Rate Adaptation Games in Wireless LANs: Nash Equilibrium and Price of Anarchy

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Abstract. In Wireless LANs, users may adapt their transmission rates depending on the observed radio conditions on their links to maximize their throughput. Recently, there has been a significant research effort in developing distributed rate adaptation schemes offering better performance than that of the current ARF (Automatic Rate Fallback). Unlike previous works, this paper characterizes the optimal reaction of a rate adaptation protocol to the contention information received from the MAC. We formulate this problem analytically. We study both competitive and cooperative user behaviors: In the case of competition, users selfishly adapt their rates so as to maximize their own throughput, whereas in the case of cooperation they aim at adapting their rates to maximize the overall system throughput. We show that the Nash Equilibrium realized in the case of competition can be inefficient (i.e., the price of anarchy is high, up to 50% of the social optimum), and provide insightful properties of the socially optimal rate adaptation schemes. We also show that RTS/CTS does not make the competitive scenario more efficient. We then apply the same analysis to recently proposed collision-aware rate adaptation algorithms and observe similar conclusions. Finally, we propose a novel collision-aware rate adaptation algorithm that significantly reduces the price of anarchy in many scenarios of interest.

I. Introduction

Radio sharing policy and rate adaptation are two key parts of the IEEE 802.11 MAC. Users share the radio resources in a distributed manner using the mandatory contention-resolution scheme DCF (Distributed Coordination Function). This scheme specifies how users should adapt their channel access probability when they experience transmission failures. When the network is perceived as congested, under the DCF, users *cooperatively* decrease their access probability, which in turn should limit the number of collisions and keep the overall network efficiency at a satisfying level. DCF design is time-critical and it is implemented in hardware. Users cannot modify it.

Transmission failures are not due to collisions only, but may also be caused by a noisy channel. In IEEE 802.11 systems, users may adapt their modulation and coding rate to identify the optimal trade-off between the transmission rate and the packet losses due to channel errors. An objective of a rate adaptation algorithm is to estimate the channel's quality and to find the optimal trade-off. A rate adaptation algorithm is typically implemented in software and it can easily be modified. The IEEE 802.11 standard does not specify any rate adaptation algorithm. Today, the great majority of commercial devices implement a common simple rate adaptation algorithm, ARF (Automatic Rate Fallback). Many rate adaptation algorithms have been recently proposed and evaluated in the literature to replace ARF, see Section II.

Rate adaptation can be done in a cooperative or competitive way. In the former scenario, the rate adaptation scheme is designed so as to maximize the total throughput of the network while guaranteeing a certain degree of fairness among users, thus achieving a social optimum of the network. In the latter scenario, each user designs its own rate adaptation strategy with the aim of maximizing its own throughput and without accounting for its impact on the performance of other users. If there is a single greedy user, selfishly trying to adapt its rate, this user could potentially receive a higher throughput than that obtained in the social optimal. However, if all users compete, the system may evolve to an inefficient Nash Equilibrium where all users would receive a lower throughput than that in the social optimum. The performance gap between the social optimum and Nash Equilibria is called the *price of anarchy*.

An important challenge in designing the rate adaptation schemes comes from the fact that transmitters may not be able to distinguish between the causes of transmission failures. Transmission failures are caused by collisions and/or channel errors. Without this *loss differentiation* capability, both scheduling DCF and rate adaptation schemes may make wrong decisions. Channel errors can lead to an unnecessary access probability decrease when the network is lightly loaded. When the network is congested, collisions may be interpreted as channel errors and lead users to decrease their transmission rates, which in turn increases the packet transmission durations and further exacerbates the network congestion. Several schemes for loss differentiation have already been proposed; see Section II.

In this paper, we are interested in the interaction of DCF and the rate adaptation protocols. We assume that all users implement the standard DCF (with or without loss differentiation), but can modify their rate adaptation protocols. We aim at characterizing how users should optimally select a transmission rate depending on their state in the DCF schemes (i.e., the back-off stage) representing the level of congestion in the network. Indeed, selecting rates based on the back-off stage is one of the novel features that separates our work from the previous work.

To make the problem analytically tractable, we assume that the channel state does not change with time. The transmitters know the channel quality on their links, and need not estimate it. The assumption is made as our primary interest is in understanding the interaction between the DCF and the rate adaptation schemes, and not in the channel estimation part. The assumption is also partly supported by typically large channel coherence times (much larger than the time required for a packet transmission). We also assume that the network is symmetric, in the sense that all the links experience the same radio conditions, and they all interfere with each other. We leave both heterogeneous scenarios and SNR estimation for the future work.

Since SNR is known at the transmitter, one may expect that it is straightforward to determine the optimal transmission rate, and this rate should be used in all the back-off stages. We, however, prove that this is not the case. In the cooperative scenario, the optimal rate increases as back-off stage increases (which was previously observed only in scenarios with hidden terminals), whereas in the competitive scenario the optimal rate decreases. Thus, price of anarchy is significant. This illustrates counter-intuitive complex behavior of inter-action between DCF and rate adaptation even in a simple scenario in which SNR is known. We quantify these interactions and show what are the optimal rate adaptation strategies in both cooperative and competitive scenarios. In the competitive scenario, we are also interested in identifying slight modifications of the DCF scheme that could eliminate the price of anarchy, and push the Nash Equilibria of the rate adaptation game toward a socially efficient system, if such modification is possible.

The main contributions of this paper are as follows:

• We formulate the problem of designing optimal rate adaptation algorithms analytically. We develop a generic framework that enables us to consider both cooperative and competitive user behaviors, but also to account for the use of RTS/CTS and the capability or the inability of users to distinguish collisions from channel errors.

• In the cooperative scenario, we characterize the optimal algorithms and provide a distributed implementation of these algorithms. We prove that even in absence of hidden terminals a node should increase its transmission rate as the contention increases. We also show that collision-awareness or RTS/CTS offer little performance improvements.

• In the competitive scenario, we analyze the resulting rate adaptation game. We show the existence of pure symmetric Nash Equilibria, and give methods to identify them. We also compute the price of anarchy in these games and show that it is not negligible in general.

• We propose ROCE, a novel way of reacting to channel errors and we show that it consistently has smaller price of the anarchy than the other proposed schemes (although it does not eliminate it completely), while it achieves the same social optimum.

The paper is organized as follows. Section II discusses the related work. Section III defines the models. Stationary analysis for a given rate adaptation is given in Section IV. Competitive scenarios are discussed in Section V and cooperative scenarios in Section VI. Numerical results are presented in Section VII and conclusions in VIII. The proofs are in the appendix.

II. Related Work

Many rate adaptation algorithms have been recently proposed and evaluated in the literature to replace ARF, see e.g. [1], [2], [3], [4]. To solve the problems raised by the interaction of the DCF and the rate adaptation algorithm, researchers have proposed ways of differentiating the causes of transmission failures, and then collision-aware rate adaptation algorithms, see e.g. [5], [6], [7], [8], [9], [10]. Note that most often, the proposed rate adaptation algorithms are based on heuristic arguments and numerical experiments.

The authors of [11], [12] provide analytical models for the interaction of DCF and the rate adaptation scheme, but they do not propose improved adaptation algorithms. Note also that the possible non-cooperative behavior of users in adapting their rate is rarely considered (in [13], [14], the authors provide preliminary analysis of rate adaptation games in WLANs where transmission failures due to channel errors are not modeled). DCF design in a game setting is discussed in [15].

From Nash's original paper [16], we know that there always exists a mixed strategy Nash Equilibrium (NE) when the number of pure strategies are finite and moreover there always exists a symmetric mixed strategy NE when the game is symmetric. Though existence of NE is guaranteed, it is well known that obtaining a NE is computationally expensive.Schmeidler showed the existence of pure strategy NE in non-atomic games, when a player's payoff depends only on his strategy and average behavior of others [17]. The question of the existence of symmetric pure strategy NE in a symmetric game and a quasi-concave payoff is addressed in [18]. However, it is difficult to verify if the payoff in our case is quasi-concave.

III. Models

We consider a network of N links. All links interfere each other, and always have packets to send in their buffer. All transmitters implement the same distributed random back-off scheduling mechanism to access the channel, e.g. DCF. This mechanism is modeled as follows. There are $I+1$ back-off stages: stage $i \in \{0, \dots, I\}$ indicates that i consecutive collisions have been experienced. In stage i a node transmits a packet with a fixed probability p_i such that $p_i \geq p_{i+1}$ for every $i < I$ (in DCF $p_i = 2^{-i} p_0$). We optimize the rate adaptation algorithm for a fixed scheduling mechanism.

The radio conditions for a link are characterized by the signal-to-noise (SNR) ratio at the receiver. We consider symmetric networks where all links have the same SNR, known at the transmitters. To send a packet, each transmitter can select a rate from a set $\mathcal{R}$ that can be either finite (as in the case of IEEE802.11g which has 8 rates) or infinite.

Definition 1 (Rate Adaptation Strategy): A rate adaptation strategy ρ is a map from $\{0, \dots, I\}$ to $\mathcal{R}$. A strategy ρ is said to be *constant* if $\rho(i) = \rho(i+1)$ for all $i < I$.

Note that $\rho(i)$ indicates the transmission rate under strategy ρ in i^{th} back-off stage.

Packet transmission can fail either due to a collision (several transmitters access the channel simultaneously) or due to a channel error. The probability that a packet sent at rate R is lost due to a channel error is a function $e(R, \mathrm{SNR})$ of the rate and of the SNR. A widely used model consists in defining e by $e(R, \mathrm{SNR}) = 1 - \gamma \mathrm{e}^{-\kappa \mathrm{R}}$, and we use this model unless specified otherwise. Since the SNR is assumed to be fixed and known at a transmitter, we denote: $e_R = e(R, SNR)$.

All packets have a fixed size σ, and thus the time to transmit a packet at rate R is $T_R = \sigma/R$. Systems with or without RTS/CTS mechanisms are analyzed. If T_{RTS} denotes the duration of the RTS/CTS signaling procedure, then with RTS/CTS, the effective transmission of a packet at rate R lasts $T_R + T_{RTS}$, and the duration of collisions reduce to T_{RTS}.

In the analysis, we consider both competitive and cooperative scenarios, and the cases where collisions and channel errors can or cannot be distinguished.

A. *Without loss differentiation (WoLD)*

When collisions and channel errors can not be distinguished (like in all of the 802.11 standards), a transmitter chooses a transmission rate $\rho(i)$ as a function of the number of consecutive transmission failures, denoted by i. Note that after i successive failures, the transmitter is in i^{th} back-off stage.

B. *With loss differentiation (WLD)*

We also consider the case where collisions and channel errors can be differentiated. This model is inspired by several proposals for collision-aware rate adaptation (e.g. [5], [6], [7], [8], [9], [10]). A transmitter chooses a transmission rate $\rho(i)$ as a function of the number of successive collisions i. Here, we do not keep track of the number of previous successive channel errors as these errors are assumed to be independent over various transmissions.

