# Performance Preserving Controller Reduction in the $\mu$ Setting

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*Abstract*— **This paper is concerned with the order reduction of controllers produced by $\mu$-synthesis controller design paradigm. A new proof has been proposed for the additive perturbation reduction technique with sufficient conditions to guarantee the closed loop structured singular value to remain less than unity. Reduction of unstable controllers is achieved through a new coprime factor reduction technique with sufficient conditions to guarantee preservation of closed loop performance. The coprime factor perturbations to the controller have been shown to have a block diagonal structure. The proposed algorithms have been tested on a widely studied benchmark HIMAT aircraft and have been found to work satisfactorily producing more than 50% reduction in the controller order without optimization.**

*Keywords*— **Controller Reduction, Structured Singular Value, Coprime factors**

## I. Introduction

THE most widely used controller design techniques like $\mathcal{H}_\infty$ loop shaping and $\mu$-synthesis typically produce very high order controllers. But practical controllers must be simple, linear and of low order. High complexity controllers are not only difficult to understand but they may turn out impossible to implement in hardware and software. Also from the point of integrity and reliability, simple controllers are preferable.

The problem of reducing complexity via reducing the state dimension of the controller has been the subject of extensive research for the last forty years. See [1] and [2] for a complete list of references. In the controller reduction literature open loop methods such as weighted and unweighted balanced truncation [3], [4], [5] and optimal Hankel norm approximation [6] have been widely studied. However these open loop methods do not guarantee closed loop stability or performance with the reduced order controller. These drawbacks were overcome among others by Enns[4] and Anderson and Liu[1] who derived weights for controller approximations which guaranteed closed loop stability. Bounds on $\mathcal{H}_\infty$-performance degradation were derived by Lenz et al[7], which were improved by Goddard and Glover[8] and Wang et al[9] to achieve performance preserving controller reduction.

All these methods in general do not consider any structure in the uncertainty. However consideration of structure in the uncertainty makes the reduction schemes less conservative. Thus a greater reduction in the controller order is possible. In this paper we treat the controller reduction problem in a $\mu$-synthesis framework and derive a set of sufficient conditions for preserving the closed loop $\mu$ with the reduced order controller less than unity.

One of the main drawbacks of the above methods is that it assumes the controller to be approximated is stable. Robust controller design techniques like $\mathcal{H}_\infty$ loop shaping or $\mu$-synthesis do not guarantee stability of the controllers. A solution is to consider a fractional description, the basis of which is multiplicative decomposition rather than additive decomposition. Fraction approaches to controller reduction have received much attention in the literature. These are generally termed as coprime factor approaches. Various sufficient conditions for the stability of the closed loop system when the controller is subjected to coprime factor perturbation has been derived in [10], [11]. $\mathcal{H}_\infty$ performance preserving techniques with multiplicative uncertainty have been studied in Goddard and Glover [8]. In this paper we propose a coprime factor based controller reduction scheme that keeps the closed loop $\mu$ value with the reduced order controller to remain less than one, thus guaranteeing robust stability and performance of the closed loop system. We also show that the perturbations have a block diagonal structure facilitating a greater reduction in the controller order than possible with the additive schemes. Section 2 reviews relevant $\mu$-analysis results while the main contributions of this paper are presented in Section 3.

## II. Preliminaries

The notation used in this paper is very standard and follows [12]. For any square matrix $M$ we denote the complex conjugate transpose by $M^*$. The largest singular value and the structured singular value(SSV) with respect to the uncertainty structure $\mathbf{\Delta}$ are denoted by $\bar{\sigma}(M)$ and $\mu(M, \mathbf{\Delta})$. $\mathbb{R}$ denotes the set of real numbers while $\mathbb{R}_e = \mathbb{R} \cup \infty$ and $\mathbb{R}_+$ denotes the set of positive real numbers. The definition of structured singular value($\mu$) is dependent on the underlying uncertainty structure. We define a few common uncertainty structures. Let S and F be nonnegative integers, not all zero, and let $n$, $s_1, ..., s_S$, $f_1, ..., f_F$ be positive integers, such that $n = \sum s_i + \sum f_i$. Consider the subspace of $n \times n$ complex matrices

