On Achieving Marton's Region for Broadcast Channel using Feedback

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***Abstract*—We consider a two user discrete memoryless broadcast channel (DMBC) with perfect feedback from both the receivers. The best known achievable region for the general DMBC *without* feedback is known as the Marton's region, where the achievable strategy employs random coding. By effectively using the available feedback, we construct explicit coding schemes which achieve any rate-pair in the Marton's region.**

I. Introduction

It is well known that feedback does not increase the capacity of a point to point discrete memoryless channel (DMC). However, it can drastically improve the error performance. Furthermore, feedback enables the construction of explicit coding strategies at all rates less than the capacity for many point to point (single user) channels of interest [1], [2].

When it comes to multi-user models, feedback offers potential improvement of the capacity region, see for example [3], where the Gaussian multiple access channel (MAC) with feedback is considered. The enlargement of capacity region is also possible for the Broadcast channel (BC), where a single transmitter conveys multiple streams of information to different users/receivers. In [4], the capacity enlargement with feedback from both the receivers in a 2 user Gaussian BC is demonstrated. Similar results with feedback from just one of the links appear in [5]. The purpose of the current paper is not in demonstrating such enlargements in the capacity region. Rather, we propose constructive coding schemes which achieves good rate-pairs in a two user BC. In particular, we construct coding schemes for all rate-pairs which are known to be achievable by random coding methods.

The best known achievable rate for the general BC is known as the Marton's region. To achieve this region using feedback, we combine several techniques available in the information theory literature. The key ones are,

1) a feedback coding scheme for the DMC proposed in [1].
2) a coding scheme with feedback and side-information, which appeared in [6].
3) an interleaving strategy which makes non-causal side-information available at the transmitter [5].

It should however be noted that the original Marton bound without its generalization to common messages is not always tight, as recently shown by [7]. We are only considering the original bound, i.e. transmission of private messages.

The organization of the paper is as follows. Section II introduces the system model and the objectives. In Section III, we will describe Ahlswede's feedback coding scheme [1] for the DMC and its generalization to a channel with transmitter side information and feedback [6]. We combine the above-mentioned techniques with a careful interleaving strategy, and demonstrate in Section IV that all rate-pairs in Marton's region are achievable using our scheme.

We employ the following notational conventions. The subscript i is an index taking value in the non-negative integers $\mathbb{Z}^+$. We use k to identify the receivers in a BC, for example $k \in \{a, b\}$, where the receivers in consideration are user a and user b. The notation U_{ai} stands for the i^{th} instantiation of the variable U_a, which is related to user a. For any variable U, the notation U_i^j stands for the sequence $\{U_i, U_{i+1}, \cdot\cdot, U^{j-1}, U^j\}$, whenever $j > i$. We will not explicitly mention the above subscript if the sequence starts from index 1.

II. System Model

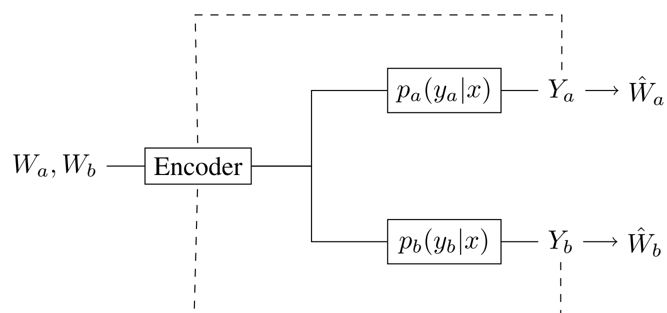

Fig. 1. DMBC with feedback

Consider the model depicted in Figure 1, where $(W_a, W_b) \in (\{1, \cdot\cdot, M_a\} \times \{1, \cdot\cdot, M_b\})$ is a pair of uniformly chosen message index. The encoder, depending on (W_a, W_b) and any other available information on the channel (for example, using feedback), will choose the transmitted symbols $X \in \mathcal{X}$. We assume that the input alphabet $\mathcal{X}$ is discrete and of finite cardinality. There are two receivers, observing the output symbols Y_a and Y_b respectively. The output symbols $Y_k \in \mathcal{Y}_k$, $k = a, b$ are also discrete and $|\mathcal{Y}_k| < \infty, \forall k$.

