On the Capacity of Multiplicative Multiple Access Channels with AWGN

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***Abstract*—We consider a multiplicative multiple access channel in the presence of additive white Gaussian noise. Under individual average power constraints at each of the transmitters, we characterize the capacity region of this channel. The structure of the region reveals some fundamental characteristics related to time-sharing, power constraints and the auxiliary random variables present in the converse theorems. As an example, it is shown that to achieve the capacity region of a two user multiplicative MAC for certain power levels, time-sharing of 3 strategies/rate-pairs is required, as opposed to the sufficiency of time-sharing between at most 2 rate-pairs in a discrete memoryless MAC.**

I. Introduction

In a multiple access channel (MAC), several transmitters communicate to a receiver, using a shared medium. We describe a MAC channel with 2 users, generalization to 3 or more is possible, but is not covered here. In our model, the transmissions from the two users at any instant are first multiplied with each other. Additive white Gaussian noise of variance σ^2 is then added to this product. More precisely, if X_1, respectively X_2 are the transmitted symbols (complex scalars in baseband) from the two users, the channel output is given by,

$$Y = X_1 X_2 + Z. \tag{1}$$

Here Z, the additive noise, is independent of X_i, and is assumed to be circularly symmetric Gaussian of variance σ^2, denoted as $\mathcal{N}(0, \sigma^2)$. The model is depicted in Figure 1.

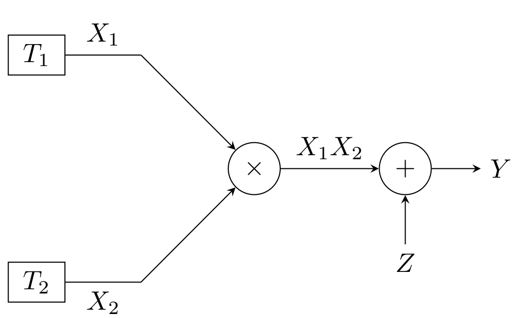

Fig. 1. Multiplicative MAC with AWGN

We will call this model a multiplicative MAC with AWGN, and write it shortly as M-MAC. The channel characteristic can be written in a manner similar to that of a discrete memoryless MAC (DM-MAC) or the standard AWGN-MAC. In particular, in the absence of any feedback,

$$p(y^n | x_1^n, x_2^n) = \prod_{i=1}^{n} p(y_i | x_{1i}, x_{2i}), \tag{2}$$

where $p(y_i | x_{1i} x_{2i})$ is given by the Gaussian distribution $\mathcal{N}(x_{1i}x_{2i}, \sigma^2)$. We assume that each user has to obey an average power constraint over its transmissions, i.e. $E|X_k|^2 \leq P_k$, $k \in \{1, 2\}$.

At this point, we digress our discussion a bit to convey the relevance of multiplicative MAC models in information theory. A binary multiplicative MAC without noise has been long used as one of the introductory models in network information theory [1]. This is also known as the logical-AND MAC. Apart from possessing a simple and clean characterization of the capacity region, it also introduces the key concept of time-sharing. The model in (1) also appears in [2], but the main motivation there is not in computing the capacity region, rather an information-lossy transformation is presented. In the absence of the additive noise term, the channel becomes a deterministic MAC [3]. The M-MAC model has some resemblance to multiplicative fading channels. In particular, if one of the transmissions does not carry any information, but contributes as iid multiplicative fading, the model becomes a fading channel. The capacity for point to point Rayleigh fading model was evaluated in [4], and the optimal distribution was shown to be discrete in nature. On a similar note, we will also see that there is a certain discrete nature to one of our input distributions.

It will turn out that the model in (1) is of significance even from just a pedagogical standpoint. Somewhat like a push-to-talk channel [5], which conveys the necessity of time-sharing in a MAC, the M-MAC defies some popular notions on computing the capacity region of continuous valued MAC channels under power constraints. As an example, we will show that for certain power levels, time-sharing between three rate-pairs is required to achieve the boundary (almost all points there) of the capacity region of our model. This should be contrasted with the DM-MAC capacity region given in Theorem 4, where at most 2 points are needed for time-sharing.

The paper is organized as follows. Section II will present some definitions and our main result. Section III presents an upperbound to the sum-rate of M-MAC. A discussion on time-sharing in the M-MAC context is presented in Section IV. We further simplify our upper bound in Section V and furnish a lower bound which equals the upper one. Some of the detailed computations here are relegated to the appendix. Section VI presents a concluding discussion.

