Intermittent Feedback Based Rejection of Persistent Bounded Disturbance

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Abstract— **In this paper, an intermittent feedback based control policy is proposed for disturbance rejection in a continuous, linear time invariant (LTI) system with distinct and rational eigenvalues, subject to a bounded disturbance signal. According to the proposed policy, the feedback link is intermittently turned on and off, to maximize the average feedback-off time while maintaining the state trajectory inside a pre-specified safe region. When the feedback is on, the state trajectory is steered to the origin and then the feedback is turned off for some pre-computed duration. The feedback control and the open-loop control in the proposed policy, which are applied during feedback-on and feedback-off intervals respectively, are obtained by solving certain time-optimal control problems. The optimal control problems in the proposed policy need to be solved only once offline and hence, do not add to the burden of real-time computation.**

I. INTRODUCTION

The presence of disturbance and model uncertainty can severely hamper the performance of a control system. Such performance degradation is usually prevented by feedback. However, continuous availability of the feedback signal cannot always be guaranteed. While disruption in feedback may occur due to technical failure, it can also be intentionally induced to reduce operational costs [5]. In absence of feedback, the deviation of the system's states from the desired operating point can exceed safe limits [10].

Consider the example of a networked control system (NCS) [18], in which a large number of spatially distributed control systems share a communication network for the transmission of feedback signals. NCS has applications in automobiles, power generation, transmission and transportation engineering etc. [19]. As the bandwidth of the shared network is finite, continuous transmission of feedback signals from all subsystems is not always possible [20], [21]. In order to avoid congestion, it is necessary that each system transmits feedback signal intermittently. However, in absence of feedback, individual subsystems can become unstable under the influence of disturbances [20].

Another example of intermittent feedback is a group of surveillance drones controlled from the ground station by feedback signal transmitted over a wireless communication channel [22], [23]. Here, the operational cost includes communication cost and power consumption. The communication cost is proportional to the duration over which the channel is used [26]. The power consumption of a drone's transponder is also proportional to the duration of communication [27]. This consumption limits the life of drone's battery and as a result, its flight time [32]. In order to reduce the operational cost, it is necessary to use intermittent communication. However, in absence of feedback, disturbances such as wind gusts can drive drones away from their desired trajectories [30].

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It is clear from the above-mentioned examples that there is a trade-off between feedback transmission related operational cost and system performance/safety. In order to maintain the balance between these two, the designer needs to develop a control policy based on an intermittent feedback which guarantees that system's states remain within safe limit, while maximizing the average feedback-off time.

Intermittent feedback based control is a well studied topic. The problem of state estimation with finite communication bandwidth is studied in [1]. The problem of stabilization under the constraint of finite communication bandwidth is investigated in [2], where a necessary condition for stabilization in terms of required data rate and the rate of change of system's state is obtained. In [3], a stabilization method based on feedback quantization is developed. In [4], conditions for stabilization of a discrete system under the constraint of limited data rate are obtained. An *event-triggered* control technique is developed in [5], [6], where control action is initiated only when certain error state reaches a threshold. This reduces the duration over which feedback is needed. However, there is no explicit maximization of feedback-off time. Hence, the minimization of feedback cost is not explicit. The problem of optimal control in the event of feedback failure is studied in [10]. The problem of faster reduction of operational errors after feedback disruption is investigated in [11]. In [10] and [11], the system dynamics has no external disturbances.

The problem of rejection of persistent bounded disturbance was formulated in [7]. It has been solved using various approaches such as dynamic programming [8], a linear matrix inequality (LMI) based approach [9] etc.. In these approaches, however, continuous feedback is necessary.

To the best of our knowledge, the problem of rejection of persistent bounded disturbance under the constraint of intermittent feedback has not been investigated. In this paper, for the first time we develop an intermittent feedback based control policy for LTI systems with distinct and rational eigenvalues, which explicitly maximizes the average feedback-off time. While maximizing the average feedback-off time, the state trajectory is steered inside a prespecified safe region and is confined inside this region thereafter. We characterize all initial conditions which can be steered inside the safe region under the proposed policy. The feedback control in the proposed policy is obtained by solving a certain min-max time-optimal control problem. During feedback-on interval, the state trajectory is controlled to reach the origin in min-max time, irrespective of the disturbance signal in that interval. The open-loop control in the proposed policy is obtained by solving a certain max-min time-optimal control problem in order to maximize the feedback-off time, i.e. the duration over which feedback can be kept off without allowing the trajectory to escape the safe region, irrespective of the disturbance signal during this period. The time-optimal control problems in the proposed policy need to be solved only once offline and hence, they do not add to the burden of real-time computation. Thus, the proposed policy is suitable for time-critical applications.

The remaining part of the paper is organized as follows. The problem of developing intermittent feedback based control policy is formulated in Section II. In Section III, the time-optimal feedback control is obtained. The time-optimal open-loop control is obtained in Section IV. In Section V, it is shown that under the proposed policy, the state trajectory remains confined in a prespecified safe region. Simulation results are presented in Section VI. In Section VII, paper is concluded with future directions.

