Partial Pole Placement with Minimum Norm Controller

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***Abstract*— The problem of placing an arbitrary subset (m) of the (n) closed loop eigenvalues of a n^{th} order continuous time single input linear time invariant(LTI) system, using full state feedback, is considered. The required locations of the remaining $(n-m)$ closed loop eigenvalues are not precisely specified. However, they are required to be placed anywhere inside a pre-defined region in the complex plane. The resulting non-uniqueness is utilized to minimize the controller effort through optimization of the feedback gain vector norm. Using a variant of the boundary crossing theorem, the region constraint on the unspecified $(n-m)$ poles is translated into a quadratic constraint on the characteristic polynomial coefficients. The resulting quadratically constrained quadratic program can be approximated by a quadratic program with linear constraints. The proposed theory is demonstrated for power oscillation damping controller design, where the eigenvalues corresponding to poorly damped electro-mechanical modes are critical for performance and hence are specified precisely by the designer, whereas the remaining eigenvalues are non-critical and need not be specified precisely. Acceptable closed loop pole placement is achieved for this example along with a 51% reduction in controller norm.**

I. INTRODUCTION

Conventional state feedback control can place closed loop poles of a controllable linear time invariant system at arbitrary locations in the complex plane. However, in some applications, only a given subset of the closed loop poles are of interest and it is enough to place this subset of poles at precise pre-specified locations. The other closed loop poles can assume any positions in (or in a specified subset of) the stable region of the complex plane. It is well known [2] that if the desired locations of *all* the closed loop poles are specified then for a SISO system the required feedback gain vector is unique. However if only a subset of the closed loop poles are specified, the extra degrees of freedom associated with the unspecified poles can be utilized to minimize the control effort associated with the controller. It is shown in this article that this problem can be posed as a quadratic program which can in turn be solved efficiently by standard numerical methods.

The proposed theory is demonstrated for inter-area oscillation damping controller design in power system applications. Inter-area oscillations (associated with inter-area modes) are electro-mechanical oscillations of order 0.1-0.8 Hz and are inherently present in large inter connected power systems [5]. Hence, it is required to carefully place only those inter-area oscillation modes to ensure desired performance following disturbances. There is no need to worry about the remaining modes as long as their settling times do not exceed those in open loop. Due to the very nature of the non-critical modes, higher control efforts are required unless they are left alone to take their natural course. This results in an overall increase in the norm of the feedback gain vector and hence, costlier actuators.

The conventional way to address such a problem would be to : (i) choose closed loop critical pole locations according to the design specifications (ii) choose the non-critical pole locations same as their open loop positions. However, in established pole placement methods [12], [13] the compensation of feedback gain vector is a coupled procedure which depends both on the critical and non-critical closed loop pole choices. As a result it often turns out in practice that this strategy leads again to a high value of the resulting feedback gain resulting in higher control effort requirements and more costly actuators.

It is proposed in this article that the state feedback controller can be designed in such a way that the critical poles will be placed precisely at the pre-specified locations. The controller norm is minimized explicitly under these constraints and the remaining non-critical poles are left to assume whatever natural position they might assume due to the optimization process. It is assumed that the non-critical modes are already stable in the open loop, and that they may be allowed to move freely about their open loop locations as long as they do not lose stability. Additionally, it is often required that all closed loop poles should have a minimum damping which implies that they should be located to the left of a given vertical line in the left half of the complex plane. Using a version of the boundary crossing theorem [1] it shown that (stability or minimum damping) requirements on the closed loop non-critical poles can be translated into constraints in the coefficient space of the characteristic polynomial corresponding to the non-critical poles. These requirements define a quadratic constraint on the subsequent minimization problem. It is shown that this problem can be posed as a direct minimization of the feedback gain vector norm with two types of constraints: (i) linear equality constraints arising out of the precise placement requirement of the critical closed loop poles, and (ii) quadratic inequality constraints arising out of the regional placement requirement of the closed loop non-critical poles. By standard results on semi-definite programming [3], [4] and [15], it is shown that this problem has a unique minimum which in turn can be

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computed by semi-definite programming methods.

This formulation can be further simplified by approximating the quadratic constraints on the characteristic polynomial coefficients by inscribing hypercubes in the coefficient space. This reduces the quadratically constrained quadratic program to a quadratic program with linear constraints which can be solved efficiently using a variety of optimization techniques [3], [4].