II. Definitions and Results

For completeness, and to better explain our objectives, we will introduce some notations and definitions. Throughout the paper, $\log(\cdot)$ stands for natural logarithm. Let $X_1^n \triangleq X_{11} \cdots X_{1n}$ represent the codeword transmitted by user 1. In general, $X_k^n(W_k)$ is the codeword corresponding to message W_k at user k, $k \in \{1, 2\}$. We assume that user k communicates one among M_k messages in a block of n transmissions. Furthermore, each message is equi-probable and is chosen independently from block to block and across users. We further assume that synchronization among transmitters can be done at the block level. After observing the received symbols Y^n, the receiver tries to estimate the transmitted pair $W \triangleq (W_1, W_2)$ as $\hat{W}$. The objective is to minimize the average error probability P_e, computed as $P(\hat{W} \neq W)$.

Definition 1. *We call a communication strategy feasible if it obeys the power constraints, i.e.*

$$\sum_{i=1}^{n} |X_{ki}|^2 \leq nP_k\,,\ \ k = 1, 2. \tag{3}$$

Definition 2. *A rate-pair (R_1, R_2) is achievable if there exists a feasible strategy with $M_k = 2^{nR_k}$, $k \in \{1, 2\}$, such that the corresponding average error probability P_e can be made arbitrarily small, possibly by taking n large enough.*

Definition 3. *The capacity region $\mathcal{C}_{MMAC}$ of a M-MAC is the convex closure of all achievable rate-pairs.*

It is well known that we can replace[1] the constraint on the energy of each codeword, by an average power constraint (in expectation), where the average is taken by considering the empirical distribution induced on the transmitted symbols by a uniform choice over the messages [1]. Thus the power constraints can be modified to $E|X_k|^2 \leq P_k$, $k \in \{1, 2\}$. Let us now recall the capacity region $\mathcal{C}_{DMAC}$ of a conventional two-user discrete memoryless MAC.

$\mathcal{C}_{DMAC}$ is well known[6]. Consider the pentagonal region R_{mac} defined by

$$0 \leq R_1 \leq I(X_1; Y_1 | X_2, Q)\,;\, 0 \leq R_2 \leq I(X_2; Y | X_1, Q)$$
$$R_1 + R_2 \leq I(X_1, X_2; Y | Q)$$

for some distribution on $p(q)p(x_1|q)p(x_2|q)p(y|x_1, x_2)$.

Theorem 4. *The capacity region of a DM-MAC is the union of all R_{mac}, where the auxiliary random variable Q has cardinality at most* 2.

[1] the new constraints are less restrictive

A natural question is whether we can evaluate $\mathcal{C}_{MMAC}$ by substituting appropriate input distributions in the equations for C_{DMAC}. The answer in general turns out to be NO[2].

We now state the capacity region of a $2-$user M-MAC.

Theorem 5. *$\mathcal{C}_{MMAC}$ is a triangle with corners $(0, 0)$, $(0, R^*)$ and $(R^*, 0)$, where*

$$R^* = \log\left(1 + \frac{P_1 P_2}{\sigma^2}\right) \mathbb{I}_{\{P_1 P_2 \geq c\}} + \sqrt{\frac{P_1 P_2}{c}} \log\left(1 + \frac{c}{\sigma^2}\right) \mathbb{I}_{\{P_1 P_2 < c\}}, \tag{4}$$

in which $c = \beta\sigma^2$, $\mathbb{I}_{\{\cdot\}}$ is the indicator function, and β is the unique solution to the transcendental equation

$$\left(1 + \frac{1}{u}\right) \log_e(1 + u) = 2. \tag{5}$$

Remark 6. *In Theorem 4, β is approximately* 3.9215 .

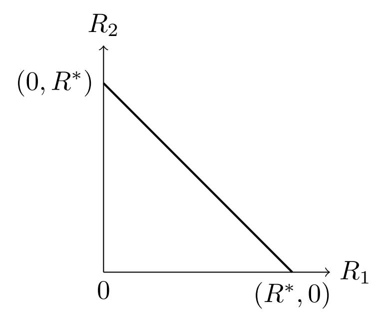

Fig. 2. Capacity Region of M-MAC

The theorem, though simple in its appearance, has some peculiarities from a capacity region point of view. It is instructive to compare the differences between the last two theorems. As we mentioned earlier, the evaluation of the region given in Theorem 4, albeit for possible continuous distributions on (X_1, X_2, Y) satisfying the input power-constraints, may not yield the capacity region presented in Theorem 5 (see Lemma 9 – 10). This shows a pit-fall in blindly applying the discrete memoryless channel results to continuous channels with input power-constraints. More specifically, we will show that the cardinality bound on Q has to be relaxed to 3 for the M-MAC. Let us now proceed to the proof of Theorem 5.