A discrete memoryless broadcast channel (DMBC) is defined by a collection of probability laws on the output product space $(\mathcal{Y}_a \times \mathcal{Y}_b)$, one pair for each transmitted symbol x, denoted as $\{P_a(y_a|x), P_b(y_b|x)\}$. We will call this a $(X, \{p_a, p_b\}, Y_a \times Y_b)$ channel. Our aim is to convey the index W_k to its respective receiver Y_k for $k \in \{a, b\}$. Suppose

Receiver k declares $\hat{W}_k$ after n uses of the channel. The probability of error is defined as,

$$P_{error} = \frac{1}{M_a M_b} \sum_{i,j} P(W_{ai} \neq \hat{W}_{ai}, W_{bj} \neq \hat{W}_{bj}). \quad (1)$$

Definition 2.1: In the presence of feedback, we say a rate-pair

$$R_a = \frac{1}{n} \log M_a, R_b = \frac{1}{n} \log M_b,$$

is achievable if we have a coding scheme which transmits $X_i = f(W_a, W_b, Y_a^{i-1}, Y_b^{i-1})$, $1 \leq i \leq n$, and the corresponding P_{error} in (1) can be made as small as possible.

In this sequel we are also concerned about the rates in the absence of feedback.

Definition 2.2: A $(n, M_a, M_b, R_a, R_b, \epsilon)$ coding scheme for the DMBC $(X, \{p_a, p_b\}, Y_a \times Y_b)$ without feedback consists of an encoder $\mathcal{E} : (\{1, \cdot\cdot, M_a\} \times \{1, \cdot\cdot M_b\}) \rightarrow \mathcal{X}^n$ and two decoding functions $g_a : \mathcal{Y}_a^n \rightarrow \{1, \cdot\cdot, M_a\}$ and $g_b : \mathcal{Y}_b^n \rightarrow \{1, \cdot\cdot, M_b\}$ such that average error probability P_{error} (defined in (1)) is less than ϵ when n is large enough.

Definition 2.3: The achievable rate region for a DMBC $(X, \{p_a, p_b\}, Y_a \times Y_b)$ without feedback consists of all pairs

$$R_a = \frac{1}{n} \log M_a, R_b = \frac{1}{n} \log M_b,$$

such that a $(n, M_a, M_b, R_a, R_b, \epsilon)$ coding scheme exists for all $\epsilon > 0$.

In the absence of feedback, the best known achievable rate-region is due to Marton [8].

Theorem 2.4: (Marton [8]) For the DMBC $(X, \{p_a, p_b\}, Y_a \times Y_b)$, any rate pair in the convex closure of the set $\mathcal{R}_{BC}$ is achievable, where

$$\mathcal{R}_{BC} = \bigcup (R_a, R_b)$$

in which

$$\begin{aligned} R_a &\leq I(U; Y_a) \\ R_b &\leq I(V; Y_b) \\ R_a + R_b &\leq I(U; Y_a) + I(V; Y_b) - I(U; V) \end{aligned}$$

for some

$$p(u, v, x, y_a, y_b) = p(u, v) p(x|uv) p(y_a, y_b|x).$$

The proof of the above theorem employs random coding arguments, where the existence of good coding schemes are proved, without providing a constructive mechanism. In the remaining part of this sequel, we will present a constructive scheme which achieves any rate-pair given in Theorem 2.4. This is made possible by efficient utilization of the available noise-less feedback. The key ingredients in this are Ahlswede's feedback scheme for the DMC and its extension to channels with transmitter side information [6].

III. Ahlswede's Feedback Scheme

A feedback coding scheme for the DMC with finite alphabets was proposed in [1]. This coding scheme builds on the duality between source and channel coding. In fact, the central results for discrete-memoryless source as well as channel coding use the formulation of typical sequences [9]. Concisely, a sequence X^n generated according to the law $\prod_{i=1}^n p(x_i)$ belongs to the typical set $A_\epsilon^n(x)$ if,

$$\left|\frac{1}{n} \sum_{i=1}^n I_{\{X_i = a\}} - p(a)\right| \leq \frac{\epsilon}{|\mathcal{X}|}, \forall a \in \mathcal{X},$$

where $I_{\{\cdot\}}$ is the indicator function. Furthermore,

$$2^{n(H(X)-\epsilon)} \leq |A_\epsilon^n(x)| \leq 2^{n(H(X)+\epsilon)},$$

which we will denote as $|A_\epsilon^n(x)| \approx 2^{nH(X)}$. It is well known that about $nH(X)$ bits suffice to represent X^n on the average in a loss-less fashion. The encoding strategy is to assign an index to every sequence $x^n \in A_\epsilon^n(x)$, this is illustrated below.