Pole placement algorithms have been studied intensively in the literature. Various researchers (e.g. [11], [14], [10], [12], [13] and [9]) have focused on finding numerically stable and efficient algorithms for multi input multi output pole placement by minimizing the condition number of a related eigenvector matrix. In many practical applications, pole placement within a desired subset of the complex plane is relevant. This approach, called regional pole placement, has been studied by [8] and references therein. In this work we pose a novel problem with the following features: (a) precise placement of a (critical) subset of the closed loop poles (b) regional pole placement for the (non-critical) remaining closed loop poles within a specified region of complex plane (c) minimization of the norm of controller under the above constraints.

II. Notation and Problem Formulation

A. Problem Formulation

Consider a continuous time LTI single-input system defined by the following state space equation

$$\dot{x} = Ax + bu \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}$, $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$. Assume that the pair (A,b) is controllable and define $k := \begin{bmatrix} k_1 & k_2 \cdots k_n \end{bmatrix}^T \in \mathbb{R}^n$. It is well known that, a linear state feedback control law of the form $u = -k^T x$ can be designed to place *all* the eigenvalues of the closed loop system

$$\dot{x} = (A - bk^T)x \tag{2}$$

at any arbitrary locations of the complex plane $\mathbb{C}$.

However, we are interested in applications where only a few critical closed loop eigenvalues are specified. As described in the introduction, the non-critical eigenvalues are allowed to assume any value in (or in a pre-specified subset of) the stable region of the complex plane. Without loss of generality, we assume that the first m eigenvalues of A are critical and their closed loop positions are specified. Let us denote the n eigenvalues of A by $\{\lambda_1, \lambda_2, \ldots, \lambda_m, \lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n\}$ $(m \leq n)$, which, in addition, are assumed to be distinct. Of these, $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ are critical and are required to be placed at $\{-\mu_1, -\mu_2, \ldots, -\mu_m\}$ in the closed loop, whereas the remaining $(n-m)$ eigenvalues $\{\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n\}$ are non-critical and are not associated with any desired closed loop location. However it is often the case that a minimum damping (or maximum acceptable settling time) requirement exists on all the closed loop poles. This is conventionally specified by restricting the pole locations to the left of a given vertical line in the complex plane. In general we will assume that $(n-m)$ eigenvalues of $(A - bk^T)$ are required to be located in a subset S of the left half of the complex plane. In this article we will define S as follows:

$$S = \{s \in \mathbb{C} : \ \mathrm{Re}(s) < \gamma\} \tag{3}$$

where the value of $\gamma \leq 0$ (signifying the maximum allowable settling time for the non-critical poles) is specified by the designer. Then the problem described in the introduction can be simply formulated as:

Problem 1: Find $\inf \|k\|_2$ such that the eigenvalues of $(A - bk^T)$ have the following properties:

1) m out of the total n eigenvalues are placed at $\{-\mu_1, -\mu_2, \ldots, -\mu_m\}$.
2) remaining $n-m$ eigenvalues are placed anywhere in a subset S of the left half of $\mathbb{C}$.

B. Additional Notation

According to the design specifications, m of the closed loop poles are required to be placed at $\{-\mu_1, -\mu_2, \ldots, -\mu_m\}$ while the remaining $(n-m)$ poles can assume any location in the region S. Denote the unspecified closed loop eigenvalues as $\{-p_1, -p_2, \ldots, -p_{n-m}\}$. Without loss of generality, let the closed loop poles be ordered as follows

$$\{-\mu_1, -\mu_2, \ldots, -\mu_m, -p_1, -p_2, \ldots, -p_{n-m}\} \tag{4}$$

Further define

$$\prod_{i=1}^{m} (s + \mu_i) := s^m + \alpha_{m-1} s^{m-1} + \ldots + \alpha_1 s + \alpha_0$$

$$\prod_{j=1}^{n-m} (s + p_j) := s^{n-m} + \beta_{n-m-1} s^{n-m-1} + \ldots + \beta_1 s + \beta_0$$

where

$$\begin{aligned}
\alpha_{m-1} &= \sum_{i_1=1}^{m} \mu_{i_1} & \beta_{n-m-1} &= \sum_{i_1=1}^{n-m} p_{i_1} \\
\alpha_{m-2} &= \sum_{i_1<i_2}^{m} \mu_{i_1}\mu_{i_2} & \beta_{n-m-2} &= \sum_{i_1<i_2}^{n-m} p_{i_1} p_{i_2} \\
&\vdots & &\vdots \\
\alpha_1 &= \sum_{i_1<\ldots<i_{m-1}}^{m} \mu_{i_1}\ldots\mu_{i_{m-1}} & \beta_1 &= \sum_{i_1<\ldots<i_{n-m-1}}^{n-m} p_{i_1} p_{i_2}\ldots p_{i_{n-m-1}} \\
\alpha_0 &= \mu_1\mu_2\ldots\mu_m & \beta_0 &= p_1 p_2 \ldots p_{n-m}
\end{aligned}$$