We consider two families of collision-aware rate adaptation strategies: The first family, called *WLDS* (WLD Standard), includes the strategies proposed in the literature, e.g. in [5], [6], [7], [8], [9], [10]. If a transmission fails due to a collision, then the back-off stage is incremented. If it fails due to a channel error, then the back-off stage remains the same.

We propose a second new family of rate adaptation strategies, referred to as ROCE (Return to 0 On Channel Error). Here, unlike WLDS, if a transmission fails due to a channel error, the back-off stage is reset to the minimum value ($i = 0$). The intuition behind this is that since the loss was not caused by collision, there is not reason to remain in a high-contention DCF state. If a transmission fails due to a collision, the back-off stage is increased, as in all other schemes.

C. *Competition vs. Cooperation*

Competition. Since all links are assumed to be equivalent in the systems considered, the competitive behavior of transmitters is modeled as a pure strategic symmetric rate adaptation game. This means that each transmitter adopts a deterministic strategy ρ. We emphasize on symmetric strategy for fairness, and on pure strategy for it is easy to implement. When all transmitters use the same strategy ρ, and when one of the transmitter updates its strategy, the latter becomes $B(\rho)$, the best response to the others' strategy

ρ. A symmetric Nash Equilibrium is reached when all transmitters use the same strategy ρ such that $\rho = B(\rho)$.

Cooperation. In the cooperation scenario, all transmitters use the same strategy ρ, hence all links achieve the same throughput. We want to find the social optimum, that is to characterize ρ that maximizes the total system throughput (or equivalently the throughput of each link since all the links are the same).

IV. Stationary analysis

We start the analysis by studying the steady state behavior of systems where all transmitters use a given rate adaptation strategy ρ. In such systems, we denote by π_i the stationary probability that a transmitter is in the back-off stage i, by $p = \sum_i \pi_i p_i$ the average transmission probability, and by $c = 1-(1-p)^{N-1}$ the collision probability.

A. Average slot duration

We consider virtual *slots*, as defined in [19]: a slot may correspond to a slot where the channel is idle (no transmission occurs), to a successful transmission, or to a collision. Denote by $S^N(\rho)$ the expected slot duration when N transmitters use the same rate adaptation strategy ρ. Also denote by $S^N_R(\rho)$ the expected slot duration in a system with $N+1$ transmitters using strategy ρ, and given that one transmitter sends a packet at rate R, which is (in case when RTS/CTS is not used):

$$S^N_R(\rho) = \sum_{r\in\mathcal{R}} Pr(\text{min TX rate of } N \text{ users is } r) \max(T_r, T_R).$$

When RTS/CTS is used, the expressions for $S^N(\rho), S^N_R(\rho)$ simplify to:

$$\begin{aligned} S^N(\rho) &= (1-p)^N + N(1-p)^{N-1} \sum_j \pi_j p_j T_{\rho(j)} \\ &\quad + \left(1-(1-p)^N\right) T_{RTS}, \\ S^N_R(\rho) &= (1-p)^N (T_R + T_{RTS}) + \left(1-(1-p)^N\right) T_{RTS}. \end{aligned}$$

When RTS/CTS is not used, the expressions are more complex, and given in the following proposition.

Proposition 1: Let $t(X) = 1 - \sum_{i\in X} \pi_i p_i$ be the probability that a node is not transmitting in any of the back-off stages in $X \subseteq \{0,\dots,I\}$. Then in the cases without RTS/CTS we have

$$\begin{aligned} S^N(\rho) &= (1-p)^N \\ &\quad + \sum_{r\in\mathcal{R}} \left(t(\{j:\rho(j)<r\})^N - t(\{j:\rho(j)\le r\})^N\right) T_r, \\ S^N_R(\rho) &= ((1-p)^N + t(\{j:\rho(j)<R\})^n) T_R \\ &\quad + \sum_{r<R} \left(t(\{j:\rho(j)<r\})^N - t(\{j:\rho(j)\le r\})^N\right) T_r \end{aligned}$$

The proof is in [20].

B. Link Throughput

From the average slot duration, we can compute the stationary throughput of a link using the similar analysis as that in [19]. The throughput is

$$\phi(\rho) = \frac{\sum_i \pi_i p_i (1-e_{\rho(i)})(1-c)}{S^N(\rho)}. \qquad (1)$$

C. Stationary distributions

To compute the link throughputs, we need to evaluate the stationary distribution π and the collision probability c. We compute these, first for WoLD systems, and then for WLDS and ROCE systems.

1) Without loss differentiation: Given that a transmitter is in stage i, it can either successfully transmit and move to state 0 with probability $p_i(1-c)(1-e_i)$, or experience of transmission failure with probability $p_i(1-(1-c)(1-e_{\rho(i)}))$ or remain idle with probability $1-p_i$. Then we classically deduce that:

$$\pi_i = \frac{p_0 \prod_{k=0}^{i-1}(1-(1-c)(1-e_{\rho(k)}))}{p_i}\pi_0, \text{ for } 0 < i < I,$$
$$\pi_I = \frac{p_0 \prod_{k=0}^{I-1}(1-(1-c)(1-e_{\rho(k)}))}{p_I(1-c)(1-e_{\rho(I)})}\pi_0.$$

2) With loss differentiation, WLDs: Given that a transmitter is in stage i, it can either move to stage $i+1$ if it encounters a collision with probability $p_i c$, or it can remain silent with probability $1-p_i$, or it can return to stage 0 with probability $p_i(1-c)$ (regardless of channel errors). We then have

$$\pi_i = \frac{p_0 c^i}{p_i}\pi_0, \text{ for } 0 < i < I; \qquad \pi_I = \frac{p_0 c^I}{p_I(1-c)}\pi_0.$$

3) With loss differentiation, ROCE: Given that a transmitter is in stage i, it can either move to stage $i+1$ if it encounters a collision with probability $p_i c$, or it can remain in stage i with probability $1-p_i+p_i(1-c)e_{\rho(i)}$ (either it remains silent or it encounters a channel error), or it can return to stage 0 with probability $p_i(1-c)(1-e_{\rho(i)})$. We have:

$$\pi_i = \frac{p_0 c^i}{p_i \prod_{k=1}^{i}(1-(1-c)e_{\rho(k)})}\pi_0, \text{ for } 0 < i < I,$$
$$\pi_I = \frac{p_0 c^I}{p_I(1-c)(1-e_{\rho(I)})\prod_{k=1}^{I-1}(1-(1-c)e_{\rho(k)})}\pi_0.$$

In each of the cases, we use the above expressions to obtain c for the system of N users. The collision probability is obtained as a fixed point of the following two expressions: First, $p = \sum_{i=0}^{I}\pi_i p_i$, and second $c = 1-(1-p)^{N-1}$. As in [19], it can be shown that the fixed point is unique.

V. Competition

Here, we study the system performance when users are competing with each other and our aim here is to determine whether a symmetric pure strategy Nash Equilibrium (NE) exists, and if it exists, then how to compute it. We also want to determine properties of the Nash Equilibrium so as to curtail the computational efforts required to compute it.

A. Analytical Results

First, we present the existence result of a symmetric pure NE for all the variations we have considered, viz., with or without RTS/CTS, and with or without loss differentiation, and their combinations. Existence of pure strategy NE is primarily studied when the number of players is large so that choice of one player's strategy does not affect the payoffs of the others. These games are referred to as non-atomic games. For precise mathematical definition, see [17]. In this section, we only consider a system with

RTS/CTS mechanism. We believe that all the results also hold when RTS/CTS is not used. For notational brevity, we assume that $e_R = 1 - e^{-\kappa R}$, i.e. $\gamma = 1$, but all the results hold for every $\gamma > 0$.

Proposition 2: If each user can select rate at every back-off stage from a closed interval $[0, R_{\max}]$, then in the non-atomic settings, a symmetric pure NE exists for all the variations of the system considered in this paper.

The proof is given in the appendix and it follows along the similar lines as that of the original proof by Nash (using Kakutani's fixed point theorem).

Though the existence is known, computing a symmetric pure NE is computationally expensive. Hence, we show certain properties a NE should satisfy in various cases, and wherever possible give an explicit procedure to compute the NE.

1) Properties of NE for WoLD and WLDs: First, we explain how the best response correspondence is obtained in these cases. Fix a user n, and let other users use rate adaptation ρ. Then the best response for n is computed using Markov decision process formulation, which allows n to determine rate adaptation strategy that minimizes the expected time to transmit a packet successfully given that other users use ρ. Let $J(i)$ denote the minimum expected time to transmit a packet successfully given that it is in i^{th} back-off stage. Then Bellman's equations for WoLD are as follows:

$$\begin{aligned} J(i) &= \frac{1-p_i}{p_i} S^{N-1}(\rho) \\ &\quad + \min_R \left\{ S_R^{N-1}(\rho) + (1-(1-c)e^{-\kappa R}) J(i+1) \right\}, \\ J(I) &= \min_R \left\{ \frac{\frac{1-p_I}{p_I} S^{N-1}(\rho) + S_R^{N-1}(\rho)}{(1-c)e^{-\kappa R}} \right\}. \end{aligned}$$

where $\frac{1-p_i}{p_i} S^{N-1}(\rho)$ is the average time the link is idle before transmitting for the first time, $S_{\rho(i)}^{N-1}(\rho)$ is the average time the link transmits (regardless of the success of the transmission) and if the transmission is unsuccessful (with probability $(1-(1-c)e^{-\kappa R})$ the average time to transmit the packet is $J(i+1, \rho)$ as the link moves to $i+1$. Note that the best response can be computed easily by back-tracking as $J(i)$ can be computed in a single step once $J(i+1)$ is known. Thus, $J(I)$ is computed first, the $J(I-1)$ and so on.

Now, in the case of WLDs, the best response is given as follows:

$$\begin{aligned} J(i) &= \min_R \left\{ \frac{\frac{1-p_i}{p_i} S^{N-1}(\rho) + cJ(i+1) + S_R^{N-1}(\rho)}{1-(1-c)(1-e^{-\kappa R})} \right\}, \\ J(I) &= \min_R \left\{ \frac{\frac{1-p_i}{p_i} S^{N-1}(\rho) + S_R^{N-1}(\rho)}{(1-c)e^{-\kappa R}} \right\}. \end{aligned}$$

Again the best response can be calculated easily by back-tracking.

Note that these relations provide the best response rate adaptation for a user only for the non-atomic game. This is because the choice of rate adaptation for a user n affects π and c, and thus in turn affects the average behavior of the system as perceived by n. In the above expressions, a key assumption is that the average system behavior does not change when n changes its rate adaptation. Using the above relations, we show the following result.

Proposition 3: Consider non-atomic setting. Let ρ be a strategy used by any user n in any symmetric NE. Then, $\rho(i) \geq \rho(i+1)$ for every i.

