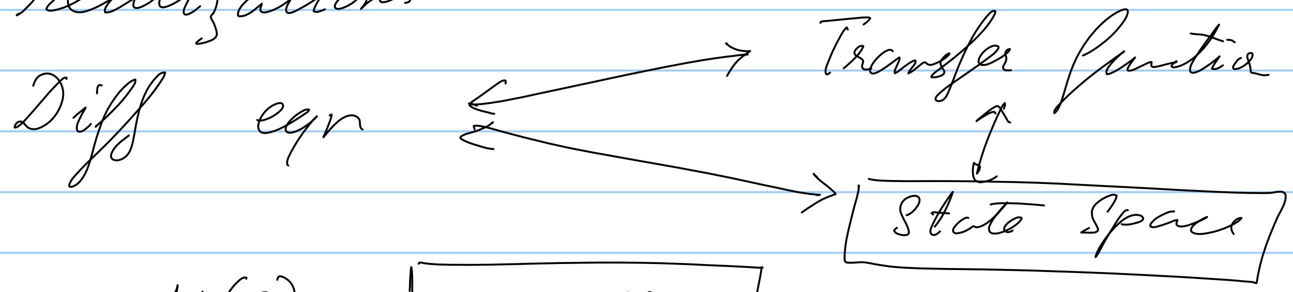
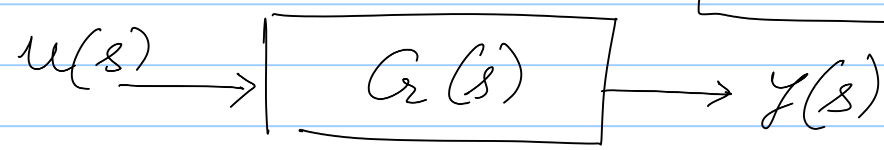
# Lecture 11: State Feedback

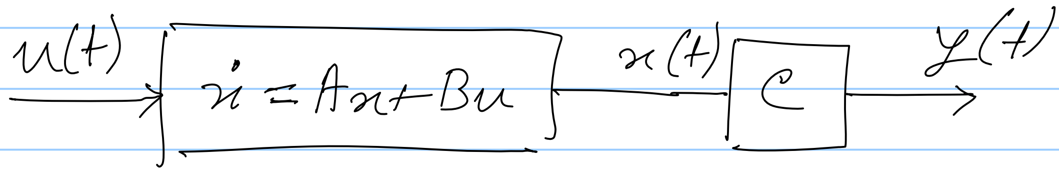
Note Title 12-04-2010

We have studied state space realizations




$G(s) \equiv \begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx \end{array}$




Known correspondances:

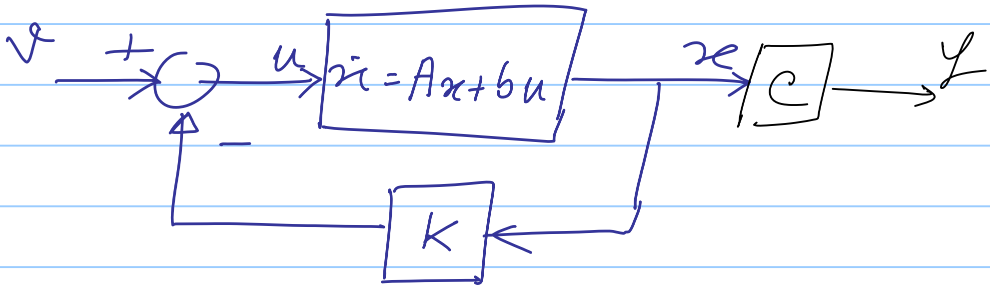
* $G(s) = C(sI - A)^{-1} B$
* Poles of $G(s)$ are eigenvalues of $A$.
* There can be many S.S. $(A, B, C)$s realizations corresponding to one $G(s)$
  → All such $A$'s have the same eigenvalues.
* Dynamic performance is determined by eigenvalues of $A$.

Q. Can we design a feedback controller which will move the eigenvalues of $A$ to favourable position?

* YES if we have access to ALL the states.