Then the characteristic equation of the closed loop system will be

$$\sigma(s) = \underbrace{\left[\prod_{i=1}^{m}(s+\mu_i)\right]}_{\text{Q}} \underbrace{\left[\prod_{j=1}^{n-m}(s+p_j)\right]}_{\text{R}} \tag{5}$$

Using the above notation, the characteristic polynomial (5) is divided into two parts: (Q) - polynomial of known coefficients and (R) - polynomial of unknown coefficients. Clearly the Q polynomial is completely defined from the problem specification. However the only requirement of the

R polynomial is that its roots should be located in a pre-specified region $S \in \mathbb{C}$ defined in (3).

Our objective is to pose Problem 1 as a quadratic program so as to make it numerically tractable. For this purpose it would be convenient to translate the requirement on the poles $(-p_i \in S,\ i = 1,..,n-m)$ of polynomial R into requirements on the coefficients $(\beta_0, \beta_1, ..., \beta_{n-m-1})$ of the polynomial R. For this purpose let us denote the set of all $(n-m)^{th}$ degree monic polynomials with real coefficients as $P_{n-m}(s)$. Further assume that the region $S \in \mathbb{C}$ corresponds to a set $C_s \subset P_{n-m}(s)$. In other words, we assume that each $(n-m)^{th}$ degree monic polynomials with roots in S will belong to the set C_s. Then problem 1 is equivalent to the following problem:

Problem 2: Find $\inf \|k\|_2$ such that $(A - bk^T)$ has the following properties:

1) m out of the total n eigenvalues are placed at $\{-\mu_1, -\mu_2, \ldots, -\mu_m\}$.
2) the polynomial $R \in C_s$.

Note that the set C_s is open [1] and hence the minimum may not be achieved. This issue is addressed in the next section by replacing C_s with a compact subset of C_s.

III. The stability ball in coefficient space

Let us consider a $(n-m)^{th}$ degree monic polynomial $\beta(s) \in P_{n-m}(s)$, represented as follows

$$\beta(s) := s^{n-m} + \beta_{n-m-1}s^{n-m-1} + \ldots + \beta_1 s + \beta_0 \qquad (6)$$

Define $\beta := \begin{bmatrix} \beta_{n-m-1} & \beta_{n-m-2} & \ldots & \beta_0 \end{bmatrix}^T$ and denote the vector space of $n-m$ tuples formed by the non-leading coefficients of the elements of $P_{n-m}(s)$ as $\mathscr{V}_p$, such that $\beta \in \mathscr{V}_p$. On $\mathscr{V}_p$, the Euclidean norm is defined as $\| \beta \|_2^2 = \beta_0^2 + \beta_1^2 + \beta_2^2 + \ldots + \beta_{n-m-1}^2$, while a ball of monic polynomials with center at $\widehat{\beta}(s) \in P_{n-m}(s)$ and radius r is given by

$$B\left(\widehat{\beta}(s), r\right) := \left\{\beta(s) \in P_{n-m}(s) \ : \ \| \beta - \widehat{\beta} \|_2 < r\right\}$$

where β and $\widehat{\beta}$ are in $\mathscr{V}_p$.

Consider the set $S \subset \mathbb{C}$ defined in (3) and denote by ∂S and ∂B the boundary of the sets S and $B\left(\widehat{\beta}(s), r\right)$ respectively. Then a maximal ball of polynomials having roots in S can be characterized by the following theorem.

Theorem 1: [1, Theorem 3.1] Given a polynomial $\widehat{\beta}(s) \in P_{n-m}(s)$, having all its roots in S, there exist a positive real number r such that every polynomial contained in $B\left(\widehat{\beta}(s), r\right)$ has all its roots in S and at least one polynomial in ∂B has one of its roots in ∂S.