$$\mathcal{C}\mathbf{\Delta}_{S,F} = \{diag(\delta_1 I_{s_1}, ...., \delta_S I_{s_S}, \Delta_1, ...., \Delta_F) : \delta_i \in \mathbb{C} \; and \; \Delta_i \in \mathbb{C}^{f_i \times f_i}\} \quad (1)$$

and denote its unit ball by $\mathbf{\Delta}_{S,F} = \{\Delta \in \mathcal{C}\mathbf{\Delta}_{S,F} : \bar{\sigma}(\Delta) \leq 1\}$. Now we define the time invariant uncertainty set using functions that have continuous extensions on the right half s-plane:

$$\mathbf{\Delta}_{TI} = \{\Delta \in \mathcal{L}(L_2) : \Delta \; is \; LTI \; and \; \Delta(s) \in \mathbf{\Delta}_{S,F} \; for \; every \; Re(s) \geq 0\} \quad (2)$$

*Definition 1:* Given a matrix $M \in \mathbb{C}^{n \times n}$, we define the matrix structured singular value or SSV of $M$ with respect to $\mathbf{\Delta}_{S,F}$ by

$$\mu(M, \mathbf{\Delta}_{S,F}) = \frac{1}{\min\{\bar{\sigma}(\Delta) : \Delta \in \mathcal{C}\mathbf{\Delta}_{S,F} \; and \; I - M\Delta \; is \; singular\}}$$

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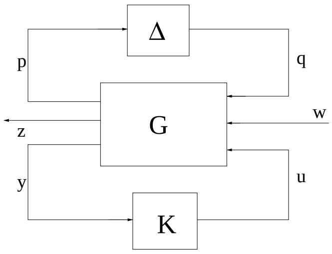


Fig. 1. Feedback System with Uncertainty

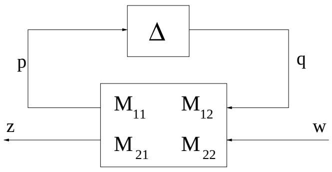


Fig. 2. Equivalent Feedback System

when the minimum is defined. Otherwise $\mu(M, \mathbf{\Delta}_{S,F})$ is defined to be zero.

The SSV is an useful tool to compute robust stability and performance for LTI systems with structured uncertainty. It is thus used to check whether a controller is able to provide desired closed loop characteristics. First we define Robust stability and performance with respect to figures 1 and 2. Let $M = \mathcal{F}_l(G,K) = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ be partitioned compatibly with the input and output signals.

*Definition 2:* With reference to figure 2 and with the extra assumption that the uncertainty class $\mathbf{\Delta}$ is causal, the uncertain system $(M_{11}, \mathbf{\Delta})$ is *robustly stable* if $(I - M_{11}\Delta)^{-1}$ exists in $\mathcal{L}(L_2)$ and is causal for each $\Delta \in \mathbf{\Delta}$.

*Definition 3:* With reference to figure 2 and with the extra assumption that the uncertainty class $\mathbf{\Delta}$ is causal, the uncertain system $(M, \mathbf{\Delta})$ has *robust performance* if $(M_{11}, \mathbf{\Delta})$ is robustly stable and $\|\mathcal{F}_u(M, \Delta)\| < 1$, for every $\Delta \in \mathbf{\Delta}$.

The next theorem gives the major result of $\mu$-synthesis providing us with a test of the $\mu$ norm to guarantee robust performance and stability.

*Theorem 4* (Proposition 9.10 in [12]) Suppose the configuration of Figure 2 is nominally stable, and the uncertainty class is $\mathbf{\Delta}_{TI}$ defined above. Define the perturbation set

$$\mathbf{\Delta}_{TI,p} = \left\{ \begin{bmatrix} \Delta_u & 0 \\ 0 & \Delta_p \end{bmatrix} : \Delta_u \in \mathbf{\Delta}_{TI}, \Delta_p \in \mathcal{L}(L_2) \ LTI \ causal, \ \|\Delta_p\| \leq 1 \right\}$$

Then the following are equivalent:

1. The uncertain system $(M, \mathbf{\Delta}_{TI})$ satisfies robust performance: it is robustly stable and $\|\mathcal{F}_u(M, \Delta_u)\| < 1$, for every $\Delta_u \in \mathbf{\Delta}_{TI}$.
2. The uncertain system $(M, \mathbf{\Delta}_{TI,p})$ is robustly stable.
3. $\sup_{\omega \in \mathbb{R}} \mu(M(j\omega), \mathbf{\Delta}_{S,F+1}) < 1$

Thus for any controller $K$, if we can show that $\mu(M(j\omega), \mathbf{\Delta}_{S,F+1}) < 1 \ \forall\omega$, then the uncertain feedback system $(M, \mathbf{\Delta}_{TI})$ is robustly stable and has robust performance. This observation is the key to to the following controller reduction procedure.