Given a realization $\dot{x} = Ax + bu$

$y = cx$




$K$ is a row vector $= [k_1 \ k_2 \ \cdots \ k_n]$

$$Kx = [k_1 \ \cdots \ k_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = k_1 x_1 + k_2 x_2 + \cdots + k_n x_n$$

$$u = v - Kx$$

Hence, $\dot{x} = Ax + b(v - Kx)$

$= Ax - bKx + bv$

$$\boxed{\begin{aligned} \dot{x} &= (A - bK)x + bv \\ y &= cx \end{aligned}}$$

$$bK = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} [k_1 \ \cdots \ k_n] = \begin{bmatrix} k_1 b_1 & k_2 b_1 & \cdots & k_n b_1 \\ k_1 b_2 & k_2 b_2 & & k_n b_2 \\ \vdots & & & \\ k_1 b_n & k_2 b_n & \cdots & k_n b_n \end{bmatrix}$$

$$= [k_1 b \quad k_2 b \quad \cdots \quad k_n b]_{n \times n}$$

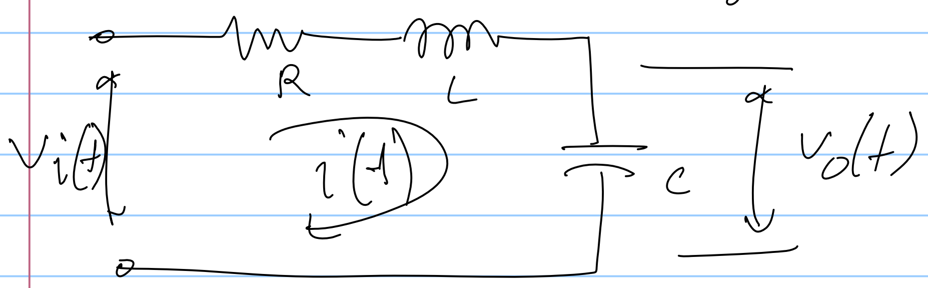
Write $A = [A_1 \ A_2 \ \cdots \ A_n]$

Then $A - bk = [A_1 \; A_2 \cdots A_n] - [k_1 b \; k_2 b \cdots k_n b]$

$= [A_1 - k_1 b \quad A_2 - k_2 b \; \cdots\cdots \; A_n - k_n b]$

Q. Is this configuration (access to all the states) practical?

In some cases: e.g.




$$LC\frac{d^2 v_o}{dt^2} + RC\frac{dv_o}{dt} + v_o(t) = v_i(t)$$

$x_1(t) = v_o(t) \qquad x_2 = \frac{dv_o}{dt}$

Matrix form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{R}{L} & -\frac{1}{LC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_i(t)$$

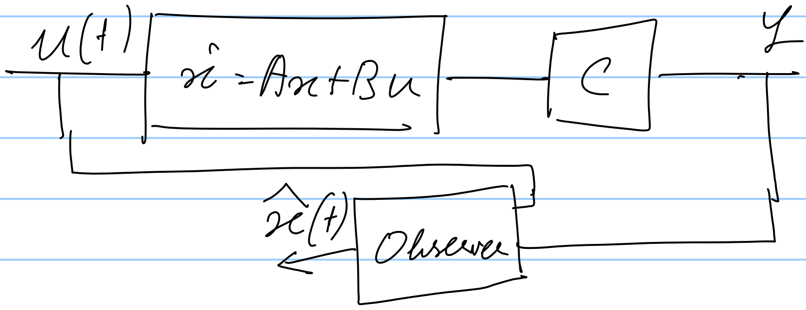
$$v_o(t) = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

But $\frac{dv_o}{dt} = \frac{1}{C} i(t)$ [$i(t)$ can also be measured]

In this example having access to all states $\equiv$ $v_o(t)$ & $i(t)$ both are being measured.

* There are however other methods

to get around this problem for applications where such measurement are not available.




* Not to be discussed here.

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The resolvent formula

$$\mathrm{adj}(sI - A) = A^{n-1} + (s + a_1)A^{n-2}$$
$$+ (s^2 + a_1 s + a_2) A^{n-3} + \cdots$$
$$\cdots + (s^{n-1} + a_1 s^{n-2} + \cdots + a_{n-1}) I$$

OR, re-arranging the terms:

$$\mathrm{adj}(sI - A) = s^{n-1} I + s^{n-2}(A + a_1 I)$$
$$+ s^{n-3}(A^2 + a_1 A + a_2 I) + \cdots$$
$$\cdots + (A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I)$$