Recall that Problem 1 and 2 are equivalent if the set of polynomials C_s corresponding to the desired pole region S (see (3)) can be computed. The above theorem shows that corresponding to any $S \in \mathbb{C}$ the corresponding set $B\left(\widehat{\beta}(s), r\right) \subseteq C_s$. However to compute $B\left(\widehat{\beta}(s), r\right)$ explicitly we still need *a priori* a polynomial $\widehat{\beta}(s) \in C_s$ and subsequently we need to compute the radius of stability r. For our choice of $\widehat{\beta}(s)$, we propose to use the polynomial formed out of the open-loop non-critical poles as follows:

$$\widehat{\beta}(s) = \left[\prod_{j=m+1}^{n} (s + \lambda_j)\right] \qquad (7)$$

Usually, those open loop eigenvalues, which are stable and already have adequate damping, are classified as non-critical. Hence in most practical scenarios, $\lambda_{m+1}, \lambda_{m+2}, \ldots, \lambda_n \in S$ and hence $\widehat{\beta}(s) \in C_s$.

The second step in calculating the maximal stability ball $B(\widehat{\beta}(s), r)$ is to estimate the radius r. For this purpose we can use the following result where the set S is assumed to be the entire left half of $\mathbb{C}$ [1]. A slight modification in the definition of $\beta(s)$ and $\widehat{\beta}(s)$ extends Theorem 2 to cases where S is defined for $\gamma < 0$ in (3).

Denote the monic polynomial corresponding to (7) as $\widehat{\beta}(s) := s^{n-m} + \widehat{\beta}_{n-m-1}s^{n-m-1} + \ldots + \widehat{\beta}_1 s + \widehat{\beta}_0$ and define the $n-m$ tuple $\widehat{\beta}$ as $\widehat{\beta} := \begin{bmatrix} \widehat{\beta}_{n-m-1} \ldots \widehat{\beta}_1 & \widehat{\beta}_0 \end{bmatrix}^T \in \mathscr{V}_p$

Theorem 2: [1, Theorem 3.2] The radius of largest stability hypersphere around a stable polynomial $\widehat{\beta}(s)$ is given by

$$r = \min(d_o, d_{min}) \qquad (8)$$

where $d_o\left(\widehat{\beta}(s)\right) = \mid \widehat{\beta}_0 \mid$ and $d_{min} := \inf_{\omega \geq 0} \bar{d_\omega}\left(\widehat{\beta}(s)\right)$. The quantity $\bar{d_\omega}\left(\widehat{\beta}(s)\right)$ can be computed as follows.

$$\bar{d_\omega}\left(\widehat{\beta}(s)\right) = d_\omega\left(\widehat{\beta}_{n-m-1}s^{n-m-1} + \ldots + \widehat{\beta}_1 s + \widehat{\beta}_0\right)$$

while $d_\omega(.)$, is given by

(i) For $n-m-1 = 2q$

$$d_\omega^2(.) = \frac{\left[\widehat{\beta}^e(\omega)\right]^2}{1 + \omega^4 + \ldots + \omega^{4q}} + \frac{\left[\widehat{\beta}^o(\omega)\right]^2}{1 + \omega^4 + \ldots + \omega^{4(q-1)}} \qquad (9)$$

(ii) For $n-m-1 = 2q+1$

$$d_\omega^2(.) = \frac{\left[\widehat{\beta}^e(\omega)\right]^2 + \left[\widehat{\beta}^o(\omega)\right]^2}{1 + \omega^4 + \ldots + \omega^{4q}} \qquad (10)$$

where

$$\mid \widehat{\beta}^e(\omega) \mid = \widehat{\beta}_0 - \widehat{\beta}_2\omega^2 + \widehat{\beta}_4\omega^4 + \ldots + (-1)^q \widehat{\beta}_{2q}\omega^{2q}$$
$$\mid \widehat{\beta}^o(\omega) \mid = \widehat{\beta}_1 - \widehat{\beta}_3\omega^2 + \widehat{\beta}_5\omega^4 + \ldots + (-1)^{(q-1)} \widehat{\beta}_{2q-1}\omega^{2q-2}$$

After calculating $\bar{d_\omega}$, d_{min} can be found out in following way

(1) Compute real positive ω's satisfying $\frac{d(\bar{d_\omega})}{d\omega} = 0$.

(2) With these ω's evaluate $\bar{d_\omega}$ and take the minimum value of $\bar{d_\omega}$.