II. Problem Formulation and Preliminaries

A. Problem formulation

Consider a linear time invariant (LTI) system

$$\dot{x}(t) = Ax(t) + B\Big(u(t) + d(t)\Big) \tag{1}$$

where, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, u is the control input and d is the disturbance signal. We make the following assumptions:

1) The pair (A, B) is controllable.
2) Eigenvalues of the matrix A are distinct and rational.
3) The control input u and disturbance d belong to sets $\mathcal{U} := \Big\{u \in \mathcal{L} \;\Big|\; |u(t)| \leq u_{max}, \quad \forall t \geq 0\Big\}$ and $\mathcal{D} := \Big\{d \in \mathcal{L} \;\Big|\; |d(t)| \leq d_{max}, \quad \forall t \geq 0\Big\}$ respectively, where $\mathcal{L}$ is the set of measurable functions.
4) The relation between bounds on the control input and disturbance is $0 < d_{max} < u_{max} < \infty$.

We denote a state-feedback control by a function $f_u : \mathbb{R}^n \to [-u_{max}, u_{max}]$ and its open-loop representation by $u_f(t) := f_u(x(t))$. We denote an open-loop control by u_o. The state trajectory of system (1) at time t, under control $u \in \mathcal{U}$, disturbance $d \in \mathcal{D}$ and an initial condition x_0, is denoted by $\bar{x}(t, x_0, u, d)$. Define $\bar{t}(x_0, u, d) := \{\inf\ t \;\;|\;\; \bar{x}(t, x_0, u, d) = 0\}$. Let $W \subseteq \mathbb{R}^n$ be a prespecified safe region with safety limit $\alpha > 0$. It is defined as $W := \Big\{y \in \mathbb{R}^n \;\Big|\; ||y||_2 \leq \alpha\Big\}$.

In the proposed intermittent feedback based policy, we propose that the feedback be turned on and off over alternating time intervals (see Figure 1) starting at $t = 0$. Let $t^{i,d}_{on}$ be the end point of the ith time interval $I^{i,d}_{on}$ over which feedback is on and $t^{i,d}_{off}$ be the end point of the ith time interval $I^{i,d}_{off}$ over which feedback is off , in presence of the realization $d \in \mathcal{D}$ of the disturbance. Then, $I^{1,d}_{on} = [0, t^{1,d}_{on}]$, $I^{i,d}_{off} = (t^{i,d}_{on}, t^{i,d}_{off}], i \geq 1$ and $I^{i,d}_{on} = (t^{i-1,d}_{off}, t^{i,d}_{on}], i \geq 2$. We

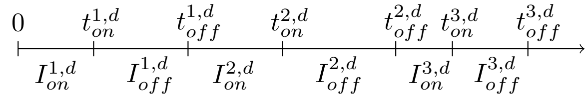

Fig. 1. Intermittent feedback

define the ith *feedback-on time* $T^{i,d}_{on}$ and the ith *feedback-off time* $T^{i,d}_{off}$ as the lengths of intervals $I^{i,d}_{on}$ and $I^{i,d}_{off}$ respectively, in presence of the realization $d \in \mathcal{D}$ of the disturbance.

Fix an integer $N < \infty$ and then define the *average feedback-off time* as

$$T_{avg} := \min_{d\in\mathcal{D}} \frac{\sum\limits_{k=1}^{N} T^{k,d}_{off}}{N} \tag{2}$$

Our objective is to develop an intermittent feedback based policy which confines the state trajectory of system (1) inside the set W for every realization $d \in \mathcal{D}$ of the disturbance, while maximizing T_{avg}. We formalize it as follows

Problem 1: Develop an intermittent feedback based policy i.e. a control signal $u^* : [0, \infty] \to [-u_{max}, u_{max}]$ such that

$$u^*(t) := \begin{cases} f^*_u(x(t)), & t \in I^{i,d}_{on} \\ u^*_o(t), & t \in I^{i,d}_{off} \end{cases}$$

where, f^*_u and u^*_o are feedback and open-loop controls respectively,

1) under which the state trajectory of system (1) satisfies $x(t) \in W, \forall t \in [t^d_1, \infty)$ for every realization $d \in \mathcal{D}$ of the disturbance where, $t^d_1 < \infty$ depends on the particular d which gets realized.
2) which maximizes T_{avg}.

B. Preliminaries

We briefly review definitions and properties of an attainable set and a reachable set of an LTI system. Consider an LTI system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{3}$$

where, matrices A, B are as in system (1) and $u \in \mathcal{U}$.

Definition 2: [15]An initial condition of system (3) which can be steered to the origin in time t by using an admissible control $u \in \mathcal{U}$ is called as a *null-controllable* state in time t.

Definition 3: The reachable set of system (3) at time t is defined as the set of all null-controllable states in time t. It is characterized ([17], [15]) as

$$R(t) = \Big\{x \;\Big|\; x = \int_0^t e^{-A\tau}Bu(\tau)d\tau,\ \forall u \in \mathcal{U}\Big\} \tag{4}$$

The following lemma enlists properties of the set $R(t)$.

