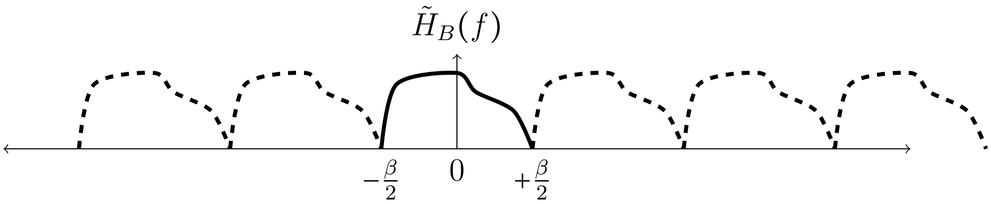
$$\tilde{H}_B(f) = H_B(f)\,,\ -\frac{\beta}{2} \le f \le \frac{\beta}{2}. \tag{24}$$

Thus, we can choose any $\tilde{H}_B(f)$ with the above property, and our output $Y_B(f)$ stays unchanged. ON taking IFT, this implies

$$y_B(t) = x_B(t) \star \tilde{h}_B(t),$$

where $\tilde{h}_B(t)$ is the IFT of $\tilde{H}_B(f)$. Let us now choose a $\tilde{H}_B(f)$ which satisfies (24), but has a convenient inverse. In particular, we will periodically repeat $H_B(f)$ over the entire frequency axis, i.e.

$$\tilde{H}_B(f) = \sum_{k \in Z} H_B(f - k\beta).$$



The periodic nature of $\tilde{H}_B(f)$ implies that $\tilde{h}_B(t)$ is of discrete in nature. Furthermore, since $\tilde{H}_B(f)$ is the periodic repetition of $H_B(f)$ in frequency, the time domain response $\tilde{h}_B(t)$ is nothing but a time-sampled version of $h_B(t)$. In other words,

$$\tilde{h}_B(t) = \sum_k \frac{1}{\beta} h_B(\frac{k}{\beta}) \delta(t - \frac{k}{\beta}). \tag{25}$$

The scaling $\frac{1}{\beta}$ is due to the fact that a periodic impulse train in time will have a Fourier Transform of a scaled impulse train, with their periods in inverse relationship. Equation (25) assures that the effective channel can be visualized as a tapped delay line between the transmitter and the receiver in the baseband itself. We will term the value of the taps as fading coefficients and denote them by $h_k, k \in \mathbb{Z}$, clearly $h_k = \frac{1}{\beta} h_B(\frac{k}{\beta})$. With this,

$$y_B(t) = \sum_{k \in \mathbb{Z}} h_k \, x_B(t - \frac{k}{\beta}). \tag{26}$$

While this model is remarkably simple, we did not loose any generality in obtaining this. For any communication system with an ideal front-end filter, our discussion holds true, and the channel can be visualized for all purposes as a tapped delay-line. This simplicity should be compared with the practical case where a continuum of paths, at almost all permitted delays, exist between the transmitter and receiver over a wireless channel.

1.2 Discrete-time Communication System

In the last section, we constructed an equivalent discrete-time channel, which gives us both conceptual and analytic simplicity. Can we convert the communication model itself to an equivalent discrete-time system. The advantages are many fold, as several DSP books will advertise in the first chapter. How to do this is explained in the second chapters there, the premise of Sampling Theorems.

Exercise 2. *What is the Nyquist sampling rate to preserve all the information of $y_B(t)$ in (26)?*

Let us sample $y_B(t)$ at the rate of β samples per second.

$$y_B(\frac{m}{\beta}) = \sum_k h_k x_B(\frac{m}{\beta} - \frac{k}{\beta}).$$

Denoting $u(\frac{m}{\beta}) = u_m$, and dropping the subscripts,

$$y_m = \sum_k h_k x_{m-k},$$

or in our usual DSP notation,

$$y[m] = h[m] * x[m].$$

1.3 Additive Noise Model

So far we have kept the noise out of consideration to obtain a simple system representation. Now let us put the noise back. While this can be done in a rigorous fashion (see for example Gallager's book), we will simply assume that the additive noise is a discrete time stochastic process with iid zero-mean Gaussian samples of variace σ^2. Thus,

$$y[m] = h[m] * x[m] + z[m],$$

or, in short, $y = h * x + z$, where $z \sim \mathcal{N}_c(0, \sigma^2)$.