An algorithm is given in [1] for calculating d_{min}. Using theorem 2, we can calculate the radius of stability r around any $\widehat{\beta}(s)$ for $S = \{s \in \mathbb{C} : \ \mathrm{Re}(s) < 0\}$. This requirement is however not restrictive. To apply theorems 1 and 2 to stability regions of the form $S = \{s \in \mathbb{C} : \ \mathrm{Re}(s) < \gamma\}$, $\gamma \leq 0$, shift the imaginary axis to γ so that the variable "s" in a polynomial (6) will become "$s-\gamma$". So the new polynomial $\tilde{\beta}(s)$ corresponding to the polynomial $\widehat{\beta}(s)$, will of the form:

$$\tilde{\beta}(s) = s^{n-m} + \tilde{\beta}_{n-m-1}s^{n-m-1} + \ldots + \tilde{\beta}_1 s + \tilde{\beta}_0$$

The roots of the polynomial $\tilde{\beta}(s)$ will be obtained by shifting the roots of polynomial $\widehat{\beta}(s)$ by amount γ. Since, $\tilde{\beta}(s) \in C_s$ we can consider this as our nominal polynomial $\widehat{\beta}(s)$ and apply Theorem 2 to find the radius of the stability ball. Every polynomial belonging to this stability ball has all its roots in $S = \{s \in \mathbb{C} : \ \mathrm{Re}(s) < \gamma\}$.

IV. Main Results

We show that Problem 2 is equivalent to a quadratically constrained quadratic program and subsequently into a quadratic program using a linear approximation of the constraints.

A. An Equivalent Quadratically Constrained Quadratic Program

Denote the open loop characteristic polynomial of (1) by:

$$\begin{aligned} a(s) &= \det(sI - A) \\ &= s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \ldots + a_1 s + a_0 \end{aligned} \quad (11)$$

Define $a := \begin{bmatrix} a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix}^T$. The characteristic polynomial of the system (2) is defined as follows

$$\begin{aligned} \sigma(s) &= \det(sI - A + bk^T) \\ &= s^n + \sigma_{n-1}s^{n-1} + \sigma_{n-2}s^{n-2} + \ldots + \sigma_1 s + \sigma_0 \end{aligned} \quad (12)$$

Define $\sigma := \begin{bmatrix} \sigma_{n-1} & \sigma_{n-2} & \cdots & \sigma_1 & \sigma_0 \end{bmatrix}^T$

$$\mathscr{C} := \begin{bmatrix} b & Ab & A^2 b & \cdots & A^{n-1}b \end{bmatrix} \quad (13)$$

$$\mathscr{A}^T := \begin{bmatrix} 1 & a_{n-1} & a_{n-2} & \cdots & a_1 \\ 0 & 1 & a_{n-1} & \cdots & a_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \quad (14)$$

It is well known [2] that if the system is controllable i.e. controllability matrix (13) is non-singular, the closed loop eigenvalues (of $A - bk^T$) can be placed at any arbitrary locations in $\mathbb{C}$. Moreover, the corresponding feedback gain vector k is unique and can be calculated from the following equations:

$$\mathscr{A}\mathscr{C}^T k + a = \sigma$$

Denote $\bar{k} = \mathscr{A}\mathscr{C}^T k$ and let $\bar{k} = [\bar{k}_n \ \bar{k}_{n-1} \ldots \bar{k}_1]^T$. Clearly each $\bar{k}_i (i = 1, \ldots, n)$ is a linear combination of $k_1, \ldots, k_n$. Then:

$$\begin{aligned} \sigma_{n-1} &= \bar{k}_n + a_{n-1} \\ \sigma_{n-2} &= \bar{k}_{n-1} + a_{n-2} \\ &\vdots \\ \sigma_0 &= \bar{k}_1 + a_0 \end{aligned} \quad (15)$$

Recalling the expression for the required closed loop characteristic polynomial (5), and equating coefficients with (12), we get:

$$\begin{aligned} \sigma_{n-1} &= \alpha_{m-1} + \beta_{n-m-1} \\ \sigma_{n-2} &= \alpha_{m-2} + \alpha_{m-1}\beta_{n-m-1} + \beta_{n-m-2} \\ &\vdots \\ \sigma_2 &= \alpha_0\beta_2 + \alpha_1\beta_1 + \alpha_2\beta_0 \\ \sigma_1 &= \alpha_0\beta_1 + \alpha_1\beta_0 \\ \sigma_0 &= \alpha_0\beta_0 \end{aligned} \quad (16)$$