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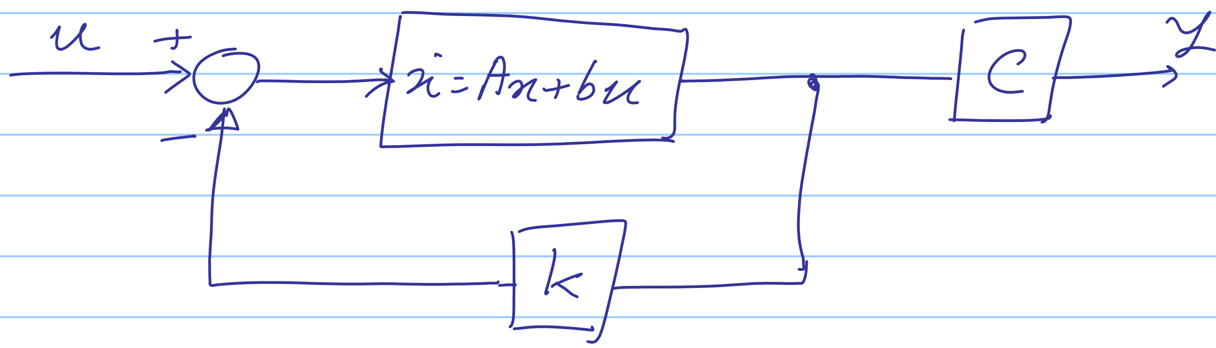
# Stabilizing Linear Systems by state feedback

Given the realization:

$$\dot{x} = Ax + bu$$
$$y = cx$$

Assume we have access to the states




The new system : $\begin{cases} \dot{x} = (A - bk)x + bu \\ y = cx \end{cases}$

## The process of pole assignment

We assign desirable eigenvalues to the matrix $(A - bk)$

Step 1 : Select $n$ eigenvalues $\lambda_1, \dots, \lambda_n$
If they are complex, their complex conjugate is included.

Step 2 : Compute the new characteristic polynomial

$$\alpha(s) = (s - \lambda_1)(s - \lambda_2) \dots (s - \lambda_n)$$
$$= s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$$

$(\alpha_1, \cdots, \alpha_n)$ are real numbers.

<u>Step 3</u>: The characteristic polynomial of the system with feedback is

$$a_K(s) = \det(sI - (A - bK))$$

Find $K$ such that

$$a_K(s) = \alpha(s).$$

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<u>A property of the determinant</u>

Suppose we have two matrices

$A$ is $n \times m$

$B$ is $m \times n$

Then, $\det(I_n + AB) = \det(I_m + BA)$

<u>Proof</u>:
$$\left[\begin{array}{c|c} I_n & -A \\ \hline B & I_m \end{array}\right]_{(n+m)\times(n+m)}$$

Perform the following block column operations:

1) Multiply the first $n$-columns by $A$ and add to the last $m$-columns

Result: $\begin{bmatrix} I_n & 0 \\ B & I_m + BA \end{bmatrix}$

Clearly, this column block operation can be achieved by multiplying on the right by $\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix}$

Check: $\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix}\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ B & I_m+BA \end{bmatrix}$

Elementary column operations does not affect the determinant

$$\det\begin{bmatrix} I_n & 0 \\ B & I_m+BA \end{bmatrix} = \det\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix} \cdot \underbrace{\det\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix}}_{1}$$

$$= \det\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix}$$

So $\det\begin{bmatrix} I_n & -A \\ B & I_m \end{bmatrix} = \det\begin{bmatrix} I_n & 0 \\ B & I_m+BA \end{bmatrix}$

$$= \det\left(I_m+BA\right)$$

Lets go back to our "trick" matrix, multiply the bottom m rows by A & add to the top n rows

Result: $\left[\begin{array}{c|c} I_n+AB & 0 \\ \hline B & I_m \end{array}\right]$

This operation can be achieved by left multiplication by $\begin{bmatrix} I_n & A \\ 0 & I_m \end{bmatrix}$

$$\therefore \det\left[\begin{array}{c|c} I_n+AB & 0 \\ \hline B & I_m \end{array}\right] = \det\left[\begin{array}{c|c} I_n & -A \\ \hline B & I_m \end{array}\right] \cdot 1$$

So, $\det\begin{bmatrix} I_n & -A \\ \hline B & I_m \end{bmatrix} = \det(I_n + AB) \cdot \det(I_m)$

$= \det(I_n + AB)$

Hence proved.