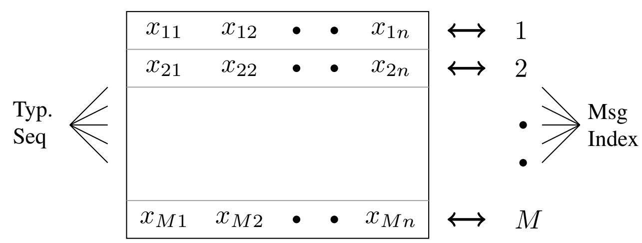

Fig. 2. Duality of Source and Channel Coding

A dual approach in channel coding is to consider a set of index $\{1, \cdot\cdot, M\}$. For sending message i, we will use the i^{th} row in Figure 2 as the channel code. Let us call this codebook as the **base code**. If the channel is error-free, the receiver will find the transmitted message using the observed *base* code-word. The only pit-fall is that in the presence of noise, the transmitted sequence may not be received as it is. Here is where feedback can help us, to progressively reduce the uncertainty at the receiver. Ahlswede's coding scheme does this, which we describe below. We use the following conventions to describe this feedback scheme.

- Fix an input distribution $p(x^n) = \prod_i p(x_i)$.
- Let $\mathcal{D}(u^n)$ denote the smallest set in $\mathcal{Y}^n$ such that, $p(y^n \in \mathcal{D}(u^n)) > 1 - \epsilon$, where u^n is the input to the DMC, and y^n the corresponding output.
- Consider a reverse channel from the output to the input, defined by the law $Q(x|y) = \frac{p(x,y)}{\sum_x p(x,y)}$.
- Let $\mathcal{E}(v^n) \subset \mathcal{X}^n$ be the smallest set such that $p(x^n \in \mathcal{E}(v^n)) > 1 - \epsilon$, where v^n is the input to the reverse DMC $Q(x|v)$, and x^n the corresponding output.
- Every time we use the sets $\mathcal{D}(\cdot)$ and $\mathcal{E}(\cdot)$, we will not explicitly say that the associated event has probability close to one. This is chosen for a simple presentation, the reader can substitute each of those deterministic statements in the algorithm with 'probability close to one'.

- Consider an indexed set A. Let $Ind_A(x)$ be an operator on A which returns a distinct index for each $x \in A$. Similarly, define $Ind_A[i]$ as an operator which returns the member of set A with index i.
- We use lexicographical ordering of the elements to form an indexed set.
- The coding scheme is iterative. In the l^{th} iteration n_l symbols from the alphabet $\mathcal{X}$ are sent. We denote $s = \sum_{i=1}^{l-1} n_i$ as the total number of transmissions performed before the l^{th} iteration, with $n_0 = 0$. We choose,
$$n_l = \frac{n_{l-1}H(X|Y)}{H(X)}, \ l \geq 2. \tag{2}$$
The reasons behind this choice will be clear as we progress.
- The symbols are chosen in such a manner that the remaining uncertainty about the transmitted symbols $X_{s-n_{l-1}}^{s}$ (in previous iteration) after receiving $Y_{s-n_{l-1}}^{s}$ is resolved by the symbols $X_s^{s+n_l}$ (in current iteration). Thus, if there is no error in the l^{th} iteration, we can find all the transmitted symbols $X_{s-n_{l-1}}^{s}$ in the previous iteration. Continuing this backwards, we can recover all the transmitted symbols with probability close to one.