In (16) above, the coefficients $\alpha_0, \alpha_1, \ldots, \alpha_{m-1}$ are known quantities (since $(-\mu_1, -\mu_2, \ldots, -\mu_m)$ are specified by the designer), where as $\beta_0, \beta_1, \ldots, \beta_{n-m-1}$ are unknown quantities due to dependency on $p_1, p_2, \ldots, p_{n-m}$ which are unspecified. First note that $\sigma_0, \sigma_1, \ldots, \sigma_{n-1}$ can be eliminated from equations (15) and (16) to get n linear equations:

$$\begin{aligned} \alpha_{m-1} + \beta_{n-m-1} &= \bar{k}_n + a_{n-1} \\ &\vdots \\ \alpha_0\beta_1 + \alpha_1\beta_0 &= \bar{k}_2 + a_1 \\ \alpha_0\beta_0 &= \bar{k}_1 + a_0 \end{aligned} \quad (17)$$

From (17), $(\beta_0, \ldots, \beta_{n-m-1})$ can be expressed in terms of $n-m$ linear equations in $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n$. A simple inductive method to get such equations is shown below:

$$\begin{aligned} \beta_0 &= \frac{1}{\alpha_0}\left(\bar{k}_1 + a_0\right) \\ \beta_1 &= \frac{1}{\alpha_0}\left(\left(\bar{k}_2 + a_1\right) - \frac{\alpha_1}{\alpha_0}\left(\bar{k}_1 + a_0\right)\right) \\ &\vdots \\ \beta_{n-m-1} &= \ldots.. \end{aligned} \quad (18)$$

Then (18) can be compactly written as follows

$$\beta = \mathscr{F}\bar{k} + g \quad (19)$$

where $\beta \in \mathbb{R}^{n-m}$, $\mathscr{F} \in \mathbb{R}^{(n-m)\times n}$, $g \in \mathbb{R}^{n-m}$. Now $\beta_0, \ldots, \beta_{n-m-1}$ from (18) can be back-substituted in the set of n equations (17) to get m linear equations in $(\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n)$ which can be written in the form:

$$\mathscr{E}\bar{k} + h = 0 \quad (20)$$

where $\mathscr{E} \in \mathbb{R}^{m\times n}$, $h \in \mathbb{R}^m$ and 0 is a zero vector of appropriate dimension. Using $\bar{k} = \mathscr{A}\mathscr{C}^T k$, and defining

$$F = \mathscr{F}\mathscr{A}\mathscr{C}^T \text{ and } E = \mathscr{E}\mathscr{A}\mathscr{C}^T \quad (21)$$

we get the following set of equations:

$$\beta = Fk + g \quad \text{and} \quad Ek + h = 0 \quad (22)$$

Further, recall the definitions of $\widehat{\beta}$ and r from (7) and (8). Then the following result holds:

Theorem 3: If for some $k \in \mathbb{R}^n$, the relations $\|Fk + g - \widehat{\beta}\|_2 \leq r$ and $Ek + h = 0$ holds, then the eigenvalues of the matrix $(A - bk^T)$ satisfy the following properties:

1) m out of the total n eigenvalues are $\{-\mu_1, -\mu_2, \ldots, -\mu_m\}$.
2) the remaining $n-m$ eigenvalues $-p_i \in S \cup \partial S$ for $i = 1, \ldots, n-m$.

Proof: Let some k satisfy $\|Fk + g - \widehat{\beta}\|_2 \le r$ and $Ek + h = 0$. Then $\|\beta - \widehat{\beta}\|_2 \le r$. Hence the polynomial $\beta(s) \in B\left(\widehat{\beta}(s), r\right) \cup \partial B$. So we can apply theorem 1 to guarantee that the roots of $\beta(s)$ lie in $S \cup \partial S$. The m equations $Ek + h = 0$ imply that the m roots of polynomial Q (see (5)) are placed at $\{-\mu_1, \ldots, -\mu_m\}$. ■

Theorem 3 defines the constraint set on the feedback gain vector k, which can be used to pose an optimization problem that minimizes the norm of k. After some simple calculations $\|Fk + g - \widehat{\beta}\|_2 \le r$ can be written as $k^T M k + 2m^T k + c \le 0$ where $M = F^T F$, $m^T = (g - \widehat{\beta})^T F$ and $c = (g - \widehat{\beta})^T (g - \widehat{\beta}) - r^2$. Here M is a positive semi-definite matrix, m is a constant vector and $c \in \mathbb{R}$. The optimization problem can be formulated as follows:

Problem 3: Find $\min_{k \in \mathbb{R}^n} \|k\|_2$ subject to

$$Ek + h = 0$$
$$k^T M k + 2m^T k + c \le 0$$

It should be noted that the above constraint set is always feasible since it is known that there is always at least one k which places the poles at arbitrary desired location. However, the optimal k might place some of the closed loop eigenvalues on the boundary ∂S of the stability region. The designer should choose the stability region S with this consideration. Another issue with problem 3 is that the optimization might not be numerically easy since the matrix M is positive semi-definite and not necessarily positive definite [3]. It is known that such a quadratically constrained quadratic program, though convex, can turn out to be NP-hard. Several relaxations of this problem have been studied (see e.g [15] and references therein). In this article, we give a simple box approximation for the spherical constraint set defined in Problem 3.

B. Approximate Linear Constraints

In this section the spherical constraint $\|\beta - \widehat{\beta}\|_2 \le r$ is approximated by a hypercube in $\mathbb{R}^{n-m}$ that inscribes the sphere. The situation is shown in Fig. 1 for $n - m = 2$. A square $EFGH$ which approximates the stability ball is obtained by projecting the vector OG onto the line AB and CD respectively. The length of the projection OM on AB will be $r/\sqrt{2}$. So each edge of the square will be $2r/\sqrt{2}$.

Similarly we can extend this notion to a higher dimensional stable ball where each edge of the inscribed hypercube will be $2r/\sqrt{n-m}$. Next define the set

$$\Sigma := \left\{\beta \in \mathbb{R}^{n-m} : \beta_i^- \le \beta_i \le \beta_i^+, i = 0, 1, \ldots, n-m-1\right\}$$

where for $i = 0, 1, \ldots, n-m-1$

$$\beta_i^- = \widehat{\beta}_i - \frac{r}{\sqrt{n-m}} \quad \text{and} \quad \beta_i^+ = \widehat{\beta}_i + \frac{r}{\sqrt{n-m}}$$

Then $\Sigma \subseteq \{\beta \in \mathbb{R}^{n-m} : \|\beta - \widehat{\beta}\|_2 \le r\}$ and a sub-optimal reformulation of Problem 3 can be written as follows:

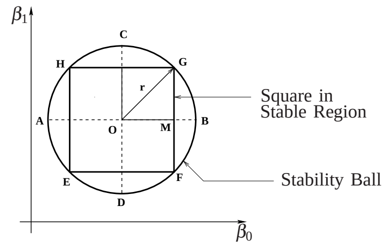

Fig. 1. Square Approximating the Stability Ball

Problem 4: Find $\min_{k \in \mathbb{R}^n} \|k\|_2$ subject to

$$Ek + h = 0$$
$$Fk + g \in \Sigma$$

Problem 4 is a quadratic program with m linear equality constraints and $n-m$ linear inequality constraints. It is well known [4] that this problem has a unique minimum and can be solved efficiently using a variety of algorithms (e.g. see [3], [4]).

C. Design Steps

It should be noted that Theorem 1 is only sufficient in guaranteeing that the corresponding eigenvalues stay in S. Consequently, in some cases, it might be possible to find a k which preserves the pole placement requirements and but has lesser norm than the solution to Problem 3. Additionally, the solution to Problem 3 might be hard in general, and one may have to depend on the linear relaxation to compute a solution. Hence, a three step design procedure is suggested below to find a controller with maximum reduction in the norm. Since, the precise location requirement on the critical eigenvalues is inflexible, the equality constraints $Ek + h = 0$ are assumed to be imposed on all the steps below:

1) Solve Problem 3 without considering the inequality constraints. If all the poles belong to S then stop; otherwise go to step 2.
2) Solve Problem 3. (Recall that the solution might be NP-hard.) If tractable, stop; otherwise proceed to step 3.
3) Solve Problem 4.

V. NUMERICAL EXAMPLES

Example 1: Consider a LTI system with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 51 & -10 & -30 & -10 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Eigenvalues of A are at $1, -3, -4 \pm i$. Only the pole at $+1$ is unstable and we assume that this pole needs to be placed at -1 using state feedback control. The remaining 3 poles are assumed to be non-critical and are allowed to be placed arbitrarily within the open left half of complex plane.