Lemma 4: [28] Let $R(t)$ be the reachable set of system (3) at time t. Then, the following hold.

1) For every $t \in [0, \infty)$, the set $R(t)$ is closed and convex.
2) For every $t \in [0, \infty)$, the set $R(t)$ has an antipodal symmetry i.e. $x \in R(t) \Leftrightarrow -x \in R(t)$.
3) $R(t_1) \subseteq R(t_2)$ if $t_1 \leq t_2$.

Definition 5: [15]The null-controllable region X_0 of system (3) is defined as

$$X_0 := \bigcup_{t\in[0,\infty)} R(t) \quad (5)$$

Definition 6: [28]The attainable set $\mathcal{A}$ of system (3) is the set of states which can be reached from the origin using an admissible control $u \in \mathcal{U}$.
It is characterized [28] as $\mathcal{A} = \bigcup_{t\in[0,\infty)} \mathcal{A}(t)$, where

$$\mathcal{A}(t) = \left\{ x \;\middle|\; x = e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau,\ \forall u \in \mathcal{U} \right\} \quad (6)$$

C. Conditions which necessitate intermittent feedback

Before attempting to solve Problem 1, we note that intermittent feedback is not always necessary to retain $x(t) \in W, \forall t \geq 0$. In particular, the necessity of intermittent feedback depends on the stability of the matrix A in (1) and the maximum magnitude d_{max} of the disturbance signal. In the absence of control action i.e. $u = 0$, (1) becomes

$$\dot{x}(t) = Ax(t) + Bd(t) \quad (7)$$

Let $\mathcal{A}_\mathcal{D}$ be the attainable set of (7) with d acting as a control input. Depending on d_{max} and eigenvalues of the matrix A, the following three situations are possible:

1) A is anti-hurwitz: In this case, $\mathcal{A}_\mathcal{D} = \mathbb{R}^n$ [28]. As $\mathcal{A}_\mathcal{D} \nsubseteq W$, some $d \in \mathcal{D}$ can drive the state trajectory of system (1) outside W, in absence of feedback control. Hence, feedback is necessary to confine the state trajectory in W.

2) A is neither hurwitz nor anti-hurwitz: In this case, the set $\mathcal{A}_\mathcal{D}$ is unbounded [28]. As $\mathcal{A}_\mathcal{D} \nsubseteq W$, some $d \in \mathcal{D}$ can drive the state trajectory of system (1) outside W, in absence of feedback control. Hence, feedback is necessary to confine the state trajectory in W.

3) A is hurwitz: In this case, the set $\mathcal{A}_\mathcal{D}$ is bounded [28] and its size depends on the disturbance bound d_{max}. If $\mathcal{A}_\mathcal{D} \subseteq W$, then disturbance cannot drive the state trajectory of system (1) outside W. Hence, once the state trajectory reaches the origin, feedback is not necessary. If $\mathcal{A}_\mathcal{D} \nsubseteq W$, then some $d \in \mathcal{D}$ can drive the state trajectory of system (1) outside W, in absence of feedback control. Hence, feedback is necessary to confine the state trajectory in W.

In summary, an intermittent feedback based control policy is necessary to solve Problem 1 if and only if $\mathcal{A}_\mathcal{D} \nsubseteq W$.

D. Control Policy

Our objective (Problem 1.1) is to retain the state trajectory of system (1) inside the safe region W. In the proposed policy, we construct a subset of W, namely *W-maximal feedback null-controllable set (W-MFNCS)*. The construction of W-MFNCS will be presented later in Section IV. It will be proved in Theorem 25 that by retaining the state trajectory of system (1) inside W-MFNCS, the average feedback-off time T_{avg} gets maximized. Hence, in the proposed policy, we retain the state trajectory of system (1) inside W-MFNCS.

The proposed policy consists of the following three elements:

1) A feedback control f_u^* to be applied in the intervals $I_{on}^{i,d}$ under which the state trajectory gets steered to the origin in min-max time, for every realization $d \in \mathcal{D}$ of the disturbance.
2) Open-loop control u_o^* to be applied in the intervals $I_{off}^{i,d}$. The open-loop control and the feedback-off time t_{off}^* are computed in such a way that the state trajectory remains inside W-MFNCS during $I_{off}^{i,d}$, for every realization $d \in \mathcal{D}$ of the disturbance.
3) An on/off rule according to which feedback is turned on/off. When the state trajectory reaches the origin under the feedback control f_u^*, the feedback is turned off for the duration t_{off}^* and then again turned back on.

The feedback control policy, open-loop control and on-off rule together ensure that T_{avg} (defined in (2)) gets maximized and for every realization $d \in \mathcal{D}$ of the disturbance, the state trajectory $x(t) \in W, \forall t \geq t_1^d$, where, $t_1^d < \infty$ depends on the particular d which gets realized.