Next we present a result that will be the basis of our controller reduction schemes. The main objective of the following result is to extend the standard mixed-$\mu$ analysis to the case when the complex uncertainties have frequency dependent upper bounds. We augment the uncertainty structures to meet the mixed-$\mu$ setup. Let R, S and F be nonnegative integers, not all zero, and let $n$, $r_1, ..., r_R$, $s_1, ..., s_S$, $f_1, ..., f_F$ be positive integers, such that $n = \sum r_i + \sum s_i + \sum f_i$. The frequency dependent bounding set is thus defined as

$$\mathcal{W} = \{\mathcal{W} = diag(\rho_1 I_{r_1}, ..., \rho_R I_{r_R}, w_1 I_{s_1}, ...w_S I_{s_S}, w_{S+1} I_{f_1}, ..., w_{S+F} I_{f_F}) : \rho_i > 0, i = 1, 2, ..., R, \ and \ some \ functions w_i : \mathbb{R}_e \rightarrow \mathbb{R}_+, i = 1, ..., (S+F)\} \tag{3}$$

Let $C\mathbf{\Delta}_{re}$ be the subspace of real $\sum r_i \times \sum r_i$ matrices defined by

$$C\mathbf{\Delta}_{re} = \{diag(\gamma_1 I_{r_1}, ..., \gamma_R I_{r_R} : \gamma_i \in \mathbb{R}\} \tag{4}$$

Next we define a structure consisting of those functions that have continuous extension on the right half s-plane. Define the set:

$$C\mathbf{\Delta}_{TI} = \{\Delta(s) \in \mathcal{H}_\infty : \Delta \ is \ LTI \ and \ \Delta(s) \in C\mathbf{\Delta}_{S,F}, \ for \ every \ Re(s) \geq 0\} \tag{5}$$

Next we define the augmented structure

$$\mathbf{\Delta}_{rc} = \{\Delta_{rc} : \Delta_{rc} \in diag(\Delta_r, \Delta), \Delta_r \in C\mathbf{\Delta}_{re}, \Delta \in C\mathbf{\Delta}_{TI}\} \tag{6}$$

Finally we define the uncertainty set bounded above by the frequency dependent functions. Given $W \in \mathcal{W}$, define

$$\mathbf{\Delta}_W = \{\Delta \in \mathbf{\Delta}_{rc} : \Delta(j\omega)^* \Delta(j\omega) \leq W(\omega)^2, \ \forall\omega \in \mathbb{R}\}$$

Then we have the following result

*Lemma 5* (Theorem 6 in [13]) Let $W \in \mathcal{W}$ and $G \in \mathcal{H}_\infty$. If

$$\mu(W(\omega)G(j\omega), \mathbf{\Delta}_W) < 1, \ \forall\omega \in \mathbb{R},$$

then $(G, \mathbf{\Delta}_W)$ is robustly stable.

## III. Main Results

### A. Additive Reduction

In this section we present a new proof of Kavranoğlu's [14] additive reduction algorithm using the above result. To reframe the performance preserving controller reduction problem in the $\mu$ setting, consider the following block diagrams.(Figures 3 and 4). Let the reduced order controller $K_r$ be expressed as a perturbation to the full order controller $K_f$ such that $K_r = K_f + \Delta_c$. We assume that $\Delta_c \in \mathbf{\Delta}_W$. For the upper bound we define a frequency