III. Upper-bound on sum-rate

In this section, we will present an upper-bound to the sum-rate of our multiplicative MAC model. We first present a structural result on $\mathcal{C}_{MMAC}$ in Lemma 7. An outerbound to the capacity region is then proposed. Our outerbound is a triangle and its edge-points later turn out to be achievable by simple communication schemes, see Figure 2.

[2] the author does not know of any prior instances where this difference appears.

From standard converse arguments [1], we can obtain the sum-rate bound for a M-MAC as

$$R_1 + R_2 \leq \frac{1}{n}\sum_{i=1}^{n} I(X_{1i}X_{2i}; Y_i). \tag{6}$$

Considering this bound alone will give an outerbound to the capacity region. We will focus on maximizing the right hand side of the above equation. Before that, let us validate the general appearance of $\mathcal{C}_{MMAC}$ presented in Theorem 5.

Lemma 7. *$\mathcal{C}_{MMAC}$ is a triangle.*

Proof: We will show that for any achievable rate-pair (R_1, R_2), the pairs $(R_1 + R_2, 0)$ and $(0, R_1 + R_2)$ are also achievable. To this end, notice that $I(X_1, X_2; Y)$ depends on (X_1, X_2) only through their product. If U has the same distribution as X_1X_2, then

$$I(U; U + Z) = I(X_1, X_2; Y)$$

Now set $X_1 = \frac{1}{\sqrt{P_2}}U$ and $X_2 = \sqrt{P_2}$ a.s. The individual power constraints are satisfied, and also the overall sum-rate remains the same. It is clear that $(R_1+R_2, 0)$ can be achieved this way. By reversing the roles of the transmitters, we can similarly achieve the rate-pair $(0, R_1+R_2)$. Thus the capacity region indeed is a triangle. ∎

Lemma 7 implies that maximizing the RHS of (6) is equivalent to maximizing R_1 while R_2 is set to zero. Notice that this does not imply that $X_2 = 0$. User 2 still transmits, so as to enable user 1 achieve its maximal rate. For given average power constraints of P_1 and P_2, let us denote the maximal rate achieved by user 1 as R_1^*.

Lemma 8. *For the purpose of evaluating R_1^*, the transmissions X_2^n from user 2 can be considered as a deterministic sequence. Furthermore, the sum-rate is bounded as*

$$R_1^* \leq \max_{P_1(\tau), P_2(\tau)} \lim_{T\to\infty} \frac{1}{T}\int_0^T \log\left(1 + \frac{P_1(\tau)P_2(\tau)}{\sigma^2}\right) d\tau$$
$$\textit{subj to} \lim_{T\to\infty}\frac{1}{T}\int_0^T P_i(\tau)d\tau \leq P_i\,,\ i \in \{1, 2\}. \tag{7}$$

Proof: The argument for choosing a deterministic X_2 is contained in the proof of Lemma 7, as all the stochastic contents can be captured in X_1. Once X_2^n is determined, X_{1i} should be chosen from appropriate Gaussian distributions. This is due to the entropy maximization property of Gaussian distribution under a variance constraint. Equation (7) covers all possible power variations that can be attempted on the two transmitters, hence will give an upper-bound to the sum-rate. ∎

Notice that (7) is about computing the convex-hull of a region in $\{\mathbb{R}^+\}^3$ of the form $(P_1, P_2, \log(1 + \frac{P_1P_2}{\sigma^2}))$. The usual trick now is to apply Caratheodory-Fenchel Theorem [7] to reduce the dimensionality of the optimization problem in (7). However, we defer this to Section V, where it is shown that the upperbound specified by (7) is indeed achievable. We will now explain the need for a ternary auxiliary random variable while time-sharing in a M-MAC.

IV. Time-sharing in M-MAC

We digress a bit, and instead of obtaining bounds to the sum-rate, let us consider maximizing $I(X_1, X_2; Y)$ for the M-MAC. The following two lemmas, though not necessary for our main theorem, will help us in explaining the significance of our results.