Remark 7: In every feedback-on interval $I_{on}^{i,d}$, we choose to transfer the state to the origin in min-max time. It might seem that this choice is arbitrary and instead we might have transferred the state to some $x(t_{on}^{i,d}) \neq 0$. To justify our choice of $x(t_{on}^{i,d}) = 0$, we show that the corresponding average feedback-off time (i.e. T_{avg}) cannot be increased even if we had chosen $x(t_{on}^{i,d}) \neq 0$ (see Theorem 23).

Secondly, if we start every feedback-off interval from the origin (i.e. $x(t_{on}^{i,d}) = 0$), then the corresponding optimal open-loop control turns out to be particularly simple ($u_o^*(t) = 0, \forall t \in I_{off}^{i,d}$, see Theorem 19).

Thirdly, if we had chosen $x(t_{on}^{i,d}) \neq 0$, then the corresponding optimal open-loop control would have been hard to compute and would require the solution of a max-min time-optimal control problem separately for each initial condition. This would be prohibitively expensive computationally for many applications, with no guaranteed convergence for known algorithms [31].

Remark 8: In Remark 7, we have justified why we choose $x(t_{on}^{i,d}) = 0$. However, the transfer of the state to the origin during feedback-on interval might have been done in multiple ways [12]. However, in each of these methods, one would need to solve a differential game [12] since $x(t_{on}^{i,d}) = 0$ is required no matter what value of $d(t)$ gets realized. Solutions of such differential games rarely lead to nice closed form feedback solutions [33]. Due to recent contributions in [13], explicit feedback solutions are available if we pose the feedback-on control synthesis problem as a min-max time-optimal problem (see Problem 9).

III. CONTROL POLICY: FEEDBACK ON

A. Formulation of the feedback-on optimization problem

As pointed out in Remark 7 and Remark 8, it is necessary to steer the state trajectory of system (1) to the origin in min-max time during feedback-on intervals, for any realization $d \in \mathcal{D}$ of the disturbance. Hence, the feedback control f_u^* in the proposed policy is obtained by solving the following problem:

Problem 9: For system (1), find a feedback control $f_u^* : \mathbb{R}^n \to [-u_{max}, u_{max}]$ whose open-loop representation $u_f^* \in$

$\mathcal{U}$ solves the following optimization problem:

$$\arg\min_{u\in\mathcal{U}}\max_{d\in\mathcal{D}} \quad t_f \qquad \text{s.t.} \quad t_f = \bar{t}(x_0, u, d)$$

Henceforth, we refer to Problem 9 as the *feedback-on min-max problem (FOMMP).*

B. Solution of the FOMMP

Define $v_{max} := u_{max} - d_{max}$, where u_{max} and d_{max} are the bounds on control input and disturbance, respectively. Then, define $\mathcal{V} := \left\{ v \in \mathcal{L} \middle| |v(t)| \leq v_{max} < \infty, \forall t \geq 0 \right\}$. Consider the following system

$$\dot{x}(t) = Ax(t) + Bv(t) \tag{8}$$

where, matrices A, B are as in (1) and $v \in \mathcal{V}$. Let X_0 be the null-controllable region (see Definition 5) of system (8). Now, consider the following problem:

Problem 10: Find a feedback control $f_v^* : X_0 \to [-v_{max}, v_{max}]$ whose open-loop representation $v_f^* \in \mathcal{V}$ solves the following optimization problem:

$$\min_{v\in\mathcal{V}} \quad t_f \qquad \text{s.t.} \quad t_f = \bar{t}(x_0, v, 0)$$

Henceforth, we refer to Problem 10 as the *time minimization problem (TMP).*

Assume that the feedback solution f_v^* of the TMP is known. It is known that the open-loop representation v_f^* of f_v^* is bang-bang i.e. it switches between v_{max} and $-v_{max}$[16]. Define

$$f_u^*(x(t)) := \begin{cases} u_{max}\text{sign}(f_v^*(x(t))) & \text{if } x(t) \in X_0 \\ 0 & \text{if } x(t) \notin X_0 \end{cases} \tag{9}$$

$$f_d^*(x(t)) := \begin{cases} -d_{max}\text{sign}(f_v^*(x(t))) & \text{if } x(t) \in X_0 \\ 0 & \text{if } x(t) \notin X_0 \end{cases} \tag{10}$$

Then, it follows that

$$f_v^*(x(t)) = f_u^*(x(t)) + f_d^*(x(t)), \quad \forall x(t) \in X_0$$

The following result from ([13] and [14]) establishes the equivalence between the TMP and the FOMMP.

Theorem 11: ([13],[14]) Consider the FOMMP for system (1) and the TMP for system (8). Then, the following hold.

1) For every $x_0 \in X_0, \forall u \in \mathcal{U}$ and $\forall d \in \mathcal{D}$,

$$\bar{t}(x_0, f_u^*, d) \;\leq\; \bar{t}(x_0, f_u^*, f_d^*) \;\leq\; \bar{t}(x_0, u, f_d^*)$$

2) For every $x_0 \in X_0$, the pair (f_u^*, f_d^*) is the unique solution of the min-max optimization in the FOMMP for system (1) if and only if f_v^* is the unique feedback solution of the TMP for system (8).