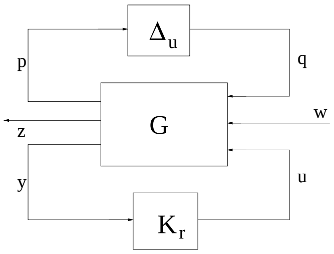


Fig. 3. Uncertain system with the reduced order controller

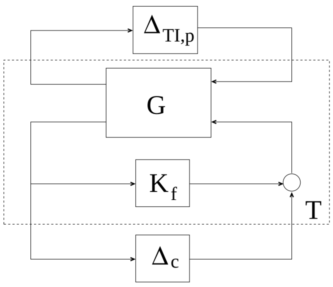


Fig. 4. Plant and controller interconnection for additive reduction

dependent function $W_a = \gamma(\omega)I$ where $\gamma(\omega) : \mathbb{R} \to \mathbb{R}_+$. Clearly $W_a \in \mathcal{W}$. Now define

$$T = \mathcal{S}(G, \begin{bmatrix} K_f & I \\ I & 0 \end{bmatrix}) \tag{7}$$

where $\mathcal{S}(\cdot,\cdot)$ denotes the Redheffer star product,

$$\tilde{T} = \begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix} T \tag{8}$$

and

$$\mathbf{\Delta}_{upc} = \left\{ \begin{bmatrix} \Delta_{TI,p} & 0 \\ 0 & \Delta_c \end{bmatrix} : \Delta_{TI,p} \in \mathbf{\Delta}_{TI,p} \textit{ and } \Delta_c \in \mathbf{\Delta}_W \right\} \tag{9}$$

A main drawback of this method is that this characterization of the set of reduced order controller limits $K_r(s)$ to have the same number of unstable poles as $K_f(s)$. This problem is addressed in the next section.

*Proposition 6:* If $\bar{\sigma}(\Delta_c(j\omega)) < \gamma(\omega),\ \forall\omega$ , and $\mu(\tilde{T}(j\omega), \Delta_{upc}) < 1,\ \forall\omega$, then the closed loop uncertain system $(\mathcal{F}_l(G, K_r), \mathbf{\Delta}_{TI})$ is robustly stable and has robust performance.

*Proof:* Clearly $\begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix} \in \mathcal{W}$ and by hypothesis for each $\omega$

$$\bar{\sigma}(\Delta_c(j\omega)) < \gamma(\omega) \Rightarrow\ \Delta_c(j\omega)^*\Delta_c(j\omega) < W_a^2$$
$$\Rightarrow \Delta_{upc}(j\omega)^*\Delta_{upc}(j\omega) < \begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix}^2$$

Now $\tilde{T} = \begin{bmatrix} I & 0 \\ 0 & W_a \end{bmatrix} T$. Also by hypothesis we have $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upc}) < 1$. Thus by direct application of Lemma 2.5 we have the system $(T(j\omega), \mathbf{\Delta}_{upc})$ is robustly stable. Now for any particular $K_f$ and $K_r$ we have

$$\begin{aligned} (T(j\omega), \mathbf{\Delta}_{upc}) &\equiv (\mathcal{S}(G(j\omega), \begin{bmatrix} K_f & I \\ I & 0 \end{bmatrix}), \mathbf{\Delta}_{upc}) \\ &\equiv (\mathcal{F}_l(G(j\omega), K_r), \mathbf{\Delta}_{TI,p}). \end{aligned}$$

Thus we have shown that the closed loop uncertain system $(\mathcal{F}_l(G, K_r), \mathbf{\Delta}_{TI,p})$ is robustly stable. But by theorem 2.4 this is equivalent to $(\mathcal{F}_l(G, K_r), \mathbf{\Delta}_{TI})$ being robustly stable and having robust performance. Hence proved. ∎

*Remark 7:* From the above theorem it is evident that if we can find an $\gamma(\omega)$ such that $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upc}) < 1,\ \forall\omega$ then any $K_r$ satisfying $\bar{\sigma}(K_f - K_r) < \gamma(\omega)$ is a robustly stabilizing controller and also provides robust performance to the closed loop system.