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Using this property we can get a useful simplification of the characteristic polynomial for systems with feedback

$$a_k(s) = \det[sI - (A - bk)]$$

$$= \det[(sI - A) + bk]$$

$$= \det[(sI-A)\{I + (sI-A)^{-1}bk\}]$$

$$= \underbrace{\det(sI-A)}_{a(s)} \det[I_n + (sI-A)^{-1}bk]$$

$a(s) \equiv$ characteristic poly of the original realization.

Now, $\det[I_n + \underbrace{(sI-A)^{-1}b}_{\substack{\text{"}A\text{"} \\ n\times 1}} \underbrace{k}_{\substack{\text{"}B\text{"} \\ 1\times n}}]$

$$= \cancel{\det}[1 + k.(sI-A)^{-1}b]$$

So, $a_k(s) = a(s)[1 + k(sI-A)^{-1}b]$

$$= a(s) + a(s)k(sI-A)^{-1}b$$

Recall, $a(s)(sI-A)^{-1} = \text{adj}(sI-A)$

So, $a_k(s) - a(s) = k\,\mathrm{adj}(sI - A)\,b$

We will now use the resolvent formula for the adj:

$$\mathrm{adj}(sI-A) = s^{n-1} I + s^{n-2}(A + a_1 I) + ----$$

Substituting:

$$a_k(s) - a(s) = k\left[s^{n-1} I + s^{n-2}(A + a_1 I) + s^{n-3}(A^2 + a_1 A + a_2 I) + \cdots\right] b$$

Recall that we wanted

$$a_k(s) = \alpha(s) = s^n + \alpha_1 s^{n-1} + --- + \alpha_n$$
$$a(s) = s^n + a_1 s^{n-1} + --- + a_n$$

$$a_k(s) - a(s) = \alpha(s) - a(s)$$
$$= s^{n-1}(\alpha_1 - a_1) + s^{n-2}(\alpha_2 - a_2) + \cdots$$
$$---- + (\alpha_n - a_n)$$

Lets compare coefficients of corresponding powers of s.

Columns of controllability matrix

$s^{n-1}$ : $\alpha_1 - a_1 = k b$

$s^{n-2}$ : $\alpha_2 - a_2 = k(Ab) + a_1 k b$

$s^{n-3}$ : $\alpha_3 - a_3 = k(A^2 b) + a_1 k A b + a_2 k b$

We know: $\mathcal{C} = [\,b \;\; Ab \;\; A^2 b \; ----\,]$

Let us try to write the above formulae in matrix form:

$$\begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix} \underbrace{\begin{bmatrix} 1 & a_1 & a_2 & \cdots & \cdots \\ 0 & 1 & a_1 & \cdots & \\ 0 & 0 & 1 & \cdots & \\ \vdots & & & \ddots & \\ \vdots & & 0 & & \ddots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}}_{(a_-)^T}$$

$$= \left[\, b \;\middle|\; a_1 b + Ab \;\middle|\; a_2 b + a_1 Ab + A^2 b \;\middle|\; \cdots \,\right]$$

Define the following quantities:

$$a_- := \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & \cdots & \\ a_2 & a_1 & 1 & 0 \cdots & 0 \\ & & & \ddots & 0 \\ a_{n-1} & a_{n-2} & \cdots & \cdots & 1 \end{bmatrix}$$

$$a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} ; \quad \alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$$

So the vector form for the coefficient equations is:

$$\alpha - a = K \mathcal{C} a_-^T$$

Now note that $\det(a_-^T) = 1$. So $a_-^T$ is always invertible.

So when the realization is controllable we can write:

$$\boxed{K = (\alpha - a)\,(\underline{a}^T)^{-1}\,\mathcal{C}^{-1}}$$

<u>Conclusion</u>: Arbitrary pole assignment by state feedback is possible if and only if the realization is controllable.

<u>Example</u>: $\dot{x} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

<u>Problem</u>: Use state feedback to assign the eigenvalues : $(-2, -3)$

(1) The controllability matrix $\mathcal{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
So realization is controllable.

$$a(s) = \det(sI - A) = \det\begin{pmatrix} s & -1 \\ -1 & s \end{pmatrix} = s^2 - 1$$

$$= s^2 + 0.s - 1$$

So $a = [0 \;\; -1]$

$$\alpha(s) = (s+2)(s+3) = s^2 + 5s + 6$$

$$\alpha = [5, 6]$$

$$\underline{a}^T = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

So $K = (\alpha - a)(\underline{a}^T)^{-1}\mathcal{C}^{-1}$

$$= \left[\begin{pmatrix} 5 & 6 \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix}\right]\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1}$$

So $K = [5, 7]$