The proof is in the appendix. Intuitively, the expected time spent in stage i waiting for the first opportunity to transmit ($\frac{1-p_i}{p_i}S^{N-1}(\rho)$) is monotone increasing in i. Thus, the best response rate adaptation is discouraged to enter the higher stages. For the WoLD case, the user moves into a higher back-off stage w.p. $1-(1-c)e^{-\kappa R}$. In this, rate adaptation can affect only the second term. Thus, for a higher i, rate adaptation decreases rate $\rho(i)$ to decrease the probability of moving to $i+1$. Similarly, for WLDS, the user stays in the same back-off stage in case of a channel error. Thus, here as well, for a larger i, rate adaptation chooses smaller rate so as to increase its chances of successful transmission given that nobody else is simultaneously transmitting.

B. Computing Symmetric Pure Strategy NE for ROCE

The best response correspondence in this case when other users use rate adaptation strategy ρ is given as follows. In this section, we do not need to assume non atomic game.

$$
\begin{aligned}
J(i) &= \frac{1-p_i}{p_i}S^{N-1}(\rho) + cJ(i+1) \\
&\quad + \min_R \left\{ S_R^{N-1}(\rho) + (1-c)(1-e^{-\kappa R})J(0) \right\}, \\
J(I) &= \frac{1-p_i}{p_i(1-c)}S^{N-1}(\rho) \\
&\quad + \frac{1}{1-c}\min_R \left\{ S_R^{N-1}(\rho) + (1-c)(1-e^{-\kappa R})J(0) \right\}.
\end{aligned}
$$

Contrary to the previous section, these relations provide the best response rate adaptation for user n even when the number of users is finite (we do not need non-atomicity). This is because here π and c do not depend on rate adaptation strategies of the users. Thus, given (ρ, π, c), the average behavior of the system as perceived by n does not change even when it changes its rate adaptation strategy. Next, we can show the following property of the best response correspondence.

Proposition 4: Let $\hat{\rho}$ be a rate adaptation strategy used by all the users in the system, and let ρ be a rate adaptation strategy obtained from the best response correspondence. Then, ρ is constant ($\rho_i = \rho_{i+1}$ for every i.)

The proposition immediately follows by observing that the function that is optimized over R is the same for every i, hence assuming all users use $\hat{\rho}$ and user n uses ρ, the throughput of the system is

$$
\begin{aligned}
&\phi(\rho, \hat{\rho}) = \\
&\frac{p(1-c)(1-e_\rho)}{(1-p)(1-c) + p(1-c)T_\rho + (1-p)cT_{\hat{\rho}} + pc\max\{T_\rho, T_{\hat{\rho}}\}}.
\end{aligned}
$$

and the best response for user n is simply $B(\hat{\rho}) = \mathrm{argmax}_{\rho \in \mathbb{R}}\, \phi(\rho, \hat{\rho})$.

Note that since we also allow for finite number users, Proposition 2 does not guarantee the existence of symmetric pure NE. But, in this case, we are able to show that symmetric pure NE exists and we also propose the following procedure to compute it. The main assumption we make is that all the users update their rate adaptation strategies synchronously based on the above best response correspondence. We start with an initial state, where each user has rate adaptation $\rho^{(0)}$. Since the system is symmetric, the best response obtained by each user is the same. Thus, all of them synchronously change their rate adaptation from $\rho^{(0)}$ to $\rho^{(1)}$, and the procedure continues in the same fashion. For this iterative procedure, we make the following claim.

Proposition 5: Let $\rho^{(0)} = R_{\max}\mathbf{1}$. Then, $\lim_{m\to\infty} \rho^{(m)} = \rho^*$ such that ρ^* is a symmetric pure NE.

The proof is in the appendix. In the key step in the proof of above result, we show that $\rho^{(m)}$ is a monotone decreasing sequence. Intuitive explanation for this phenomenon is as follows. Clearly, $\rho^{(1)} \leq \rho^{(0)}$. When the rate decreases, time required to transmit the packet increases, which in turn results in increase in $J(i)$ for all i. As $J(i)$ increases, best response chooses a rate adaptation that reaches termination state (successful packet transmission) with a higher probability. This can be achieved only by reducing the rates as π and c are independent of the rate adaptation strategy. Thus, $\rho^{(2)}$ will be smaller than $\rho^{(1)}$, and this continues.

VI. Cooperation

Here, we derive key properties of the socially optimal rate adaptation strategy, and present its distributed implementation.

A. Optimal rate adaptation for ROCE

We first show that the optimal rate adaptation has to be non-decreasing.

Proposition 6: Let ρ be the socially optimal rate adaptation strategy. Then, in absence of RTS/CTS, for every $i < I$, $\rho(i) \leq \rho(i+1)$; and with RTS/CTS, $\rho(i) = \rho(i+1)$.

The proof of the above proposition is presented in appendix. The intuition behind this result is as follows: the reward for successful transmission in state i is proportional to $\pi_i p_i$. The probability that a slot will last $T_{\rho(i)}$ is proportional to different collision probabilities and it increases faster with i than $\pi_i p_i$.

Using this result, we can derive a gradient-descent algorithm to enable transmitters to identify the optimal rate adaptation strategy in a distributed manner (in the sense that all variables can be measured locally). Here we assume for simplicity that the set of rates is continuous. The algorithm is described in the following proposition where the update of the rate adaptation strategy from step m to step $m+1$ is described through the duration of a packet transmission in any given stage (these durations uniquely define the rate adaptation strategy). The algorithm classically uses at step m a step size ϵ_m, where $\epsilon_m \to 0$ as $m \to \infty$ and $\sum_{m=0^\infty} \epsilon = \infty$. For convenience, we denote the error function e as follows: $e(T_{\rho(i)}) = e_{\rho(i)}$.

Proposition 7: Assume that at step $m+1$, all transmitters update their rate adaptation strategy from ρ_m to ρ_{m+1} as:

$$\begin{aligned} T_{\rho_{m+1}(i)} = T_{\rho_m(i)} - \epsilon_m \pi_i p_i (1-c) e'(T_{\rho_m(i)}) \\ - \epsilon_m (t(\{k : k \leq i-1\})^N - t(\{k : k \leq i\})^N)\phi(\rho_m). \end{aligned} \tag{2}$$

where $t(\{k : k \leq i\}) = 1 - \sum_{k\leq i} \pi_k p_k$ and $e'(T) = \frac{\partial e}{\partial T}$. Then, when m grows large, ρ_m converges to a socially optimal rate adaptation strategy.

The proof is in the appendix. Notice that $\pi_i p_i$ is the fraction of time a transmitter transmits in back-off stage i, c is the collision probability for a packet and $t(\{k : k \leq i-1\})^N - t(\{k : k \leq i\})^N$ is the fraction of slots that last $T_{\rho(i)}$. All of these can easily be measured locally hence (2) can be evaluated locally at a transmitter. Similarly, we can deduce an update algorithm for the case with RTS/CTS.

Proposition 8: Assume that at step $m+1$, all transmitters update their rate adaptation strategy from ρ_m to ρ_{m+1} as:

$$\begin{aligned} T_{\rho_{m+1}(i)} = T_{\rho_m(i)} - \epsilon_m \pi_i p_i (1-c) e'(T_{\rho_m(i)}) \\ - \epsilon_m N (1-p)^{N-1} \pi_i p_i \phi(\rho_m). \end{aligned} \tag{3}$$

Then, when m grows large, ρ_m converges to the socially optimal rate adaptation strategy.

The proof of this proposition is similar to that of the previous proposition, so we omit it. Notice that here as well all the quantities involved in the update can be calculated locally. In particular $N(1-p)^{N-1}\pi_i p_i$ is the fraction of slots with transmissions without collisions.

B. Optimal rate adaptation for WoLD

The major difficulty for systems without loss differentiation is that the stationary probability π depends on ρ. The problem becomes hard, if not intractable. We do not have explicit solution in a general case. However, if only constant strategies are allowed, then we obtain the socially optimal strategy. In Section VII, we evaluate a socially optimal strategy for the general case using numerical computations. Let us denote by $e = e_\rho$. We have:

Proposition 9: The optimal constant rate adaptation strategy ρ in absence of RTS/CTS satisfies:

$$\begin{aligned}\frac{\partial \pi_0}{\partial e} p_0 S^N(\rho) + N(1-p)^{N-1}(1-T)\frac{\partial p}{\partial e}\pi_0 p_0 \\ -(1-(1-p)^N)\frac{\partial T}{\partial e}\pi_0 p_0 = 0,\end{aligned} \tag{4}$$

where

$$\frac{\partial \pi_0}{\partial e} = -\frac{2(1-c)^2(1-e)}{(1-c)(1-e)+(N-1)\frac{p_0}{1-p}}, \tag{5}$$

$$\frac{\partial p}{\partial e} = -\frac{p_0}{(1-c)(1-e)+(N-1)\frac{p_0}{1-p}}\frac{1}{1-e}. \tag{6}$$

The proof of the proposition is in the appendix. Uniqueness of the solution for (4) is difficult to establish. Numerical computations, however, point toward unique solution. Similarly, for systems with RTS/CTS we have:

Proposition 10: The optimal constant rate adaptation ρ with RTS/CTS satisfies:

$$\begin{aligned}0 = \frac{\partial \pi_0}{\partial e} p_0 S^N(\rho) - Np(1-p)^{N-1}\frac{\partial T}{\partial e}\pi_0 p_0 + N\frac{\partial p}{\partial e}\times \\ \times(1-p)^{N-2}((1-p)(1-T_{RTS}) + (1-2p+Np)T)\pi_0 p_0.\end{aligned} \tag{7}$$

The proof is similar to that of Proposition 9, and is omitted.

C. Optimal rate adaptation for WLDS

Like WoLD systems, these systems are also difficult to analyze. Hence, here as well we only consider constant rate adaptation strategies. We obtain the same results as in Propositions 9 and 10, but need to substitute the following derivatives:

$$\begin{aligned}\frac{\partial \pi_0}{\partial e} &= -\frac{2c}{(1-c)(1-e)+(N-1)\frac{p_0}{1-p}}\frac{1}{1-e}, \\ \frac{\partial p}{\partial e} &= -\frac{p_0 c}{(1-c)(1-e)+(N-1)\frac{p_0}{1-p}}\frac{1}{1-e}.\end{aligned} \tag{8}$$

VII. NUMERICAL RESULTS

In this section we present a numerical analysis of the rate adaptation algorithms previously discussed. In order to obtain realistic results, we take the error probability function $e(R, SNR)$ from the results of 802.11a measurements presented in [21, Figure 2]. We fix the number of back-off stages to 8 ($I = 7$) and we take the standard RTS/CTS signaling parameters to calculate T_{RTS}. We use brute-force approach and explore the full state space to find the social optimum of the cooperative approach where it cannot be found otherwise. We iterate over the best response to find the Nash equilibria of competitive cases. Note that we observe numerically that this process always converges, even in the cases where we cannot prove it analytically. Thus we are able to obtain the numerical results for all the cases analyzed in the paper. Also note that for the numerical calculations we do not need any a priori assumptions on the error probability $e(R, SNR)$. However, we verify that in most of the cases the measurements from [21, Figure 2] can be well fitted with a function of the form of $1 - \gamma e^{\kappa R}$.