Lemma 9. *For any $p(x_1, x_2) = p(x_1)p(x_2)$ in the M-MAC,*

$$I(X_1, X_2; Y) \leq \log(1 + \frac{P_1P_2}{\sigma^2})$$

Proof: Notice that,

$$\begin{aligned} I(X_1, X_2; Y) &= h(Y) - h(Z) \\ &\leq \log(\pi e E|Y|^2) - \log(\pi e\sigma^2) \\ &\leq \log\left(1 + \frac{P_1P_2}{\sigma^2}\right) \end{aligned} \tag{8}$$

∎

On the other hand, the bound in the above lemma is easily achieved by taking $X_1 \sim \mathcal{N}(0, P_1)$ and $X_2 = \sqrt{P_2}$ a.s. The other corner point can be achieved by reversing the roles of the transmitters. At first sight, it may appear that the RHS of the lemma is indeed an upper-bound on the sum-rate, which can be achieved but not beaten, and hence will give the capacity region. However, we will now emphasize that this is not the case.

Lemma 10. *For the M-MAC, there exist achievable rate-pairs (R_1, R_2) such that,*

$$R_1 + R_2 > \log\left(1 + \frac{P_1P_2}{\sigma^2}\right),$$

when P_1P_2 is low enough.

Proof: We will prove this by an example. Consider an alternate scheme where the transmitters employ generalized time-sharing on top of the previous strategy, i.e. transmitting half the time with double the power and no transmissions for the latter half. This will achieve a sum-rate of

$$\frac{1}{2}\log\left(1 + \frac{(2P_1)\times(2P_2)}{\sigma^2}\right) + 0 = \frac{1}{2}\log\left(1 + \frac{4P_1P_2}{\sigma^2}\right). \tag{9}$$

If P_1P_2 is low enough,

$$\frac{1}{2}\log\left(1 + \frac{4P_1P_2}{\sigma^2}\right) \geq \log\left(1 + \frac{P_1P_2}{\sigma^2}\right).$$

∎

Thus we have beaten the bound in Lemma 9. We should underline that this is not a contradiction of any sort. The bound in Lemma 9 did not account for the rate enlargement due to the presence of auxiliary random variable Q, thus ruling out the prospect of time sharing. The key observation is that even for maximizing one of the rates in a M-MAC, time-sharing is necessary at low values of power. Reversing the roles of the transmitters, and taking convex combination of the rates, it can be observed that $|Q| = 3$ is required for the auxiliary random variable. More details are provided in the next section.

V. Capacity Region ($\mathcal{C}_{MMAC}$)

We first show that the upperbound in Section III evaluates to the rate-expression given in Theorem 5. Notice that Lemma 8 is about computing the convex-hull of points in $\mathbb{R}^3$ of the form, $(P_1, P_2, \log(1+P_1P_2/\sigma^2))$. The function $\log(1+\frac{P_1P_2}{\sigma^2})$ is not in general concave, otherwise there is no need for taking the convex-hull. However, we have concavity in a restricted region, as stated below.

Lemma 11. *The function* $\log(1+\frac{P_1P_2}{\sigma^2})$ *is locally concave in* (P_1, P_2)*, if* $P_1P_2 > \sigma^2$.

Proof: Computing the Hessian H, we get

$$H = \frac{1}{(1+\frac{P_1P_2}{\sigma^2})^2}\begin{bmatrix} -\frac{P_1^2}{\sigma^4} & \frac{1}{\sigma^2} \\ \frac{1}{\sigma^2} & -\frac{P_2^2}{\sigma^4} \end{bmatrix} \tag{10}$$

Observe that the trace of the above 2×2 matrix H is negative. Furthermore, the determinant is positive when $P_1P_2 > \sigma^2$. Thus the Hessian becomes negative definite in this range, hence the lemma. ■

We have already shown in Lemma 10 that $\log(1+\frac{P_1P_2}{\sigma^2})$ is not concave when $\frac{P_1P_2}{\sigma^2}$ is small enough. This makes time-sharing necessary to obtain the convex-hull, particularly for low values of $\frac{P_1P_2}{\sigma^2}$. Since the tuple $(P_1, P_2, \log(1+\frac{1}{\sigma^2}P_1P_2))$ defines a connected region in the positive quadrant of $\mathbb{R}^3$, we can apply the Caratheodory-Fenchel theorem [7] in (7) to obtain,

$$R_1^* \le \max \sum_{i=1}^{3} \lambda_i \log\left(1+\frac{P_1(i)P_2(i)}{\sigma^2}\right),$$
$$\text{such that} \sum_{i=1}^{3} \lambda_i P_k(i) \le P_k\,,\ k \in \{1,2\}. \tag{11}$$

The maximization above is over non-negative values $P_1(j)$ and $P_2(j)$ for $j = 1,2,3$. Performing the optimization will give the RHS of (4). We have relegated the arguments to the appendix. Let us now focus on achieving the RHS of (4).