It follows from Theorem 11.2 that once we obtain the feedback solution f_v^* of the TMP, (9) directly gives the solution f_u^* of the FOMMP. Hence, in order to obtain f_u^*, it is sufficient to compute f_v^*. A technique of computing f_v^* and f_u^* is given in the next section.

According to the proposed control policy, the feedback control applied during feedback-on intervals $I_{on}^{i,d}, \forall i \geq 1$ is the solution f_u^* of the FOMMP, irrespective of disturbance realization. Let x_{on}^i be the state of system (1) in the beginning of interval I_{on}^i. Then, it follows from Theorem 11.1 that for every $x_{on}^i \in X_0$ and $\forall d \in \mathcal{D}$ other than the open-loop representation of f_d^*, the feedback control f_u^* steers the state trajectory of system (1) from x_{on}^i to the origin in time less than that of f_d^*.

Note that Theorem 11.2 gives the solution of the FOMMP if the initial condition $x_0 \in X_0$. The following lemma shows that for an initial condition $x_0 \notin X_0$, a solution of the FOMMP does not exist.

Lemma 12: For every admissible feedback control f_u and $\forall x_0 \notin X_0$, there exists $\tilde{d} \in \mathcal{D}$ such that $\bar{t}(x_0, f_u, \tilde{d}) = \infty$.

C. Computation of feedback control

It is well known that the feedback solution f_v^* of the TMP consists of switching surfaces and a state-feedback law [17]. The computation of switching surfaces and the construction of a state-feedback law using *Gröbner* basis based implicitization has been given recently in [15].

Define the set $P := \{1, 2, \ldots, n\}$. Let $M_k^{\pm}, k \in P$ be the switching surfaces of the TMP where, M_k^{+} corresponds to bang-bang inputs $v \in \mathcal{V}$ with $v(0) = v_{max}$ and at most $k-1$ switches while M_k^{-} corresponds to bang-bang inputs $v \in \mathcal{V}$ with $v(0) = -v_{max}$ and at most $k-1$ switches. Define $M_k := M_k^{+} \cup M_k^{-}$. Then, the feedback solutions of the TMP and the FOMMP are given ([13],[15]) as follows

$$f_v^*(x(t)) = \begin{cases} v_{max} & \text{if } x(t) \in M_k^{+} \setminus M_{k-1}\ , \forall k \in P \setminus \{n\} \\ -v_{max} & \text{if } x(t) \in M_k^{-} \setminus M_{k-1}\ , \forall k \in P \setminus \{n\} \end{cases}$$

$$f_u^*(x(t)) = u_{max}\text{sign}(f_v^*(x(t)))$$
$$f_d^*(x(t)) = -d_{max}\text{sign}(f_v^*(x(t)))$$

D. Feedback null-controllable sets

In this section, we characterize all initial conditions of system (1) which can be steered to the origin under the solution f_u^* of the FOMMP, for every realization $d \in \mathcal{D}$ of the disturbance.

Definition 13: The feedback null-controllable set of system (1) at time t, denoted by $S^f(t)$, is defined as

$$S^f(t) := \{x_0 \in \mathbb{R}^n \quad | \quad \bar{t}(x_0, f_u^*, d) \leq t, \;\; \forall d \in \mathcal{D}\}$$

Definition 14: The feedback null-controllable region of system (1), denoted by S_0^f, is defined as

$$S_0^f := \bigcup_{t\in[0,\infty)} S^f(t) \tag{11}$$

Define t_{max} as

$$t_{max} := \max_{[0,\infty)} \quad t \qquad \text{s.t.} \quad S^f(t) \subseteq W \tag{12}$$

where, W is the prespecified safe region.

Definition 15: The W-maximal feedback null-controllable set (W-MFNCS) of system (1), denoted by S^*, is defined as $S^* := S^f(t_{max})$.

The following lemma enlists the properties of S^*.

Lemma 16: Let W be the prespecified safe region and S^* be as in Definition 15. Then,

1) S^* is a closed, convex set with an antipodal symmetry.
2) $0 \in S^*$.
3) $S^* \subseteq (S_0^f \cap W)$.

The following lemma shows the invariance of the set S^* under the solution f_u^* of the FOMMP.

Lemma 17: Let $x_0 \in S^*$ and $\bar{x}(t_{final}, x_0, f_u^*, d) = 0$ for some $d \in \mathcal{D}$. Let t_{max} be as defined in (12). Then,

1) $t_{final} \leq t_{max}$.
2) $\bar{x}(t, x_0, f_u^*, d) \in S^*, \quad \forall t \in [0, t_{final}]$.