A. Social optima

We start by analyzing different social optima that can be achieved in different cases discussed in the paper. They are depicted in Figure 1 for SNR $= 10$ dB. As one can see, the differences among the social optima are very small. We verify that the same conclusion holds for various SNR values. Hence, it becomes irrelevant which protocol one chooses, and in particular whether a loss differentiation capability is available.

This is an apparent paradox with respect to [5], [6], [7], [8], [9], [10] which show that a collision-aware rate adaptation improves performance. There, the major reason for such improvement stems from the improved channel estimation. Our results show there is almost no impact of collision awareness on scheduling. We see that in the WoLD case, if the probability of a channel error is large, this will reduce the collision probability leading to a similar steady-state transmission probability p to the same level as in WLDS or ROCE.

We also verify that the use of RTS/CTS almost always decreases the performance of the network, as expected by a common wisdom, because the RTS/CTS packets are sent with the lowest rate and incur more overhead than benefits.

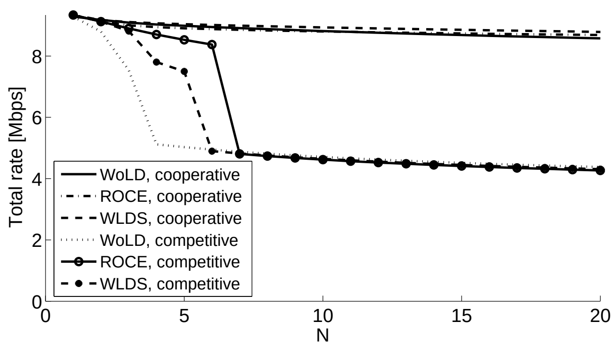


Fig. 1. The rates achieved in different cases, without RTS/CTS, when SNR= 10 dB. On the x axis is the number of users N and on the y axis is the sum of rates of all users.

Next, we look at the optimal rate adaptation strategy ρ. As already discussed in Section VI, the optimal $\rho(i)$ is not constant in general. We illustrate the shape of $\rho(i)$ in Figure 2. Restricting to the set of constant rate allocations one can loose up to 10% of the rate, as shown in Figure 3.

Type	$\rho(i)$
WoLD, cooperative	36, 36, 36, 48, 48, 54, 54
WLDS, cooperative	36, 36, 36, 48, 54, 54, 54
ROCE, cooperative	36, 36, 36, 36, 36, 36, 36
WoLD, competitive	24, 18, 18, 18, 18, 18, 18
WLDS, competitive	24, 24, 24, 18, 18, 18, 18
ROCE, competitive	24, 24, 24, 24, 24, 24, 24

Fig. 2. The socially optimal rate selections for different cases, for a network with $N = 10$ users, SNR$=20$ dB, without RTS/CTS.

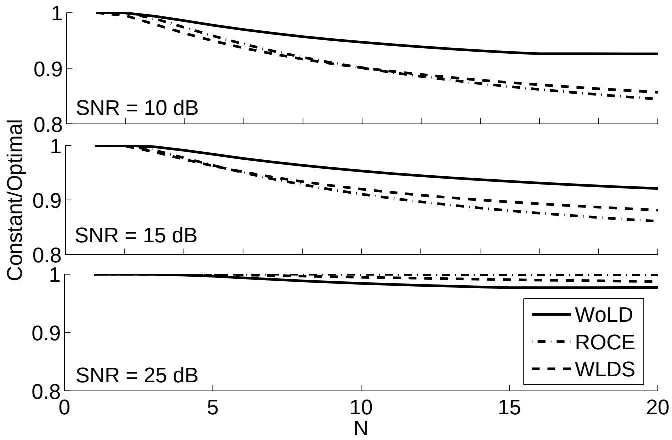


Fig. 3. The ratio of the constant rate social optimum over the global social optimum, in absence of RTS/CTS.

B. Price of anarchy

We next look at the price of anarchy in different scenarios. The numerical results are depicted in Figure 4. Already for $N > 2$ users the price of anarchy is significant, rapidly increasing to up to 50% of the social optimum. Figure 2 gives the optimal ρ. We see that ROCE consistently has smaller price of anarchy than the other two protocols. Moreover, in many cases of networks with less than 6 nodes it completely eliminates the price of anarchy. Since the three protocols do not differ much in terms of social optima, we can conclude that ROCE is the preferred protocol. We also see from Figure 4 that RTS/CTS does not decrease the price of anarchy. On the contrary, for networks of size $N = 4$ to $N = 8$ it may lead to a significant price of anarchy in cases where it otherwise would not exist.

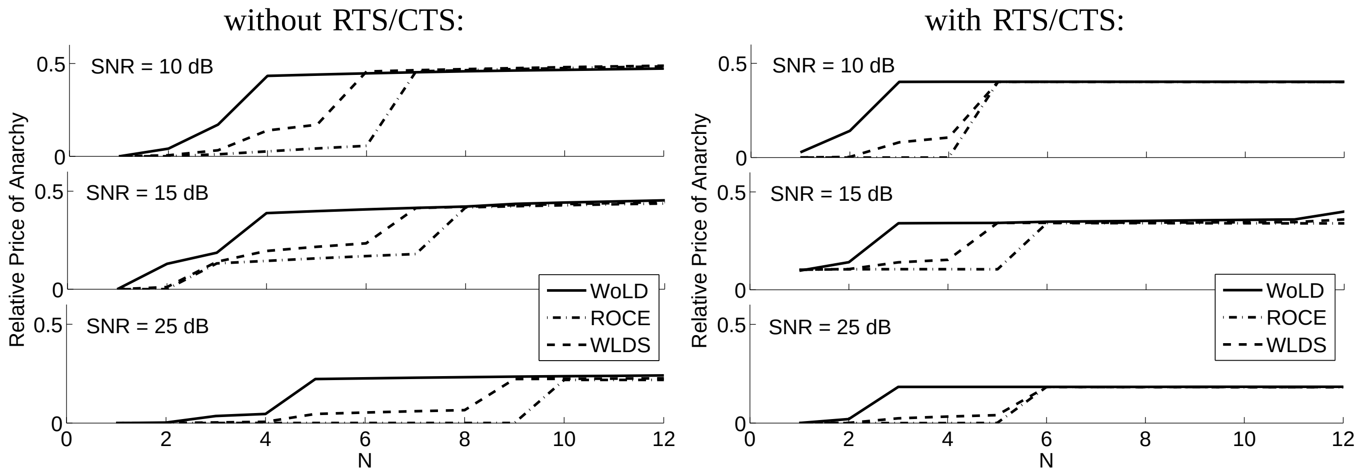


Fig. 4. Relative price of anarchy for different types of games: on the y axis is the difference between the cooperative and the competitive optima, divided by the cooperative optimum. On the x axis is the number of nodes in the network.

VIII. Conclusions

We analyze the optimal rate adaptation strategy given a fixed scheduling protocol in both competitive and collaborative scenarios. We consider protocols with and without RTS/CTS and loss differentiation. We give a generic analytical model for symmetric networks that encompasses all variations. We show that price of anarchy exists and it is significant. We also show that our modification of collision-aware medium-access protocol *ROCE* exhibits the smallest price of anarchy among other proposed schemes. We provide a local algorithm that converges to the social optimum and we show that the social optimum does not depend on loss differentiation capabilities nor on use of RTS/CTS. In future we plan to extend this work to incorporate channel estimation and heterogeneous scenarios.

References

[1] G. Holland, N. Vaidya, and V. Bahl, "A rate-adaptive MAC protocol for multihop wireless networks," in *Proceeding of ACM Mobicom*, 2001.

[2] B. Sadeghi, V. Kanodia, A. Sabharwal, and E. Knighlty, "Opportunistic media access for multirate ad hoc networks," in *Proceeding of ACM Mobicom*, 2002.

[3] M. Lacage, M. H. Manshaei, and T. Turletti, "IEEE 802.11 rate adaptation: a practical approach," in *Proceeding of ACM MSWiM*, 2004.

[4] J. Bicket, "Bit-rate selection in wireless networks," *Master thesis, MIT*, 2005.

[5] Q. Pang, V. Leung, and S. Liew, "Improvement of WLAN contention resolution by loss differentiation," *IEEE Trans. on Wireless Communications*, vol. 5-12, pp. 3605–3615, 2006.

[6] J. Kim, S. Kim, S. Choi, , and D. Qiao, "CARA: Collision-aware rate adaptation for IEEE 802.11 WLANs," in *Proceeding of IEEE Infocom*, 2006.

[7] S. H. Y. Wong, H. Yang, S. Lu, and V. Bharghavan, "Robust rate adaptationfor 802.11 wireless networks," in *Proceeding of ACM Mobicom*, 2006.

[8] J. Choi, J. Na, K. Park, and C.-K. Kim, "Adaptive optimization of rate adaptation algorithms in multi-rate WLANs," in *Proceeding of IEEE ICNP*, 2007.

[9] C. Chen, H. Luo, E. Seo, N. H. Vaidya, and X. Wang, "Rate-adaptive framing for interfered wireless networks," in *Proceeding of IEEE Infocom*, 2007.

[10] P. Acharya, A. Sharma, E. Belding, K. Almeroth, and K. Papagiannaki, "Congestion-aware rate adaptation in wireless networks: A measurement-driven approach," in *Proceedings of IEEE SECON*, 2008.

[11] J. Choi, K. Park, and C.-K. Kim, "Cross-layer analysis of rate adaptation, DCF, and TCP in multi-rate WLANs," in *Proceeding of IEEE Infocom*, 2006.

[12] A. Singh and D. Starobinski, "A semi-Markov-based analysis of rate adaptation algorithms in WLANs," in *Proceeding of IEEE SECON*, 2007.

[13] L. Chen and J. Leneutre, "On the power and rate control in ieee 802.11 wlans - a game theoretical approach," in *Proceeding of ICCCN*, 2008, pp. 450–456.

[14] ——, "Selfishness, not always a nightmare: Modeling selfish mac behaviors in wireless mobile ad hoc networks," in *Proceeding of ICDCS*, 2007.

[15] L. Chen, T. Cui, S. H. Low, and J. C. Doyle, "A game-theoretic model for medium access control," in *Proceedings of International Wireless Internet Conference (WICON)*, Austin, TX, Oct. 2007. [Online]. Available: http://netlab.caltech.edu/publications/rag07.pdf

[16] J. Nash, "Non-cooperative games," *Annals of Mathematics*, vol. 2, no. 54, pp. 286–295, 1951.

[17] D. Schmeidler, "Equilibrium points of non-atomic games," *Journal of Statistical Physics*, vol. 4, pp. 295–300, 1973.