Algorithm to send N bits

1) Choose an index $w \in \{1, \cdot\cdot, 2^N\}$ based on the input bits.
2) Let $l = 1$ and $A = A_\epsilon^{n_1}(x)$ be the **base code**.
3) Encoder sends $Ind_A[w]$ in the coming n_l channel-uses [1].
4) By observing the output $y_s^{s+n_l}$, the decoder finds the set $\mathcal{E}(y_s^{s+n_l})$ (definition before the algorithm).
5) Using perfect feedback, the encoder also finds the set $\mathcal{E}(y_s^{s+n_l})$.
6) Assign $w = Ind_{\mathcal{E}(y_s^{s+n_l})}(x_s^{s+n_l})$.
7) Set $A = A_\epsilon^{n_{l+1}}$, where A is indexed and assign $l = l+1$.
8) Go back to Step 3.

Let us find the total number of transmissions to send N bits of information. The initial number of transmissions n_1 is chosen such that,
$$n_1 H(X) = N.$$
Using (2),
$$\sum_{l\geq 1} n_l = \frac{N}{H(X)} \sum_l \left[\frac{H(X|Y)}{H(X)}\right]^{l-1} \tag{3}$$
$$= \frac{N}{H(X) - H(X|Y)} \tag{4}$$
$$= \frac{N}{I(X;Y)} \tag{5}$$
Thus, by choosing $p(x)$ as the capacity achieving distribution of the DMC, we can transmit at rates close to the capacity. In addition, a termination criteria can be judiciously chosen such that the last stage has zero error, say using a conventional low-rate block code. This will ensure that n_l remains large enough in each of the iterations, that the typicality arguments are tight.

[1]assume n_l large enough to support all the index

Notice that the termination step will **not** cause a considerable reduction in the overall data rate.

Remark 3.1: At transmission instant $s = \sum_{i=1}^{l-1} n_i$, the future transmitted values $x_s^{s+n_l}$ are completely known in Ahlswede's scheme. In other words, this information is available non-causally.

A. Transmitter Side Information

Ahlswede's feedback scheme was generalized by Merhav and Weissman [6] to do feedback coding in the presence of transmitter side information. The general DMC with transmitter side information is also known as the Gelfand-Pinsker channel, see Figure 3.

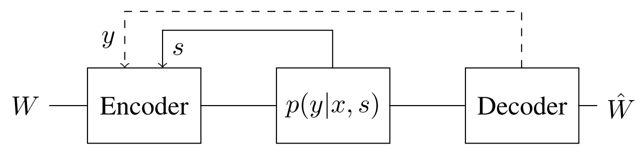

Fig. 3. Gelfand-Pinsker channel with feedback

The sequence S (see figure), which can be considered as a state of the channel, is assumed to be available non-causally at the transmitter. The output of the channel at any instant i depends on the current input and state, and independent of all other variables once the input and state is given. We will communicate an auxiliary random variable U which contains information about the message. The transmitted symbol X is chosen to be a deterministic function of U and S. In [6], it is shown that after knowing the state sequence S^n, the transmitter can jointly generate U^n using a distributed source coding approach. Effectively, $nH(U|S)$ bits can be conveyed in the absence of noise using this approach. In other words, the **base** codebook is of length n and contains $2^{nH(U|S)}$ sequences. The impairments created by noise can be iteratively rectified in the same manner as in Ahlswede's scheme, except that $n_l = n_{l-1}\frac{H(U|Y)}{H(U|S)}$ is the number of required transmissions in stage l, $l \geq 2$. Thus, the number of transmissions required to convey N bits of information is,
$$\sum n_l = \frac{N}{H(U|S) - H(U|Y)}$$
$$= \frac{N}{I(U;Y) - I(U;S)}. \tag{6}$$
The denominator in (6) can now be maximized over the distribution $p(u|s)$. We call the above scheme as the **M-W scheme** [6].

IV. DMBC WITH FEEDBACK

Let us now describe a feedback coding scheme for the DMBC. Similar in spirit to the original Marton's coding [8] for the BC, our strategy is to convey two auxiliary random variables U and V, say U to the receiver Y_a and V to Y_b. We use Ahlswede's coding scheme on the link to Y_b, along with an additional stage of interleaving. The **M-W scheme** (Merhav and Weissman [6]) is used to convey U on the link

Y_a, where the variable V is considered as side-information. The initial transmission steps are as follows.