IV. CONTROL POLICY: OPEN LOOP

A. Formulation of the open-loop optimization problem

Our objective (Problem 1.2) is to maximize T_{avg} (defined in (2)). It will be proved in Theorem 25 that in order achieve the maximize possible value of T_{avg}, it is necessary to retain the state trajectory of system (1) inside S^* (Definition 15), for every realization $d \in \mathcal{D}$ of the disturbance. Hence, in the proposed policy, we retain the state trajectory of system (1) inside S^* for every $d \in \mathcal{D}$. Further, in order maximize T_{avg}, we need to maximize the feedback-off time while retaining the state trajectory inside S^* for every $d \in \mathcal{D}$. Hence, the open-loop control and the feedback-off time in the proposed policy are obtained by solving the following problem:

Problem 18: Let $x_0 \in S^*$ be an initial condition of system (1). Find an open-loop solution $u_o^* \in \mathcal{U}$ of the following optimization problem:

$$
\begin{aligned}
u_o^*(t, x_0) &= \arg \max_{u \in \mathcal{U}} \min_{d \in \mathcal{D}} \quad t_o \\
t_{off}^*(x_0) &= \max_{u \in \mathcal{U}} \min_{d \in \mathcal{D}} \quad t_o \\
&\text{s.t.} \quad \bar{x}(t, x_0, u, d) \in S^*, \quad \forall t \in [0, t_o]
\end{aligned}
\tag{13}
$$

Henceforth, we refer to Problem 18 as the *open-loop max-min problem (OLMMP).*

B. Solution of OLMMP

In this section, first we obtain the solution $u_o^*(t, 0)$ of the OLMMP for $x_0 = 0$, in the following theorem.

Theorem 19: Consider system (1) with initial condition $x_0 = 0$. Then, the solution $u_o^*(t, 0) \in \mathcal{U}$ of the OLMMP is

$$u_o^*(t, 0) = 0, \quad \forall t \in [0, t_{off}^*(0)].$$

We will show in the following theorem that $t_{off}^*(x_0) \leq t_{off}^*(0)$ for every $x_0 \in S^*$.

Theorem 20: For every $x_0 \in S^*$, $t_{off}^*(x_0) \leq t_{off}^*(0)$.

Recall that $t_{on}^{i,d}$ are the end-points of the feedback-on intervals in presence of a realization $d \in \mathcal{D}$ of the disturbance. Let x^d be the state trajectory of system (1) under the proposed policy and disturbance $d \in \mathcal{D}$. In order to maximize T_{avg} (Problem 1.2), we need to maximize the feedback-off duration. Hence, based on Theorem 20, we ensure that $x^d(t_{on}^{i,d}) = 0, \forall i \geq 1$ and $\forall d \in \mathcal{D}$. Then, it follows from Theorem 19 that the open-loop control and the feedback-off time in the proposed policy are solutions $u_o^*(t, 0)$ and $t_{off}^*(0)$ of the OLMMP, irrespective of the realized $d \in \mathcal{D}$.

It follows from Theorem 19 that the feedback-off time $t_{off}^*(0)$ is the solution of the following optimization problem:

$$\min_{d \in D} \quad t_o \qquad \text{s.t.} \quad \bar{x}(t_o, 0, 0, d) \in \partial S^* \tag{14}$$

The optimization problem in (14) is an optimal control problem. A technique of obtaining the expression of ∂S^* is developed in [15]. By using the expression of ∂S^* and standard tools from optimal control theory [31], the optimization problem in (14) can be solved.

Remark 21: The open-loop control $u_o^*(t, 0) = 0$ in the proposed policy does not require any run-time computation. In addition, it needs zero controller effort during feedback-off intervals. Hence, it is very efficient.

V. COMPLETE CONTROL POLICY

Let f_u^* be the feedback control obtained by solving the FOMMP. Let $u_o^*(t, 0)$ and $t_{off}^*(0)$ be the open-loop control and the feedback-off time obtained by solving the OLMMP for the zero initial condition, respectively. Recall that $I_{on}^{i,d}$ and $I_{off}^{i,d}$ are the ith feedback-on and feedback-off intervals in presence of the realization $d \in \mathcal{D}$ of the disturbance, respectively. Then, for any $d \in \mathcal{D}$, the control input u^* according to the proposed policy is

$$u^*(t) = \begin{cases} f_u^*(x(t)), & t \in I_{on}^{i,d} \\ u_o^*(t, 0), & t \in I_{off}^{i,d} \end{cases}$$

The following theorem shows that for every realization $d \in \mathcal{D}$ of the disturbance, the proposed policy retains the state trajectory of system (1) inside set $S^* \subseteq W$ and hence, satisfies the requirement in Problem 1.1.

Theorem 22: Let x^d be the state trajectory of system (1) under the proposed policy, any $d \in \mathcal{D}$ and an initial condition $x^d(0) = x_0 \in S_0$. Let $t_1 < \infty$ be the first time instant at which $x^d(t) \in S^*$. Then, $x^d(t) \in S^* \subseteq W, \quad \forall t \geq t_1$.

The following theorem shows that the average feedback-off time under the proposed policy is more than that of any other intermittent feedback based policy which retains the state trajectory of system (1) inside the set S^*.

Theorem 23: Let P be any intermittent feedback based policy which retains the state trajectory of system (1) inside the set S^*. Let T_{avg}^* and T_{avg}^P be the average feedback-off times under the proposed policy and policy P, respectively. Then, $T_{avg}^* \geq T_{avg}^P$.