[18] A.-F. Cheng, D. Reeves, Y. Vorobeychik, and M. Wellman, "Notes on equilibria in symmetric games," in *AAMAS-04 Workshop on Game Theory and Decision Theory*, 2004.

[19] G. Bianchi, "Performance analysis of the IEEE 802.11 distributed coordination function," *IEEE Journal on Selected Areas in Communications*, vol. 18-3, pp. 535–547, 2000.

[20] P. Chaporkar, A. Proutiere, and B. Radunovic, "Rate adaptation games in wireless lans: Nash equilibrium and price of anarchy," http://research.microsoft.com/users/cambridge/bozidar/Publications/Infocom09-TR.pdf, Tech. Rep., 2008.

[21] J. Zhang, K. Tan, J. Zhao, H. Wu, and Y. Zhang, "SNR-guided rate adaptation for IEEE 802.11 wireless networks," in *Proceeding of IEEE Infocom*, 2008.

Appendix I
Proofs of the results in Section IV

Proof of Proposition 1: We have

$$S^n(\rho) = (1-p)^n + \sum_{R\in\mathbb{R}} Pr(\text{at least one TX with } R \wedge \text{none TX with } [0,R-1])T_R.$$

since we know that a duration of a slot is T_R when there is at least one node transmitting with rate R and no nodes transmit with rates $\{0,\dots,R-1\}$. Furthermore

$$\begin{aligned}
&Pr(\text{at least one TX with } R \wedge \text{none TX with } [0,R-1]) = \\
&= Pr(\text{no node TX with } [0,R-1]) \\
&- Pr(\text{no node TX with } [0,R]) \\
&= (t(\{j : \rho(j) < r\})^n - t(\{j : \rho(j) \le r\})^n)
\end{aligned}$$

Similarly we have

$$\begin{aligned}
S_i(\rho) &= \left((1-p)^N + Pr(\text{no node TX with } \{r : r < R\})\right) T_R \\
&+ \sum_{r<R} Pr(\text{at least one TX in } j \wedge \text{none TX in } [j+1,I])T_j.
\end{aligned}$$

and the second relation immediately follows. □

Appendix II
Proofs of the results in Section V

Let all users use strategy ρ, and let $\hat{\rho}$ be the best response of user n. Let π and c denote the steady state distribution and the collision probability, respectively, corresponding to the profile ρ. First, for analysis, let us write the best response MDP in the following way for WoLD, WLDS and ROCE systems. Let $\xi_i(\pi,c) = \frac{1-p_i}{p_i} S^{N-1}(\rho) + T_{RTS}$. For WoLD systems:

$$\begin{aligned}
J(i) &= \min_R \left\{ \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})J(i+1) \right\} + \xi_i(\pi,c), \\
J(I) &= \min_R \left\{ \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})J(I) \right\} + \xi_I(\pi,c).
\end{aligned}$$

Next, for the WLDS systems:

$$\begin{aligned}
J(i) &= \min_R \left\{ \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})J(i) \right\} + \xi_i(\pi,c) + c[J(i+1) - J(i)] \\
J(I) &= \min_R \left\{ \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})J(I) \right\} + \xi_I(\pi,c).
\end{aligned}$$

Finally, for ROCE systems:

$$\begin{aligned}
J(i) &= \min_R \left\{ \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})J(0) \right\} + \xi_i(\pi,c) + c[J(i+1) - J(0)] \\
J(I) &= \min_R \left\{ \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})J(0) \right\} + \xi_I(\pi,c) + c[J(I) - J(0)].
\end{aligned}$$

Now, consider the function $G(R,u) = \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})u$. Note that if $i < I$, then (a) for WLDS systems $\hat{\rho}(i) = \arg\min_R G(R, J(i+1))$ when $i < I$, and $\hat{\rho}(I) = \arg\min_R G(R, J(I))$, (b) for WoLD systems $\hat{\rho}(i) = \arg\min_R G_1(R, J(i))$ for every i, and (c) for ROCE systems $\hat{\rho}(i) = \arg\min_R G_1(R, J(i))$ for every i. Thus, obtaining insights into the properties of the function $G(\cdot,\cdot)$ is useful for obtaining the properties of the best response. Thus, first we obtain some properties of the function $G(R,u)$, and then, using these properties, we prove Theorems in Section V.

A. Supporting Lemmas

Lemma 1: Consider the function $G(R,u)$. We show that
1)If $\log\left(\frac{\sigma}{u\kappa}\right) > 2\left[\log\left(\frac{2}{\kappa}\right) - 1\right]$, i.e. $u < \frac{\sigma\kappa e^2}{4}$, then $G'(R,u) \stackrel{\text{def}}{=} \frac{\partial G(R,u)}{\partial R} = 0$ has no solution in $[0,\infty)$. Moreover, $G(R,u)$ is strictly decreasing.
2)If $\log\left(\frac{\sigma}{u\kappa}\right) = 2\left[\log\left(\frac{2}{\kappa}\right) - 1\right]$, i.e. $u = \frac{\sigma\kappa e^2}{4}$, then $G'(R,u) = 0$ has a unique solution in $[0,\infty)$. However, the solution is only a saddle point and $G(R,u)$ is monotone decreasing.
3)If $\log\left(\frac{\sigma}{u\kappa}\right) < 2\left[\log\left(\frac{2}{\kappa}\right) - 1\right]$, i.e. $u > \frac{\sigma\kappa e^2}{4}$, then $G'(R,u) = 0$ has two solutions, say $R_1(u)$ and $R_2(u)$ with $R_1(u) < R_2(u)$, in $[0,\infty)$. Here, $R_1(u)$ ($R_2(u)$, resp.) is a local minima (maxima, resp.) of $G(R,u)$. Moreover, $[R_1(u), R_2(u)] \in [R_1(u_1), R_2(u_1)]$ whenever $u_1 > u$.

Proof: First, we note that

$$G'(R,u) = (1-c)u\kappa e^{-\kappa R} - \frac{(1-c)\sigma}{R^2}.$$

Thus, $G'(R,u) = 0$ implies that

$$2\log(R) - \kappa R = \log\left(\frac{\sigma}{u\kappa}\right). \tag{9}$$

We consider the function $F(R) \stackrel{\text{def}}{=} 2\log(R) - \kappa R$, and note that

$$F'(R) = \frac{2}{R} - \kappa \tag{10}$$

$$F''(R) = \frac{-2}{R^2}. \tag{11}$$

Relation (11) shows that $F(R)$ is a concave function in the positive half plane, and then (10) shows that

$$F_{\max} \stackrel{\text{def}}{=} \max_{R\in[0,\infty)} F(R) = 2\left[\log\left(\frac{2}{\kappa}\right) - 1\right]. \tag{12}$$

We also note that $F(0) = F(\infty) = -\infty$. Thus, (9) and (12) along with concavity of $F(R)$ imply that
1) when $\log\left(\frac{\sigma}{u\kappa}\right) > F_{\max}$, then $G'(R,u) = 0$ does not have any root. Moreover, $G'(R,u) < 0$ for every $R \in [0,\infty)$. This proves the first statement of the lemma.
2) when $\log\left(\frac{\sigma}{u\kappa}\right) = F_{\max}$, then $G'(R,u) = 0$ has exactly one root, say $R_1(u)$. Moreover, $G'(R,u) \le 0$ for every $R \in [0,\infty)$ with equality only at $R_1(u)$. This proves the second statement of the lemma.
3)when $\log\left(\frac{\sigma}{u\kappa}\right) < F_{\max}$, then $G'(R,u) = 0$ has exactly two roots, say $R_1(u)$ and $R_2(u)$ with $R_1 < R_2$. Note that $G'(R,u) < 0$ for $R \in [0, R_1(u)) \cup (R_2(u), \infty)$, and $G'(R,u) > 0$ for $R \in (R_1(u), R_2(u))$. Thus, the last statement of the lemma follows. Moreover, from concavity of $F(R)$ along with its boundary conditions and the fact that $\log\left(\frac{\sigma}{u\kappa}\right)$ decreases as u increases, clearly imply that $[R_1(u), R_2(u)] \subset [R_1(u_1), R_2(u_1)]$ whenever $u_1 > u$. This concludes the proof. □

The function $F(R)$ is shown in Figure 5. Note that the points where horizontal lines intersect the curve

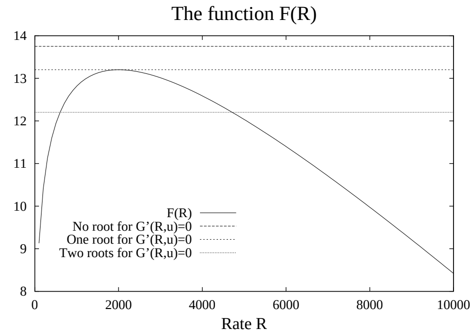
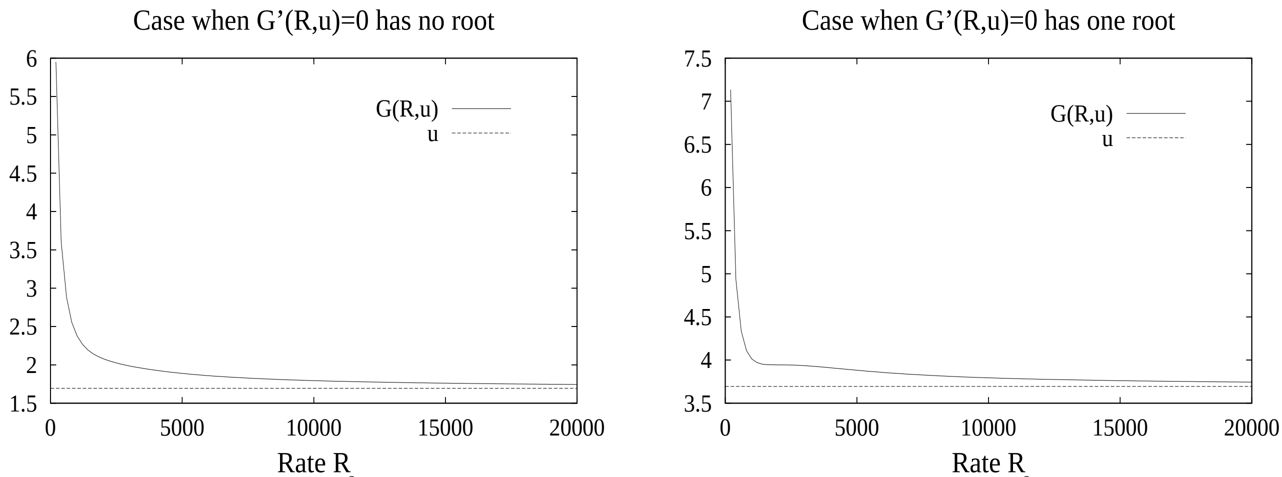


Fig. 5. The function $F(R)$ is shown. The horizontal lines correspond to various values of $\log\left(\frac{\sigma}{u\kappa}\right)$. Specifically, the top most line corresponds to the first case in Lemma 1, i.e., we chose $u < \frac{\sigma\kappa e^2}{4}$. The middle line corresponds to the second case in Lemma 1, i.e., we chose $u = \frac{\sigma\kappa e^2}{4}$. Finally, the bottom most line corresponds to the third case in Lemma 1, i.e., we chose $u > \frac{\sigma\kappa e^2}{4}$.