Now let $\hat{\gamma}(j\omega)$ be a rational function approximating $\gamma(\omega)$ such that $\hat{\gamma}(j\omega) < \gamma(\omega)$. Thus for each $\omega$ we have the following set of equivalences.

$$\begin{aligned} &\bar{\sigma}(K_f(j\omega) - K_r(j\omega)) < \hat{\gamma}(j\omega) \\ \Longleftrightarrow\ &\tfrac{1}{\hat{\gamma}(j\omega)}\bar{\sigma}(K_f(j\omega) - K_r(j\omega)) < 1 \\ \Longleftrightarrow\ &\bar{\sigma}[\tfrac{1}{\hat{\gamma}(j\omega)} I(K_f(j\omega) - K_r(j\omega))] < 1 \\ \Longleftrightarrow\ &\bar{\sigma}[W_a^{-1}(K_f(j\omega) - K_r(j\omega))] < 1 \\ \Longleftrightarrow\ &\left\|[W_a^{-1}(K_f(j\omega) - K_r(j\omega))]\right\|_\infty < 1 \end{aligned}$$

Thus we have converted the controller reduction problem into the widely studied frequency-weighted $L_\infty$ model approximation problem. This problem can be solved by a number of methods such as frequency weighted balanced truncation and optimal Hankel norm approximation.

Now the remaining problem is to calculate $\gamma(\omega)$ such that $\mu(\tilde{T}, \mathbf{\Delta}_{upc}) < 1$. To facilitate greater reduction a larger $\gamma(\omega)$ is preferable. We can find $\gamma(\omega)$ using among others the $\mu-tools$ software of MATLAB [15]. One has to perform a few bisection steps on the size of $\bar{\sigma}(K_f(j\omega) - K_r(j\omega))$ for each frequency to determine the maximum size possible such that $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upc}) = 1$.

### B. Coprime Factor Reduction

In this section we propose a coprime factor based controller reduction scheme that keeps the closed loop $\mu$ value with the reduced order controller to remain less than one, thus guaranteeing robust stability of the closed loop system. Apart from having the natural advantages of an multiplicative scheme, we also show that the perturbations have a block diagonal structure. This points to the possibility of greater reduction in the controller order than possible with the additive scheme.

We assume that the left coprime factors of $K_r = V_r^{-1}U_r$ may be represented by perturbations to the left coprime factors of the full order controller $K_f = V_f^{-1}U_f$ i.e.

$$U_r = U_f + \Delta_U \tag{10}$$
$$V_r = V_f + \Delta_V \tag{11}$$

We assume $\Delta_U, \Delta_V \in \mathbf{\Delta}_W$. Let $W_V = \gamma_V(\omega)I$ and $W_U = \gamma_U(\omega)I$ where $\gamma_V(\omega), \gamma_U(\omega) : \mathbb{R} \to \mathbb{R}_+$. Clearly $W_V, W_U \in$

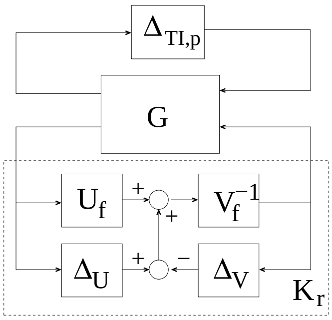


Fig. 5. Uncertain system with reduced order controller

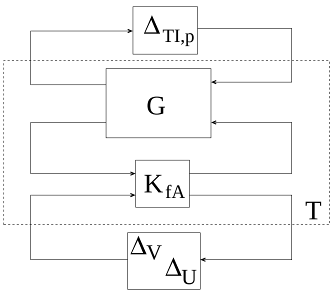


Fig. 6. Plant and controller interconnection for coprime factor reduction

$\mathcal{W}$. Figure 3 can be redrawn as in Figure 5, which after some manipulations give Figure 6. We define

$$K_{fA} = \begin{bmatrix} K_f & V_f^{-1} & V_f^{-1} \\ I & 0 & 0 \\ -K_f & -V_f^{-1} & -V_f^{-1} \end{bmatrix} \tag{12}$$

$$T = \mathcal{S}(G, K_{fA}) \tag{13}$$

and

$$\tilde{T} = \begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix} T \tag{14}$$

while

$$\mathbf{\Delta}_{upvu} = \left\{ \begin{bmatrix} \Delta_{TI,p} & 0 & 0 \\ 0 & \Delta_V & 0 \\ 0 & 0 & \Delta_U \end{bmatrix} : \Delta_{TI,p} \in \mathbf{\Delta}_{TI,p}, \right. \\ \left. \Delta_V \in \mathbf{\Delta}_W, \Delta_U \in \mathbf{\Delta}_W \right\} \tag{15}$$

We have the following proposition to motivate our controller reduction procedure.