Recall that $T_{on}^{i,d}$ and $T_{off}^{i,d}$ are the ith feedback-on and feedback-off times respectively, in presence of disturbance $d \in \mathcal{D}$. Recall that $t_{off}^*(0)$ is the feedback-off time in the proposed policy and t_{max} is defined in (12). The following lemma shows the relation between $(T_{on}^{i,d}, T_{off}^{i,d})$ and $(t_{max}, t_{off}^*(0))$.

Lemma 24: According to the proposed policy, for any realization $d \in \mathcal{D}$ of the disturbance,

1) $T_{off}^{i,d} = t_{off}^*(0), \ \forall i \geq 1$.
2) $T_{on}^{i,d} \leq t_{max}, \ \forall i \geq 2$.

It follows from Lemma 12 and Lemma 17 that only sets of the form $S^f(t) \subseteq W$ (i.e. for $t \leq t_{max}$) are invariant under the proposed policy. Define $\mathcal{Z} := \left\{ S^f(t) \;\middle|\; t \leq t_{max} \right\}$. Let $T_{S,avg}$ denote the average feedback-off time obtained when the proposed policy retains the state trajectory of system (1) inside $S \in \mathcal{Z}$. Then, the following theorem shows that S^* gives the maximum of $T_{S,avg}$ over the elements of $\mathcal{Z}$.

Theorem 25: For every $S \in \mathcal{Z}$, $T_{S,avg} \leq T_{S^*,avg}$.

VI. SIMULATION RESULTS

Let W be the prespecified safe region with safety limit $\alpha = 0.1$. Let $u_{max} = 1$, $d_{max} = 0.2$ and hence, $v_{max} =$

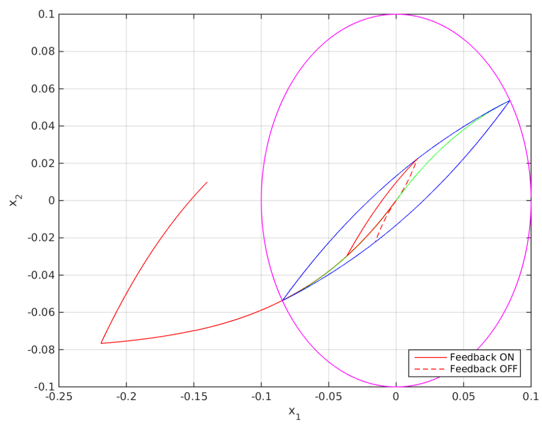

Fig. 2. State trajectory

$u_{max} - d_{max} = 0.8$. Let $x_0 = [-0.14, 0.01]$ be an initial condition of system (1) with

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The value of t_{max} (defined in (12)) is 0.1114s. The value of $t^*_{off}(0)$ (solution of the OLMMP) is 0.0749s. The state trajectory of this system under the proposed policy, some disturbance $d \in \mathcal{D}$ and initial condition x_0 is plotted in Figure 2, for the duration 0.6615s. The feedback-on and feedback-off times are $T^{1,d}_{on} = 0.4003$s, $T^{1,d}_{off} = 0.0749$s, $T^{2,d}_{on} = 0.1114$s and $T^{2,d}_{off} = 0.0749$s. For any integer N, the average feedback-off time T_{avg} is 0.0749s.

VII. CONCLUSION

In this paper, we developed an intermittent feedback based control policy for continuous LTI systems with distinct and rational eigenvalues, subject to a bounded disturbance signal. The proposed policy confines the state trajectory of system in a prespecified safe region while maximizing the average feedback-off time. As the set of rational numbers is dense in the set of real numbers, it is possible to extend the proposed policy for systems with real eigenvalues. The work of extending the proposed policy for systems with complex eigenvalues is currently in progress.

REFERENCES

[1] W. Wong and R. Brockett, "Systems with finite communication bandwidth constraints-part I: state estimation problems", *IEEE Transactions on Automatic Control*, vol 42, no. 9, pp. 1294-1299, 1997.

[2] W. Wong and R. Brockett, "Systems with finite communication bandwidth constraints-part II: stabilization with limited information feedback", *IEEE Transactions on Automatic Control*, vol. 44, no. 5, pp. 1049-1053, 1999.

[3] R. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems", *IEEE Transactions on Automatic Control*, vol. 45, no. 7, pp. 1279-1289, 2000.

[4] G. Nair and R. Evans, "Stabilization with data-rate-limited feedback: tightest attainable bounds", *Systems and Control Letters*, vol. 41, no. 1, pp. 49-56, 2000.

[5] A. Anta and P. Tabuada, "To sample or not to sample: self-triggered control for Nonlinear Systems", *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030-2042, 2010.

[6] W. Heemels, K. Johansson and P. Tabuada, "An introduction to event-triggered and self-triggered control", *Proceedings of the IEEE conference on decision and control*, Hawaii, USA, 2012.