(a) $u < \frac{\sigma\kappa e^2}{4}$ (b) $u = \frac{\sigma\kappa e^2}{4}$

Fig. 6. The function $G(R,u)$ is monotone decreasing for $u \le \frac{\sigma\kappa e^2}{4}$. For $u < \frac{\sigma\kappa e^2}{4}$, from case 2 of Lemma 1, $G(R,u)$ has a saddle point as shown in the figure (b).

of the function $F(R)$ are the roots of $G'(R,u) = 0$.

Let $R(u) = \arg\min_{R\in[0,R_{\max}]} G(R,u)$. Then, we deduce following results from above lemma.

Corollary 1: $R_{\max} = R(u)$. for every $u \in \left[0, \frac{\sigma\kappa e^2}{4}\right]$.

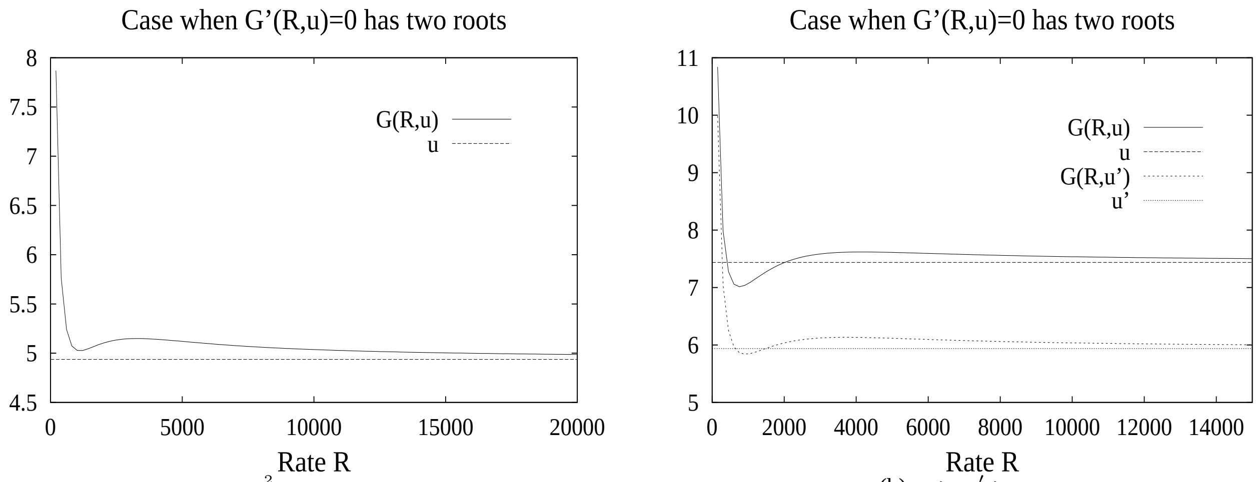
Proof: From the first two statements of Lemma 1, the function $G(R,u)$ is monotone decreasing for every u in the given range. Thus, the result follows. $\square$

Refer to Figure 6 for illustrations.

Corollary 2: For every $u > \frac{\sigma\kappa e^2}{4}$, then $R(u) \in \{R_1(u), R_{\max}\}$. Specifically, if $R_{\max} < R_1(u)$, then $R(u) = R_{\max}$; otherwise $R(u) = \arg\min_{R\in\{R_1(u),R_{\max}\}} G(R,u)$.

Proof: The proof follows from the third statement of Lemma 1. Note that the function is monotone decreasing until $R(u)$. Thus, if $R_{\max} < R_1(u)$, then the minimum is achieved at $R_{\max}$. Now, we also note that the function is monotone increasing in $(R_1(u), R_2(u))$, and it decreases monotonically after $R_2(u)$. Thus, $\min_{R\in[0,R_{\max}]} G(R,u) = \min_{R\in\{R_1(u),R_{\max}\}} G(R,u)$. $\square$

Corollary 3: For every $u > \kappa\sigma e$, then $R(u) = \min\{R_1(u), R_{\max}\}$.



(a) $\frac{\sigma\kappa e^2}{4} < u \le \kappa\sigma e$ (b) $u > u' > \kappa\sigma e$

Fig. 7. The function $G(R,u)$ has a local minima and maxima at $R_1(u)$ and $R_2(u)$, respectively. When $\frac{\sigma\kappa e^2}{4} < u \le \kappa\sigma e$ (figure (a)), $R_1(u)$ is not the global minima in the range $[0,\infty)$. The global minima occurs at $R=\infty$. But, when $u > \kappa\sigma e$ (figure (b)), $R_1(u)$ is the global minima. Moreover, as u increases $G(R,u)$ increases at every R and $R_1(u)$ decreases.

Proof: The result follows from the fact that if $u > \kappa\sigma e$, then $\min_{R\in[0,\infty)} G(R,u) < u$. We prove this using contradiction. Fix $u > \kappa\sigma e$ and let $\min_{R\in[0,\infty)} G(R,u) > u$. Then, for every $R \in [0,\infty)$

$$\begin{aligned} u &\le \min_{R\in[0,\infty)} \left\{ \frac{(1-c)\sigma}{R} + (1-(1-c)e^{-\kappa R})u \right\} \\ \Rightarrow u &\le \frac{\sigma e^{\kappa R}}{R}. \end{aligned}$$

Now, choose $R = \frac{1}{\kappa}$. Clearly, $\frac{1}{\kappa} \in [0,\infty)$. But, for this value of R, the above inequality becomes $u \le \kappa\sigma e$, which provides the required contradiction. □

For illustrations, refer to Figure 7.

Lemma 2: $R(u)$ is a monotone decreasing function of u.

Proof: By Corollary 1, it suffices to focus on $u > \frac{\sigma\kappa e^2}{4}$. Here, $R(u) \in \{R_1(u), R_{\max}\}$ by Corollary 2. Now, we show that if for some u $R(u) = R_1(u)$, then $R(u_1) = R_1(u_1)$ for every $u_1 > u$. Clearly, since $R(u) = R_1(u)$, $R_1(u) < R_{\max}$. By the third statement of Lemma 1, $R_1(u)$ is a monotone decreasing function of u. Thus, for every $u_1 > u$, $R_1(u_1) < R_{\max}$ (see Corollary 2). Thus, to show $R(u_1) = R_1(u_1)$, we need to show that $G(R_{\max}, u_1) - G(R_1(u_1), u_1) > 0$. First, note that $G(R,u_1) - G(R,u) = (1-(1-c)e^{-\kappa R})[u_1 - u]$. Thus, $G(R,u_1) - G(R,u)$ is monotone increasing function of R, and hence achieves the maximum at $R_{\max}$. Now, we claim the following:

$$G(R_1(u_1), u_1) - G(R(u), u) < G(R(u), u_1) - G(R(u), u).$$

The above relation holds as $[R_1(u), R_2(u)] \subset [R_1(u_1), R_2(u_1)]$ from Lemma 1, and since the function $G(R,u_1)$ is monotone increasing in $(R_1(u_1), R_2(u_1))$. Now, from the above expression we can conclude that

$$\begin{aligned} &G(R_1(u_1), u_1) - G(R(u), u) < G(R_{\max}, u_1) - G(R_{\max}, u) \\ &\Rightarrow G(R_{\max}, u) - G(R(u), u) < G(R_{\max}, u_1) - G(R_1(u_1), u_1) \\ &\Rightarrow G(R_{\max}, u_1) - G(R_1(u_1), u_1) > 0. \end{aligned}$$

This concludes the proof. $\square$

Let $R(u) = \arg\min_R G(R,u)$ and $\hat{R}(u) = \arg\min_R G_1(R,u)$.

Lemma 3: The functions $G(R(u),u)$ and $G_1(\hat{R}(u),u)$ are monotone increasing in u.

Proof: We prove the required for $G(\cdot,\cdot)$. The proof for $G_1(\cdot,\cdot)$ follows from the similar arguments. We note that $G(R,u) \leq G(R,u_1)$ whenever $u < u_1$ for every R. Thus, $G(R(u),u) \leq G(R(u_1),u) \leq G(R(u_1),u_1)$. This concludes the proof. $\square$

We note that from the practical perspective, we only need to consider the case with $u > \kappa\sigma e$. This is because when $u \leq \kappa\sigma e$, then the minimum value of the function $G(R,u)$ in the positive half plane is u, and it is achieved at $R = \infty$. Since, with the proper choice of u, $\arg\min_R G(R,u)$ is the rate chosen by the best response rate adaptation policy at every back-off stage. Thus, $\arg\min_R G(R,u) = \infty$ at certain back-off stage k implies that it is suboptimal to waste time in transmitting the packet, rather it is optimal to move to the next stage as soon as possible. Specifically, in back-off stage k, the user waits for its turn to transmit, but when the turn comes, it transmits at ∞ rate. The transmission at ∞ rate is guaranteed to fail as the probability of transmission error is 1. This process continues until the user moves to the subsequent stages. Thus, the throughput of the user can be improved by eliminating such back-off stages entirely as it saves the time that user spend in these back-off stages. So, we assume that the system operates in the regime that has $u > \kappa\sigma e$. Note that in this regime, $\arg\min_{R\in[0,R_{\max}]} G(R,u)$ is unique (see Corollary 2). We need this assumption explicitly for the proof of Theorem 2, remaining results hold even otherwise.

Lemma 4: Fix any given ρ, and let (π, c) denote the steady state probability and the collision probability, respectively, when all the users use rate adaptation profile ρ. Then, $J(i) \leq J(i+1)$ for every $i \geq 0$ in WoLD and WLDS system.

Proof: We focus of WoLD systems. The proof for WLDS systems follows similarly. With some abuse of notation, let us define

$$J(i,R) = \left\{(1-c)T_R + (1-(1-c)e^{-\kappa R})J(i+1)\right\} + \xi_i(\pi,c),$$
$$J(I,R) = \left\{(1-c)T_R + (1-(1-c)e^{-\kappa R})J(I)\right\} + \xi_I(\pi,c),$$

Thus, $J(i) = \min_{R\in[0,R_{\max}]} J(i,R)$.