TABLE I

COMPARISON TABLE

Procedure	Closed loop poles	$\|k_r\|_2$	% Red. in $\|k_r\|_2$	Sys. Cond.
Step - 1	$0 \pm 6.4031, -1, 1$	20	84.93	Unstable
Step - 2	$-3.9158 \pm 1.9057i$ $-2.2363,\ -1$	117.2485	11.66	Stable
Step - 3	$-1.9161 \pm 5.0794i$ $-1.5395,\ -1$	120.0862	9.52	Stable
Step - 4	$-4 \pm 1i,\ -3,\ -1$	132.7253	–	Stable

TABLE II

COMPARISON TABLE

Procedure	Non critical closed loop poles	$\|k\|_2$	% Red. in $\|k\|_2$	Sys. Cond.
Step - 1	$-10.0009 \pm 9.0498i$ $-7.0277, -0.4076$	2.1128	51.2877	Stable
Step - 2	$-5.9726 \pm 6.8486i$ $-35.2522, -0.3078$	4.0973	5.5326	Stable
Step - 3	$-5.8745 \pm 7.0547i$ $-34.6056,\ -0.3036$	4.1432	4.4723	Stable
Step - 4	$-5.7250 \pm 7.4088i$ $-33.5344,\ -0.3078$	4.3372	–	Stable

Following the discussion in Section IV, (22) takes the form:

$$\begin{aligned}
\beta_0 &= k_1 - 51 \\
\beta_1 &= -k_1 + k_2 + 61 \\
\beta_2 &= k_4 + 9 \\
k_1 - k_2 + k_3 - k_4 - 40 &= 0
\end{aligned}$$

Next we are required to construct the polynomial $\widehat{\beta}(s)$ according to (7). Recalling that the open loop poles are at $-3, -4 \pm i$, $\widehat{\beta}(s) := s^3 + 11s^2 + 41s + 51$. Following the procedure given in Section III, $d_o = 51$ and according to (10), d_ω^2 will be

$$d_\omega^2 = \frac{\left[51 - 11\omega^2\right]^2}{1 + \omega^4} + \left[41 - \omega^2\right]^2$$

and resulting $d_{min} = 9.74$. Hence the radius of the stability ball $r = min(d_o, d_{min})$ will be 9.74.

Following the design procedure proposed in IV-C, a comparison table is shown in Table I where Steps 1 to 3 corresponds with the steps described in section IV-C, while Step - 4 evaluates $\|k\|_2$ keeping three non critical poles in their original location.

The percentage reduction in $\|k\|_2$ in Step - 1, Step - 2 and Step - 3 is compared with Step - 4. It is observed that a large reduction in $\|k\|_2$ is achieved in Step-1. However, the non-critical poles are in the unstable region. Hence this example needs the design Step - 2 which is developed in Theorem 3. In this step all the non-critical poles are in the stable region and hence we have achieved our goal. But for comparison of various norms of k under different situations we have shown the other steps also. In Step - 2, the percentage reduction in $\|k\|$ is modest i.e. about 11.6%. However, we can get a better percentage reduction in $\|k\|_2$ in practical situations as is demonstrated by the power system example next.

Example 2: In this example, the linearized model of a 16-generator, 68 bus bar power system [6] and [7] is considered around its nominal operating condition. The 133 order original model is reduced to a 10^{th} order equivalent without introducing much error within the frequency range of interest (0.1 to 0.8 Hz). Open loop poles of the reduced system are at are at $-33.5344, -5.7250 \pm 7.4088i, -0.1741 \pm 3.7981i, -0.1781 \pm 3.1604i, -0.1808 \pm 2.4535i, -0.3078$. It is required to place six poles at $-0.4000 \pm 3.7980i, -0.4000 \pm 3.1604i, -0.4000 \pm 2.4535i$ and remaining 4 poles are non critical. Following all the design procedure proposed in IV-C, a comparison table is shown in Table II. It is seen that a 51.28% reduction in $\|k\|_2$ is achieved in Step - 1. This leads to a substantial reduction in controller effort and an associated reduction in actuator cost.

VI. CONCLUSION

The article describes a general method of reducing controller effort through optimization of the state feedback vector norm. The optimization guarantees that (a) the critical poles are placed in specified locations in the complex plane (b) the non-critical poles are placed inside a pre-specified design region S. This region S typically corresponds to a maximum settling time requirement on the non-critical poles. The proposed method is shown to produce a substantial reduction in controller effort in two numerical examples.

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