[7] M. Vidyasagar, "Optimal rejection of persistent bounded disturbances", *IEEE Transactions on Automatic Control*, vol. AC-31, no. 6, pp. 527-534, 1986.

[8] N. Elia and M. Dahleh, "Minimization of the worst case peak-to-peak gain via dynamic programming: state feedback case", *IEEE Transactions on Automatic Control*, vol. 45, no. 4, pp. 687-701, 2000.

[9] J. Abedor, K. Nagpal and K. Poolla, "A linear matrix inequality approach to peak-to-peak gain minimization", *International Journal of Robust and Nonlinear Control*, vol. 6, pp. 899-927, 1996.

[10] D. Chakraborty and J. Hammer, "Optimal control during feedback failure", *International Journal of Control*, vol 82, no. 8, pp. 1448-1468, 2009.

[11] Z. Yu and J. Hammer, "Fastest recovery after feedback disruption", *International Journal of Control*, vo. 89, no. 10, pp. 2121-2138, 2016.

[12] R. Issacs, "Differential Games", Dover publications, 1999.

[13] A. Mulla and D. Chakraborty, "MinMax Time Consensus Tracking With Communication Guarantee", *IEEE Transactions on Automatic Control*, vol. 63, no. 1, 2018.

[14] M. Bardi and P. Soravia, "A PDE framework for games of pursuit-evasion type", in Differential Games and Applications, New York, USA: Springer, pp. 62-71, 1989.

[15] D. Patil, A. Mulla, D. Chakraborty and H. Pillai, "Computation of feedback control for time optimal state transfer using Groebner basis", *Systems and Control Letters*, vol. 79, pp. 1-7, 2015.

[16] L. Pontryagin, V. Boltyanskii, R. Gamkrelidze and E. Mishchenko, "The mathematical theory of optimal processes", Gordon and Breach science publications, 1962.

[17] O. Hajek, "Terminal manifolds and switching locus", *Theory of Computing Systems*, vol. 6, no. 3, pp. 289-301, 1971.

[18] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems", *Proceedings of the IEEE*, vol. 95, no. 1, pp. 138-162, 2007.

[19] F. Wang and D. Liu, "Networked control systems: theory and applications", Springer, 2008.

[20] J. Baillieul and P. Antsaklis, "Control and communication challenges in networked real-time systems", *Proceedings of the IEEE*, vol. 95, no. 1, pp. 9-28, 2007.

[21] G. Nair, F. Fagnani, S. Zampieri, and R. Evans, "Feedback control under data rate constraints: an overview", *Proceedings of the IEEE*, vol. 95, no. 1, pp. 108-137, 2007.

[22] E. Barmpounakis, E. Vlahogianni and J. Golias, "Unmanned aerial aircraft systems for transportation engineering: current practice and future challenges", *International Journal of Transportation Science and Technology*, vol. 5, no. 3, pp. 111-122, 2016.

[23] K. Kanistras, G. Martins, M. Rutherford and K. Valavanis, "Survey of unmanned aerial vehicles (UAVs) for traffic monitoring", *Handbook of Unmanned Aerial Vehicles*, Springer, pp. 2643-2666, 2014.

[24] D. Cox, J. Little and D. O'Shea, "Ideals, Varieties and Algorithms:An Indroduction to Computational Algebraic Geometry and Commutative Algebra", Springer, 2007.

[25] R. Clarke, "Understanding the drone epidemic", *Computer Law and Security Review*, vol. 30, no.3, pp. 230-246, 2014.

[26] J. Meyer, F. du Plessis, and W. Clarke, "Design considerations for long endurance unmanned aerial vehicles", *InTech*, 2009.

[27] L. Corral, I. Fronza, N. Ioini and A. Ibershimi, "A measurement tool to track drones battery consumption during flights", *Proceedings of International Conference on Mobile Web and Information Systems*, Vienna, Austria, 2016.

[28] H. Hermes and J. LaSalle, "Functional Analysis and Time Optimal Control", Elsevier, 1969.

[29] S. Boyd and L. Vandenberghe, "Convex Optimization", Cambridge University Press, 2004.

[30] A. Joshi, N. Limbu, I. Ahuja, A. Mulla, H. Chung and D. Chakraborty, "Implementation of distributed consensus on an outdoor testbed", *Proceedings of the European Control Conference*, Aalborg, Denmark, 2016.

[31] A. Bryson and Y. Ho, "Applied Optimal Control: Optimization, Estimation and Control", Taylor and Francis, 1980.

[32] A. Joshi, A. Wala, M. Ludhiyani, S. Singh, M. Gagrani, S. Hazra, D. Chakraborty, D. Manjunath and H. Chung, "Implementation of distributed consensus with guaranteed real-time communication on an outdoor quadrotor testbed", *Proceedings of the IEEE conference on decision and control*, Melbourne, Australia, 2017.

[33] T. Raivio and H. Ehtamo, "On the Numerical Solution of a Class of Pursuit-Evasion Games", *Annals of the International Society of Dynamic Games*, vol. 5, pp. 177-194, 2000.