- Fix a distribution $p(u,v)$, with marginals $p(u)$ and $p(v)$.
- Generate $2^{n_1 H(V)}$ sequences $\sim \prod_{i=1}^{n_1} p(v_i)$, and keep it as the base code for the link $V \to Y_b$. In particular n_1 is chosen to convey N_b bits of information, i.e. $n_1 H(V) = N_b$.
- Collect d blocks of information bits intended for Y_b, each block containing N_b bits. In our scheme d can be taken arbitrarily large, without any degradation in performance.
- Pick the base code-word for each block and concatenate these to get a long sequence, which forms the transmitted symbols $V^{n_1 d}$ to user b.
- Pick $m_1 = \frac{N_a}{H(U|V)}$, where N_a is the number of bits to be conveyed to user a.
- By choosing $m_1 \leq n_1 d$, the sequence V^{m_1} is non-causally available at the transmitter. It then performs M-W coding to generate U^{m_1}.
- For $1 \leq i \leq m_1$, the transmitter sends $X_i = f(U_i, V_i)$. The observations (Y_a, Y_b) are then perfectly fed-back to the transmitter.

The feedback values are used to generate the input variables U and V for future channel-uses. We will update V iteratively, where at the end of iteration $l-1$, $n_l d$ new symbols are generated using feedback, to be conveyed as the auxiliary variable V in the coming iteration. The transmissions to user a follow another loop, where m_j symbols are generated for future transmissions at the end of step $j-1$. We notice that a large enough d can be chosen to align the transmission index of both U and V to some super-block.

- Let $s = \sum_{i=1}^{l-1} n_i$ and $t = \sum_{i=1}^{j-1} m_i$. Consider the l^{th} iteration for transmitting to user b. At the end of transmission $i = d\,s + k\,n_l$, the entire sequence $V^{d(s+n_l)+k\,n_{l+1}}$ is available at the transmitter in lieu of Remark 3.1.
- By choosing d large enough, we can assume that for each iteration for user a, the information $V_t^{t+m_j}$ is available non-causally at the transmitter. The symbols $U_t^{t+m_j}$ can be generated as per the M-W scheme.
- The remaining steps are now straight-forward. Utilizing feedback, the encoder chooses auxiliary code-words of length $n_l = n_{l-1} \frac{H(V|Y_b)}{H(V)}$ to user b. The codebook contains indexed elements from $A_\epsilon^{n_l}(v)$. The chosen code-words for each of the d blocks of data are then concatenated to form a long sequence.
- At transmission instant i, the encoder generates $X_i = f(U_i, V_i)$ and sends it over the channel, similar to the M-W scheme in Section III-A.
- The number of transmissions required for every N_b bits of input information to user b is approximately $\frac{N_b}{I(V;Y_b)}$. Thus,

$$R_b \leq I(V;Y_b) \tag{7}$$

- N_a bits of information are conveyed to Y_a using the M-W scheme, then the rate obeys,

$$R_a \leq I(U;Y_a) - I(U;V) \tag{8}$$

The last two equations suffice to ensure that all rate-pairs in the Marton's region (see Theorem 2.4) are achievable [8]. However, we need to reconcile two points to support this sufficiency. The differences between the achievable rates in (7)–(8) and that in Theorem 2.4 are that

- we used a deterministic mapping $X_i = f(U_i, V_i)$ to generate the transmitted symbols, while randomization was allowed in the latter.
- the random variables U and V have to be of finite cardinality for our arguments to work.

The first point can be quickly reconciled by the following argument [10]. Let W be an independent random variable such that $X = f(U,V,W)$, and define $\tilde{V} = (V,W)$. The expressions of Theorem 2.4 can only be improved by using $\tilde{V}$ instead of V. As for the second limitation, cardinality bounds for the auxiliary random variables U and V were recently established in [11].

V. Conclusion

We presented a coding scheme which effectively utilizes feedback to achieve all rate-pairs in the so called Marton's region of a two user BC. Improving the presented scheme to achieve some of the known outerbounds for the broadcast channel is a direction which we will pursue in future.

VI. Acknowledgments

The authors thank Vinod Prabhakaran for pointing out the necessary arguments which guarantee the sufficiency of deterministic maps, in generating the transmitted symbols for a BC from the auxiliary variables.

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