We prove the required by induction. Let R_I be such that $J(I,R_I) = J(I)$. Then, we know that

$$J(I,R_I) = (1-c)T_{R_I} + (1-(1-c)e^{-\kappa R_I})J(I,R_I) + \xi_I(\pi,c).$$

Now, consider the following:

$$J(I,R_I) - J(I-1,R_I) = \xi_I(\pi,c) - \xi_{I-1}(\pi,c).$$

Since $p_I \leq p_{I-1}$, $\xi_I(\pi,c) \geq \xi_{I-1}(\pi,c)$. Thus, we conclude that $J(I,R_I) \geq J(I-1,R_I)$. Now, observe that $J(I) = J(I,R_I) \geq J(I-1,R_I) \geq J(I-1)$. By induction hypothesis, let $J(i-1) \leq J(i)$ for every $i \geq j$. Moreover, let R_j satisfy $J(j,R_j) = J(j)$. Now, consider

$$\begin{aligned} & J(j,R_j) - J(j-1,R_j) \\ = \; & (1-(1-c)e^{-\kappa R_j})(J(j+1,R_j) - J(j,R_j)) \\ & +(\xi_j(\pi,c) - \xi_{j-1}(\pi,c)). \end{aligned}$$

Again note that $\xi_j(\pi,c) \geq \xi_{j-1}(\pi,c)$, and also $J(j+1,R_j) \geq J(j,R_j)$ by induction hypothesis. Thus, it follows that $J(j) \geq J(j-1)$. $\square$

B. Proof of Proposition 2

Since at each stage, the rate is chosen from $[0, R_{\max}]$, any rate adaptation strategy $\rho \in \mathcal{A}$, where $\mathcal{A} = [0, R_{\max}]^I$. We assume that each user uses the same rate adaptation strategy. Every rate adaptation strategy ρ corresponds to a unique steady state distribution π_ρ and the collision probability c_ρ. Note that the throughput for a user is a function of (ρ, π_ρ, c_ρ). Now, let us define a correspondence $B : \mathcal{A} \hookrightarrow \mathcal{A}$ as $B(\rho) = A_\rho \subseteq \mathcal{A}$, where A_ρ is a set of the rate adaptation strategies that optimize the throughput for given (ρ, π_ρ, c_ρ) using the MDP formulation. In a non-atomic game $B(\cdot)$ is the best response correspondence. This is because if a single user deviates from the profile ρ, then it does not affect the average system behavior in the non-atomic game. The average system behavior is still defined by (ρ, π_ρ, c_ρ). Thus, to maximize the throughput assuming all the other users do not change there profile from ρ, a user has to choose a profile from A_ρ. Formally, let $\phi(\rho_2, \rho_1)$ denote the throughput of a user with profile ρ_2 given that the average system behavior is described by $(\rho_1, \pi_\rho, c_\rho)$. Here, if $\rho_2 \in B(\rho_1)$, then $\phi(\rho_2, \rho_1) \geq \phi(\rho, \rho_1)$ for every $\rho \in \mathcal{A}$. Note that to prove the existence of symmetric pure NE, it suffices to prove that the correspondence $B(\cdot)$ has a fixed point. We use Kakutani's fixed point theorem to prove the required.

First note that by construction, $\mathcal{A}$ is a compact, convex and non-empty subset of the finite dimensional Euclidean space. Moreover, $B(\rho)$ is non-empty for all $\rho \in \mathcal{A}$, and $B(\rho)$ is convex. The convexity of $B(\rho)$ follows as it contains a single point as shown in Corollary 2. Thus, to apply Kakutani's Fixed Point Theorem it suffices to show that $B(\cdot)$ has a closed graph, i.e., if $(\rho^n, \hat{\rho}^n) \to (\rho, \hat{\rho})$ such that $\hat{\rho}^n \in B(\rho^n)$ for every n, then $\hat{\rho} \in B(\rho)$. We prove the required using contradiction. Let there exist a sequence $(\rho^n, \hat{\rho}^n) \to (\rho, \hat{\rho})$ such that $\hat{\rho}^n \in B(\rho^n)$ for every n, but $\hat{\rho} \notin B(\rho)$. This implies that there exists a profile ρ' and $\epsilon > 0$ such that

$$\phi(\rho', \rho) > \phi(\hat{\rho}, \rho) + 3\epsilon. \tag{13}$$

We note that $\phi(\rho, \rho')$ is clearly a continuous function of ρ. We claim that $\phi(\rho, \rho')$ is also a continuous function of ρ'. This can be seen as follows. A small perturbation in ρ' results in a small perturbations in $\pi_{\rho'}$ and $c_{\rho'}$. Thus, the average system behavior is perturbed by a small amount, resulting in a small change in $\phi(\rho, \rho')$. Now, since $\rho^n \to \rho$, for large enough n we can conclude that $\phi(\rho', \rho^n) \geq \phi(\rho', \rho) - \epsilon$. Using this and (13), we conclude that $\phi(\rho', \rho^n) > \phi(\hat{\rho}, \rho) + 2\epsilon$. Again using continuity of $\phi(\cdot, \cdot)$, note for large enough n, $\phi(\hat{\rho}^n, \rho^n) \leq \phi(\hat{\rho}, \rho) + \epsilon$. Combining the previous two expression we obtain that $\phi(\rho', \rho^n) > \phi(\hat{\rho}^n, \rho^n) + \epsilon$, which contradicts the fact that $\hat{\rho} \notin B(\rho)$ for every n. Thus, correspondence $B(\cdot)$ has a fixed point. $\square$

C. Proof of Proposition 3

Proof: Fix ρ and the corresponding (π, c). Then, we show that $\hat{\rho}_i \geq \hat{\rho}_{i+1}$ for every $i < I$ in WoLD and WLDS systems. Again, we focus on WoLD systems, and the proof for WLDS systems follows similarly. The result follows immediately from the above lemma and Lemma 2 for all $i < I$ as here we seek $\min_R G(R, J(i+1))$. To see the result for stage I, note that $J(I)$ must satisfy

$$\begin{aligned} J(I) \;=\; & \min_{R \in [0, R_{\max}]} \left\{ (1-c)T_R + (1 - (1-c)e^{-\kappa R}) J(I) \right\} \\ & + \xi_I(\pi, c). \end{aligned}$$

Thus, here as well the required follows from Lemma 2. Since, every best response satisfies that the rates decreases monotonically with the back-off stages, the Theorem 3 follows. □

Note: From the above discussion, we can conclude that for every ρ, $\hat{\rho}_{I-1} = \hat{\rho}_I$ for WoLD systems.

D. Proof of Proposition 5

Fix any user, say n, and let all the other users use a rate adaptation policy ρ. Then the user's best response to ρ is obtained from the expressions presented in Section V-B

We refer to the $J(i)$ values obtained for $\rho^{(m)}$ as $J^{(m)}(i)$. For proving the theorem, we prove various properties of $J^{(m)}(i)$ and $\rho^{(m+1)}$, given $\rho^{(m)}$ by exploiting the value iteration method used to solve MDP. Let, for every i, $J_\ell^{(m)}(i)$ denote the value of J-function in ℓ^{th} for stage i. Then, $J_{\ell+1}^{(m)}(i)$ is computed using the following recursion.

$$J_{\ell+1}^{(m)}(i) = \min_{R\in[0,R_{\max}]} G(R, J_\ell^{(m)}(0)) + \xi_i(\rho^{(m-1)}) + c[J_\ell^{(m)}(i+1) - J_\ell^{(m)}(0)] \quad \text{for } i < I \tag{14}$$

$$J_{\ell+1}^{(m)}(I) = \min_{R\in[0,R_{\max}]} G(R, J_\ell^{(m)}(0)) + \xi_I(\rho^{(m-1)}) + c[J_\ell^{(m)}(I) - J_\ell^{(m)}(0)], \tag{15}$$

where $\xi_i(\rho^{(m-1)})$ is the value of function ξ when all the other users use rate adaptation strategy $\rho^{(m-1)}$. From results in MDP theory, $\lim_{\ell\to\infty} J_\ell^{(m)}(i) = J^{(m)}(i)$ for every i. Moreover, if we let $\rho^{(m)}(\ell)$ to be the rate adaption policy in the ℓ^{th} iteration, then $\lim_{\ell\to\infty} \rho^{(m)}(\ell) = \rho^{(m)}$.

Lemma 5: In the best response, $\rho^{(m)}(i) = \rho^{(m)}(i+1)$ for every $i = 0, \dots, I-1$ and m for any $\rho^{(m-1)}(i)$.

Proof: The result follows immediately from (14) and (15). Here, note that the optimal rate depends on the same function $G(R, J_\ell^{(m)}(0))$ for every i and ℓ. Thus, for every ℓ, $\rho_i^{(m)}(\ell) = \rho_{i+1}^{(m)}(\ell)$ for every $i = 0, \dots, I-1$. Thus, the same property will hold even in the limit as ℓ goes to ∞. □

Lemma 6: Let $\rho^{(0)} = R_{\max}\mathbf{1}$. Then, $\rho^{(m)} \geq \rho^{(m+1)}$ for every m.

Proof: The proof is by induction. Clearly, $\rho^{(0)} \geq \rho^{(1)}$. By induction hypothesis, we assume that $\rho^{(0)} \geq \rho^{(1)} \geq \cdots \geq \rho^{(m)}$. Now, we show that $\rho^{(m)} \geq \rho^{(m+1)}$. To show this, by Lemma 3, it suffices to show that $J^{(m)}(0) \leq J^{(m+1)}(0)$. We show this by showing that $J_\ell^{(m)}(i) \leq J_\ell^{(m+1)}(i)$ for every i and ℓ starting from the initial condition $J_0^{(m)}(i) = J_0^{(m+1)}(i) = 0$ for all i. Clearly, the required holds for $\ell = 0$. By induction hypothesis, let the required hold until ℓ^{th} iteration. Now, we consider the $(\ell+1)^{\text{th}}$ iteration and observe that for every $i < I$

$$\begin{aligned} & J_{\ell+1}^{(m+1)}(i) - J_{\ell+1}^{(m)}(i) \\ &= \left[\min_{R\in[0,R_{\max}]} G(R, J_\ell^{(m+1)}(0)) - \min_{R\in[0,R_{\max}]} G(R, J_\ell^{(m)}(0))\right] \\ &\quad + \left[\xi_i(\rho^{(m)}) - \xi_i(\rho^{(m-1)})\right] + c\left[J_\ell^{(m+1)}(i+1) - J_\ell^{(m)}(i+1)\right]. \end{aligned}$$

Note that for every R, $G(R, J_\ell^{(m+1)}(0)) \geq G(R, J_\ell^{(m)}(0))$ as $J_\ell^{(m+1)}(0) \geq J_\ell^{(m)}(0)$. Thus, the first term in the above expression is non-negative. The second term is also non-negative as $\rho^{(m)} \leq \rho^{(m-1)}$ by the induction hypothesis on m. Finally, the third term is also non-negative by induction hypothesis of ℓ. Thus, the required follows for all $i < I$. Also, note that using the similar arguments, it is easy to see that the required also holds for $i = I$. □

Proof of Proposition 5: Since $\rho^{(m)}$ is a monotone decreasing sequence in compact space, there exists $\rho^* = \lim_{m\to\infty} \rho^{(m)}$. Moreover, it can be easily seen that the best response to ρ^* is ρ^* itself. Thus, ρ^* is a NE. □

APPENDIX III
PROOFS OF THE RESULTS IN SECTION IV

Proof of Proposition 6: Let us assume the contrary, that for some i we have $\rho(i) = R_1 > \rho(i+1) = R_2$ (consequently $T_1 = T_{\rho(i)} < T_{\rho(i+1)} = T_2$) and let us construct two other rate allocations. In the first one, ρ^1, we increase $T_{\rho^1(i)} = T_{\rho(i+1)}$ and keep the remaining rates the same ($T_{\rho^1(j)} = T_{\rho(j)}$ for $j \neq i$). In the second one, ρ^2, we decrease $T_{\rho^2(i+1)} = T_{\rho(i)}$ and keep the remaining rates the same ($T_{\rho^1(j)} = T_{\rho(j)}$ for $j \neq i+1$).