*Proposition 8:* Let $\bar{\sigma}(\Delta_V(j\omega)) < \gamma_V(\omega)$ and $\bar{\sigma}(\Delta_U(j\omega)) < \gamma_U(\omega)$, $\forall\omega$. Now if $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upvu}) < 1, \forall\omega$,then the closed loop uncertain system $(\mathcal{F}_l(G, K_r), \mathbf{\Delta}_{TI})$ is robustly stable and has robust performance.

*Proof:*

Clearly $\begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix} \in \mathcal{W}$ and by hypothesis for each $\omega$

$$\begin{aligned} &\bar{\sigma}(\Delta_V(j\omega)) < \gamma_V(\omega) \ and \ \bar{\sigma}(\Delta_U(j\omega)) < \gamma_U(\omega) \\ &\Longleftrightarrow \begin{bmatrix} \Delta_V & 0 \\ 0 & \Delta_U \end{bmatrix}^* \begin{bmatrix} \Delta_V & 0 \\ 0 & \Delta_U \end{bmatrix} < \begin{bmatrix} W_V & 0 \\ 0 & W_U \end{bmatrix}^2 \\ &\Longleftrightarrow \Delta_{upvu}(j\omega)^* \Delta_{upvu}(j\omega) < \begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix}^2 \end{aligned} \tag{16}$$

Now $\tilde{T} = \begin{bmatrix} I & 0 & 0 \\ 0 & W_V & 0 \\ 0 & 0 & W_U \end{bmatrix} T$. Also by hypothesis we have $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upvu}) < 1$. Thus by direct application of Lemma 2.5 we have the system $(T(j\omega), \mathbf{\Delta}_{upvu})$ is robustly stable. Now for any particular $K_f$ and $K_r$ we have

$$\begin{aligned} &(T(j\omega), \mathbf{\Delta}_{upvu}) \\ &\equiv \left( \mathcal{S}\left( G(j\omega), \begin{bmatrix} K_f & V_f^{-1} & V_f^{-1} \\ I & 0 & 0 \\ -K_f & -V_f^{-1} & -V_f^{-1} \end{bmatrix} \right), \mathbf{\Delta}_{upvu} \right) \\ &\equiv (\mathcal{F}_l(G(j\omega), K_r), \mathbf{\Delta}_{TI,p}). \end{aligned} \tag{17}$$

Thus we have shown that the closed loop uncertain system $(\mathcal{F}_l(G, K_r), \mathbf{\Delta}_{TI,p})$ is robustly stable. But by theorem 2.4 this is equivalent to $(\mathcal{F}_l(G, K_r), \mathbf{\Delta}_{TI})$ being robustly stable and having robust performance. Hence proved. ■

Like in the additive case the above result implies that if we can find suitable $\gamma_V(\omega)$ and $\gamma_U(\omega)$ such that $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upvu}) < 1$, $\forall\omega$, then our controller reduction problem can be written as frequency weighted $L_\infty$ approximation problems. Let $\hat{\gamma}_V(j\omega)$ and $\hat{\gamma}_U(j\omega)$ be two rational functions such that $\hat{W}_V(j\omega) = \hat{\gamma}_V(j\omega) < \gamma_V(\omega)$ and $\hat{W}_U(j\omega) = \hat{\gamma}_U(j\omega) < \gamma_U(\omega) \forall\omega$. Then our controller reduction problem reduces to the following set of frequency weighted $L_\infty$ approximation problems following the arguments outlined in the additive case.