A. Achievable Scheme

The rate in Theorem 5 can be achieved by a simple strategy. Specifically, both the users simultaneously transmit for a fraction δ of the time, using respective powers $(\frac{P_1}{\delta}, \frac{P_2}{\delta})$. The transmitters remain silent for the rest of the time. The rate R^* can be achieved by letting one of the users employ a Gaussian codebook, the other user sends a constant value throughout its transmissions. The time-sharing parameter δ is chosen such that,

$$\delta = \begin{cases} 1, \text{ if } P_1P_2 \ge c \\ \sqrt{\frac{P_1P_2}{c}}, \text{ otherwise}\ , \end{cases} \tag{12}$$

where c is the same parameter as in (4). In its ON state, the first user transmits with a power of

$$P_1^{max} = \sqrt{\frac{cP_1}{P_2}}. \tag{13}$$

The average power expenditure is

$$\delta P_1^{max} = P_1,$$

thus satisfying the constraint. A similar argument holds true for the other user. Thus our proposed rates in (4) can be achieved, settling the capacity region $\mathcal{C}_{MMAC}$. Notice that the time-sharing of two rate-pairs is required to achieve the maximal sum-rate of each user, when P_1P_2 is low enough. Since one of these pairs is always the origin, a total of three rate-pairs suffice to achieve the capacity region.

VI. Conclusion

We computed the capacity region of a multiplicative MAC with AWGN under individual average power constraints at the transmitters. This is one of the channels where the convexification/time-sharing needs more operating modes than the number of users. On inspection, it becomes evident that almost all points in the boundary of the capacity region is obtained by time-sharing between at most three modes, viz.

- a mode in which the first user transmits information while the second one sends a constant value.
- a mode in which the users reverse their roles from the above.
- a silent mode, where both users switch themselves off.

The time-sharing parameter and operating powers are chosen in such a way so as to satisfy the average power constraints.

The results can be generalized to more than 2 users, where we again obtain a simplex like structure for the capacity region.

VII. Appendix

A. Upperbound on R_1^*

For simplicity we will assume that $\sigma^2 = 1$ and compute the convex-hull of $(P_1, P_2, \log(1+P_1P_2))$. A quick handle on this region can be obtained by looking along the $P_1 = P_2$ line. The function $f(x) = \log(1+x^2)$ can be convexified by drawing a tangent from $(0,0)$ to $f(x)$, as the function is convex for low values of x, and then inflects to a concave one, see Figure 3.

Let α be the x-coordinate where the tangent from origin grazes the function $\log(1+x^2)$, see Figure 3. The value α^2 plays a key role in performing the maximization in (11), i.e. the convex-hull of $(P_1, P_2, \log(1+P_1P_2))$. Consider the the points (P_1, P_2) such that $P_1P_2 = \alpha^2$. For any such point, the line drawn from the origin to $(P_1, P_2, \log(1+P_1P_2))$ is a tangent along this direction. To see this, consider the $P_2 = \theta P_1$ line. By computing the slope at the point where the tangent touches $\log(1+P_1P_2)$ we have,

$$\frac{\log(1+\theta P_1^2)}{P_1} = \frac{2\theta P_1}{1+\theta P_1^2}. \tag{14}$$

When θP_1^2 is a constant, the above equation is invariant with respect to P_1 and P_2. Notice that θP_1^2 is nothing but P_1P_2. It is also evident that there is only one solution for (14) when $\theta \ge 0$.