Let us call E, E^1, E^2 and D, D^1, D^2 and ϕ, ϕ^1, ϕ^2 the enumerators, the denominators and the average rate from (1) for ρ, ρ^1, ρ^2 respectively. Since in ρ^2 we have decreased the duration of a typical slot by decreasing $T_{\rho^2(i+1)}$, we have $D > D^2$, and consequently we have $\frac{E-E^2}{D-D^2} > \phi > \phi_2$. Similarly, in ρ^1 we have increased the slot duration, hence $D < D^1$ and $\frac{E^1-E}{D^1-D} < \phi_1 < \phi$. Therefore,

$$\frac{E - E^2}{D - D^2} > \frac{E^1 - E}{D^1 - D}.$$

Since $E - E^2 = \pi_{i+1} p_{i+1}(1-c)(e_i - e_{i+1})$ and $E^1 - E = \pi_i p_i (1-c)(e_i - e_{i+1})$, the inequality further simplifies to

$$\frac{D^1 - D}{\pi_i p_i} > \frac{D - D^2}{\pi_{i+1} p_{i+1}}. \tag{16}$$

Case without RTS/CTS: Observe that $\{j : \rho^1(j) < R_1\} = \{j : \rho(j) < R_1 \cup \{i\}\}$ and $\{j : \rho^1(j) \leq R_2\} = \{j : \rho(j) \leq R_2 \cup \{i\}\}$ because $\rho^1(i) = R_2$. Similarly, $\{j : \rho^1(j) < R_1\} = \{j : \rho(j) < R_1 \setminus \{i+1\}\}$ and $\{j : \rho^1(j) \leq R_2\} = \{j : \rho(j) \leq R_2 \setminus \{i+1\}\}$ because $\rho^2(i+1) = R_1$. Hence we have

$$\begin{aligned}
&D^1 - D = \\
&= [t(\{j : \rho(j) < R_1\} \cup \{i\})^n - t(\{j : \rho(j) < R_1\})^n]\, T_1 \\
&+ [t(\{j : \rho(j) \leq R_2\} \cup \{i\})^n - t(\{j : \rho(j) \leq R_2\})^n]\, T_2, \\
&D - D^2 = \\
&= [t(\{j : \rho(j) < R_1\})^n - t(\{j : \rho(j) < R_1\} \setminus \{i+1\})^n]\, T_1 \\
&+ [t(\{j : \rho(j) \leq R_2\})^n - t(\{j : \rho(j) \leq R_2\} \setminus \{i+1\})^n]\, T_2.
\end{aligned}$$

For brevity let us denote $t_1 = t(\{j : \rho(j) < R_1\})$ and $t_2 = t(\{j : \rho(j) \le R_2\})$. We then have

$$\begin{aligned}
&\frac{D^1 - D}{\pi_i p_i} - \frac{D - D^2}{\pi_{i+1} p_{i+1}} = \\
&= \left[\frac{(t_1 - \pi_i p_i)^n - t_1^n}{\pi_i p_i} - \frac{t_1^n - (t_1 + \pi_{i+1} p_{i+1})^n}{\pi_{i+1} p_{i+1}}\right] T_1 \\
&- \left[\frac{(t_2 - \pi_i p_i)^n - t_2^n}{\pi_i p_i} - \frac{t_2^n - (t_2 + \pi_{i+1} p_{i+1})^n}{\pi_{i+1} p_{i+1}}\right] T_2 \\
&= \sum_{j=1}^{n-1} \left[t_1^{n-j}(t_1 - \pi_i p_i)^j - t_1^{n-j}(t_1 + \pi_{i+1} p_{i+1})^j\right] T_1 \\
&- \sum_{j=1}^{n-1} \left[t_2^{n-j}(t_2 - \pi_i p_i)^j - t_2^{n-j}(t_2 + \pi_{i+1} p_{i+1})^j\right] T_2 \\
&= \sum_{j=1}^{n-1} t_1^{n-j}(\pi_i p_i + \pi_{i+1} p_{i+1}) \sum_{l=1}^{j-1} (t_1 - \pi_i p_i)^{j-l}(t_1 + \pi_{i+1} p_{i+1})^l T_1 \\
&- \sum_{j=1}^{n-1} t_2^{n-j}(\pi_i p_i + \pi_{i+1} p_{i+1}) \sum_{l=1}^{j-1} (t_2 - \pi_i p_i)^{j-l}(t_2 + \pi_{i+1} p_{i+1})^l T_2 \\
&< 0.
\end{aligned}$$

where the last inequality follows from the fact that $t_1 < t_2$ and $T_1 < T_2$, hence every term in the first sum is strictly smaller than the corresponding term in the second term. This leads to contradiction.

Case with RTS/CTS: Let us start again from (16). It is easy to see that

$$\frac{D^1 - D}{\pi_i p_i} = \frac{D - D^2}{\pi_{i+1} p_{i+1}} = n(1-p)^n (T_2 - T_1).$$

which is again a contradiction and we cannot have that $\rho(i) > \rho(i+1)$. Similarly we can show that we cannot have $\rho(i) < \rho(i+1)$ hence in this case we have $\rho(i) = \rho(i+1)$. □

Lemma 7: Let us define $t_j = t(\{k : k \le j\})$ and

$$\bar{\phi}(\rho) = \frac{\sum \pi_i p_i (1-c)(1 - e_{\rho(i)})}{(1-p)^N + \sum_{j \ge 0} (t_{j-1}^N - t_j^N) T_{\rho(j)}}$$

The optimal rate allocation $\rho* = \operatorname{argmax}_{\rho \in \mathcal{R}^I} \phi(\rho) = \operatorname{argmax}_{\rho \in \mathcal{R}^I} \bar{\phi}(\rho)$ and the maximum rate is $\phi^* = \bar{\phi}(\rho^*)$.

Proof: From Proposition 6 we have that the optimal $\rho^*(i)$ is an increasing function in i. It is easy to verify using Proposition 1 that $\bar{\phi}(\rho) = \phi(\rho)$ for any monotone ρ. Hence if ρ^* obtained by maximizing $\bar{\phi}(\rho)$ is increasing, it also maximizes $\phi(\rho)$ over all increasing functions $\rho(i)$, thus it also maximizes $\phi(\rho)$. It remains to show that ρ^* is increasing.

Since ρ^* maximizes $\bar{\phi}$ we have that $\nabla \bar{\phi}(\rho^*) = 0$. By simple derivation it then follows that

$$\left.\frac{\partial e}{\partial T}\right|_{\rho^*(i)} = -\frac{(t_{i-1}^N - t_i^N)}{\pi_i p_i (1-c)} \bar{\phi}(\rho^*).$$

Now $t_{i-1} - t_i = \pi_i p_i$ hence

$$\frac{t_{i-1}^N - t_i^N}{\pi_i p_i} = \sum_{j=1}^{N-1} t_{i-1}^{N-j} t_i^j,$$

which is increasing in i. Consequently we have that $\frac{\partial e}{\partial T}\Big|_{\rho^*(i)}$ is decreasing and hence $\frac{\partial e}{\partial T}\Big|_{\rho^*(i+1)} < \frac{\partial e}{\partial T}\Big|_{\rho^*(i)}$. Since $e(T)$ is a strictly convex function we have that $\frac{\partial e}{\partial T}\Big|_T$ is increasing in T hence we have $T_{\rho^*(i+1)} \leq T_{\rho^*(i)}$, hence $\rho^*(i)$ is increasing in i which concludes the proof. $\square$

Proof of Proposition 7: As lemma 7 shows, we can optimize $\bar{\phi}$ instead of ϕ. Moreover, if our initial rate allocation $\rho^0(i)$ is non-decreasing, the update (2) guarantees $\rho^m(i)$ stays non-decreasing in i, as explained in lemma 7. Hence for every m we have $\phi(\rho^m) = \bar{\phi}(\rho^m)$. Update (2) represent a gradient-descent algorithm for $\bar{\phi}$. It is well know that a gradient descent algorithm will converge to a local maximum for an appropriate choice of step sizes (choosing ϵ_m for convergence is a standard technique and we do not discuss it here.)

Finally, it remains to be proved that function $\bar{\phi}$ has only a single local maximum. Suppose the contrary, that $\rho_1 \neq \rho_2$ are two local maxima. Let us choose an arbitrary $x \in (0,1)$ and choose ρ_3 such that $e_{\rho_3(i)} = xe_{\rho_1(i)} + (1-x)e_{\rho_2(i)}$. Let us define

$$g(y) = \frac{\sum \pi_i p_i (1-c)(1 - ye_{\rho^1(i)} - (1-y)e_{\rho^2(i)})}{(1-p)^N + \sum_{j\geq 0}(t_{j-1}^N - t_j^N)(yT_{\rho^1(j)} + (1-y)T_{\rho^2(j)})}.$$

If is easy to see that $g(y)$ is either increasing or decreasing. Suppose with out loss of generality that $\bar{\phi}(\rho^1) \leq \bar{\phi}(\rho^2)$, hence $g(0) \leq g(x)$. Now since $e(T)$ is a convex function, it is easy to verify that by construction $T_{\rho_3(i)} < xT_{\rho^1(j)} + (1-x)T_{\rho^2(j)})$ and consequently $g(x) \leq \bar{\phi}(\rho^3)$ for an arbitrary small $x > 0$. Thus ρ^1 cannot be a local minimum. Since there is a single local maximum, the gradient descent will always converge to the global maximum, which concludes the proof. $\square$

Proof of Proposition 9: Since the we use the same rate in all stages, the stationary probabilities simplify significantly and we have $p = \frac{1+2(1-c)(1-e)}{(1-c)(1-e)}p_0$. We next have

$$\frac{\partial p}{\partial e} = -\frac{p_0}{(1-c)^2(1-e)^2}\left((1-c) + (1-e)\frac{\partial c}{\partial e}\right),$$
$$\frac{\partial c}{\partial e} = (N-1)(1-p)^{N-2}\frac{\partial p}{\partial e},$$

and by solving the system we derive (5) and similarly (6). Finally, we can express $\phi = \pi_0 p_0 / S^n$ and by simple derivation we obtain (4). $\square$