$$\left\| \hat{W}_V^{-1}(V_f - V_r) \right\|_\infty < 1 \tag{18}$$

$$\left\| \hat{W}_U^{-1}(U_f - U_r) \right\|_\infty < 1 \tag{19}$$

Thus our objective again reduces to finding $\gamma_V(\omega)$ and $\gamma_U(\omega)$ such that for each frequency $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upvu}) < 1$. Here also the bisection algorithm outlined in the additive case may be used. But search over two separate parameters is not practically possible as the dependance of $\mu$ on the bound of the uncertainty is not explicitly known. However we could just search over one parameter by taking $W_V = W_U = \gamma_{VU}(\omega)I$. Under this assumption we find out $\gamma_{VU}(\omega)$ by performing a few bisection steps on the size of $\bar{\sigma}\left( \begin{bmatrix} \Delta_V(j\omega) & 0 \\ 0 & \Delta_U(j\omega) \end{bmatrix} \right)$ for each frequency to determine the maximum size possible such that $\mu(\tilde{T}(j\omega), \mathbf{\Delta}_{upvu}) = 1 \ \forall\omega$. The weighted $L_\infty$ approximation problem can be solved by a number of available algorithms such as weighted balanced truncation or optimal Hankel

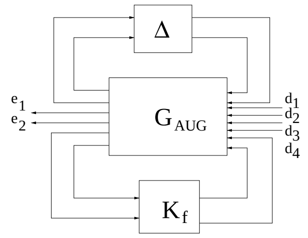


Fig. 7. The Augmented Plant and Controller

TABLE I
Results of Model Reduction

| Approach | Controller Order | $\mu(\mathcal{F}_l(P,K),\mathbf{\Delta})$ |
|---|---|---|
| Full Order | 30 | 0.966 |
| $\mu A(P)$ | 16 | 1.0567 |
| $\mu C(P)$ | 15 | 1.0536 |
| $\mu A(P)$+B.T. | 8 | 1.1219 |

norm approximation. In the example of the next section we use the frequency weighted optimal Hankel norm approximation for its tight upper bound of approximation error.

## IV. Example

In this section we examine the application of the proposed algorithms on the widely studied problem, the pitch control of an experimental aircraft model, namely HIMAT. A linearized model and detailed specifications can be found in [15]. A generalized plant incorporating all the weights is first formed. The resulting plant $G_{AUG}$ along with the uncertainty set $\mathbf{\Delta}_G$ and the controller $K_f$ is shown in figure 7. The generalized plant $G_{AUG}$ has 10 states. $K_f$ is a 30 state controller designed in [15] by D-K iteration, satisfying the performance specifications. The closed loop robust performance is conveniently measured in terms of the $\mu$ of the closed loop with respect to an augmented uncertainty set. We define

$$\mathbf{\Delta}_{AUG} := \left\{ \begin{bmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{bmatrix} : \Delta_1 \in \mathbb{C}^{2\times 2}, \Delta_2 \in \mathbb{C}^{2\times 2} \right\}$$

The Augmented plant $G_{AUG}$ has robust performance and robust stability if $\mu(\mathcal{F}_l(G_{AUG},K_f),\mathbf{\Delta}_{AUG}) < 1$. In this case $K_f$ is found to satisfy $\mu(\mathcal{F}_l(G_{AUG},K_f),\mathbf{\Delta}_{AUG}) \leq 0.996$.

The results of controller reduction are presented in Table I. The following abbreviations are used

$\mu A(P)$ - Additive Perturbation Reduction

$\mu C(P)$ - Coprime Factor Perturbation Reduction

Thus the proposed algorithms are found to produce a 50It is suspected that further reduction is possible but for the poor performance of the actual frequency weighted reduction algorithms. In fact a $8^{th}$ order controller can be found simply by balanced truncation of the reduced controller of the $\mu A(P)$ approach, which approximately satisfies the sufficient condition i.e. $\mu(\mathcal{F}_l(G_{AUG},K_r),\mathbf{\Delta}_{AUG}) <$ 1 and is also found to give good performance in time simulations.

It should further be noted that the main advantage of the proposed methods is not utilized in this example as the original uncertainty of the HIMAT system was unstructured. With the incorporation of structure in the uncertainty the proposed methods remain applicable. This is not the case with most of the currently available techniques which do not incorporate any knowledge of the uncertainty structure.

## V. Conclusion

In this paper two performance preserving techniques in the $\mu$-framework have been proposed. A new proof has been given for the Kavranoğlu's additive reduction technique [14]. A new coprime factor reduction scheme has been proposed which guarantees closed loop stability and performance with structured uncertainty. The coprime factor perturbations to the controller have been found to have a block diagonal structure thus improving the reduction algorithm.

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