Thus for every (P_1, P_2) such that $P_1P_2 < \alpha^2$, the function $\log(1+P_1P_2)$ is below the line connecting $(0,0)$ and $\log(1+$

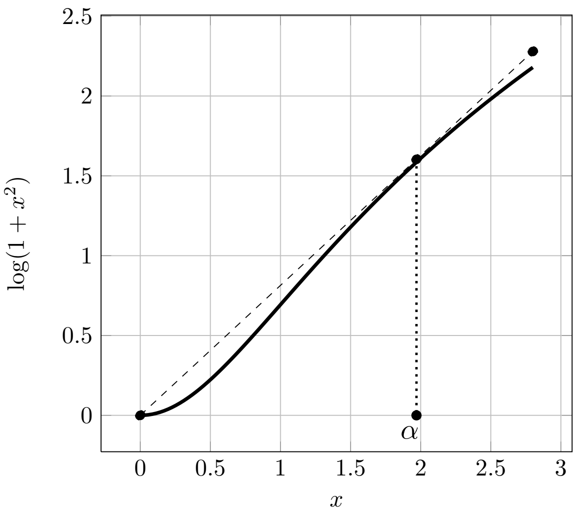

Fig. 3. Convex-Concave nature of $\log(1 + x^2)$

$\alpha^2)$ along this direction. In other words, we are just connecting each point in the level-set of $\log(1+\alpha^2)$ to the origin, and the surface thus generated will lie above $(P_1, P_2, \log(1 + P_1P_2))$ for $P_1P_2 < \alpha^2$.

Notice that the collection of all such tangent lines forms part of a cone. Specifically, the cone is formed by its vertex at the origin and the cross-section in the $z = \log(1 + \alpha^2)$-plane given by a convex curve, i.e. $\{(x, y) \in \mathbb{R}^+ \times R^+ : xy = \alpha^2\}$. Notice that the section is not a closed curve, see the illustration in Figure 4.

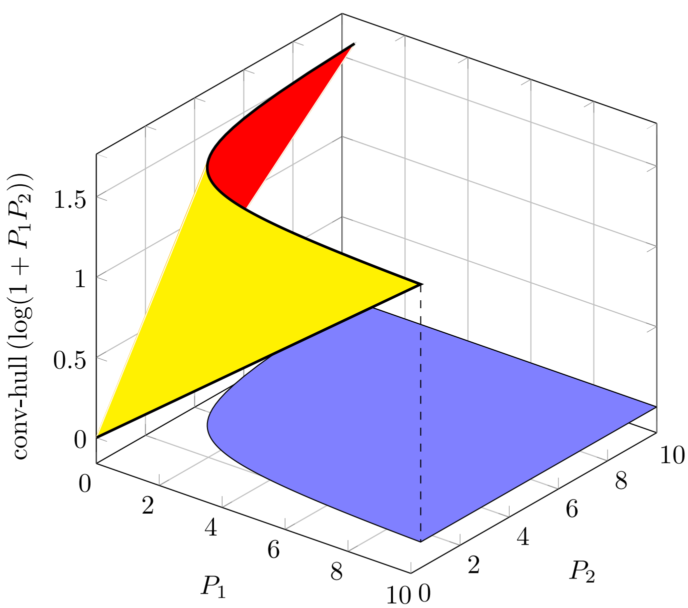

Fig. 4. $3D$ Illustration of $\mathcal{C}_{MMAC}$

In Figure 4, the shaded region in the $xy-$ plane ($z = 0$) represents pairs of average powers (P_1, P_2) for which no time sharing is necessary. Thus the convexification will yield $\log(1 + P_1P_2)$ as the value of the function in this region. On the other hand, the conical structure drawn in the $xyz-$plane represents time-sharing between the level-set of $\log(1 + \alpha^2)$ and the origin. Elementary arguments suffice to show that the union of the two proposed regions will indeed be concave.

Putting the pieces together, we obtain the bound

$$R_1^* \leq \log\left(1 + P_1P_2\right) \mathbb{I}_{\{P_1P_2 \geq \alpha^2\}} + \sqrt{\frac{P_1P_2}{\alpha^2}} \log\left(1 + \alpha^2\right) \mathbb{I}_{\{P_1P_2 < \alpha^2\}}, \quad (15)$$

Rescaling P_1P_2 by $\frac{1}{\sigma^2}$ will yield the RHS of (4). Observe that for the region corresponding to the cone, time-sharing between the origin and another point is required. In this range, if the upperbound happens to be achievable, then time-sharing is required to maximize even the $x-$coordinate of the capacity region. This will necessitate a cardinality relaxation in Theorem 4, so that it becomes applicable to the M-MAC.

VIII. Acknowledgments

Numerous discussions with Yash Deshpande, B. K. Dey, and constructive comments from the reviewers are gratefully acknowledged.

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