# Lecture - 4

## Transient Response - TF and S.S.

Note Title 26-12-2009

Given a T.F. $G(s)$ and input $R(s)$ we can calculate the output $(t \geq 0)$

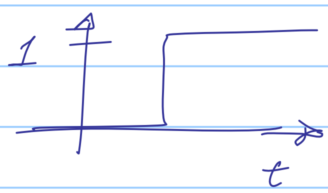
$$y(t) = \mathcal{L}^{-1}\{G(s)R(s)\}$$ (we can use MATLAB also)

$\Rightarrow$ we already know how to compute the "Transient Response".

Q) How can we predict the transient response of any t.f. without computing the sol$^{n}$?

Additional Information: Transient Response obviously depends on input, so we choose the test input

"STEP FUNCTION" $u(t)$




Example: $\frac{C(s)}{R(s)} = \frac{s+2}{s+5}$ with $R(s) = \frac{1}{s}$

$$C(s) = \frac{s+2}{s(s+5)} = \frac{A}{s} + \frac{B}{s+5} = \frac{2/5}{s} + \frac{3/5}{s+5}$$

$$A = \left.\frac{s+2}{s+5}\right|_{s=0} = 2/5 \; ; \quad B = \left.\frac{s+2}{s}\right|_{s=-5} = 3/5$$

$$c(t) = \underbrace{\frac{2}{5}u(t)}_{\text{forced response}} + \underbrace{\frac{3}{5}e^{-5t}u(t)}_{\text{natural response}}$$

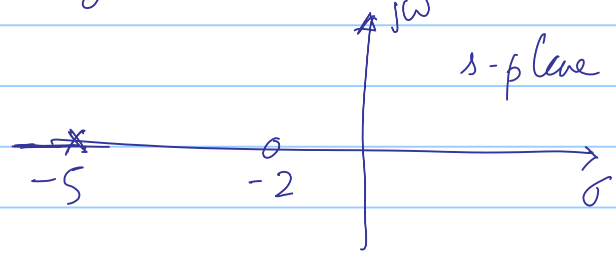
Observations

1) The root of the input denominator decides

the form of the forced response.

2) The root of the t.f. denominator ($-5$) generates the form of the natural response
   - → that form is $e^{-\alpha t}$ where $-\alpha$ is the root
   - → If the root is more $-$ve, the exp. decays faster.

3) The roots of both num & den generate the amplitude of both responses




* So the roots of both num. & den. seem important in predicting transient response: We give them names:

**Pole**: The poles of a transfer function are the roots of its denominator.

**Zeros**: The zeros of a t.f. are the roots of its numerator.

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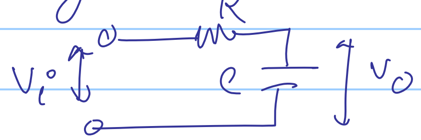
## Performance Specifications for first order systems

Recall that one of the primary objectives of designing a control system is to make the output follow the input as closely as possible.

Q) Does this happen for first order systems?

Consider an RC ckt:



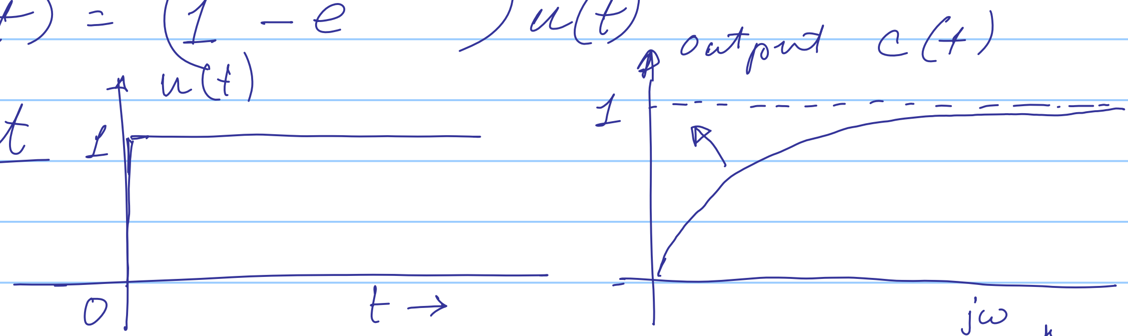

$$\frac{V_o(s)}{V_i(s)} = \frac{1/RC}{s + 1/RC} \leftarrow \text{first order system}$$

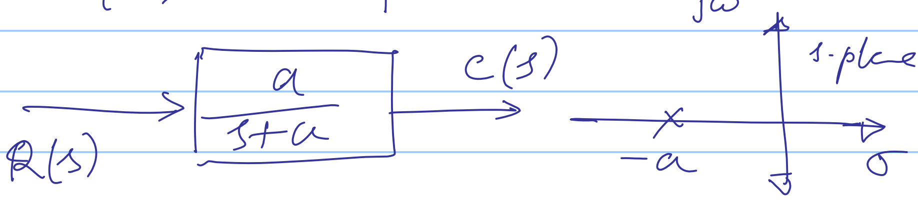
→ Can we make $V_o(t)$ follow $v_i(t) = u(t)$?

Let $G(s) = \dfrac{a}{s+a}$ → Then $C(s) = \dfrac{a}{s(s+a)}$

$$c(t) = \left(1 - e^{-at}\right) u(t)$$

Input




We would like to specify how close to $u(t)$ we want $c(t)$!

1) Time Constant: Clearly the steeper the initial slope the closer $c(t)$ is to $u(t)$.

$$\text{Initial slope} = a. \quad \left[ \frac{d}{dt}[c(t)] \Big|_{t=0} = a \right]$$

Define: Time Constant $= \dfrac{1}{a} = \dfrac{1}{\text{initial slope}}$

$$c\left(t = \frac{1}{a}\right) = 1 - e^{-1} = 0.63$$

Hence Time Constant is the time it takes for the step response to rise to 63% of its final value (= 1).

<u>Note:</u> The time constant is known just by looking at the t.f. $G(s)$.

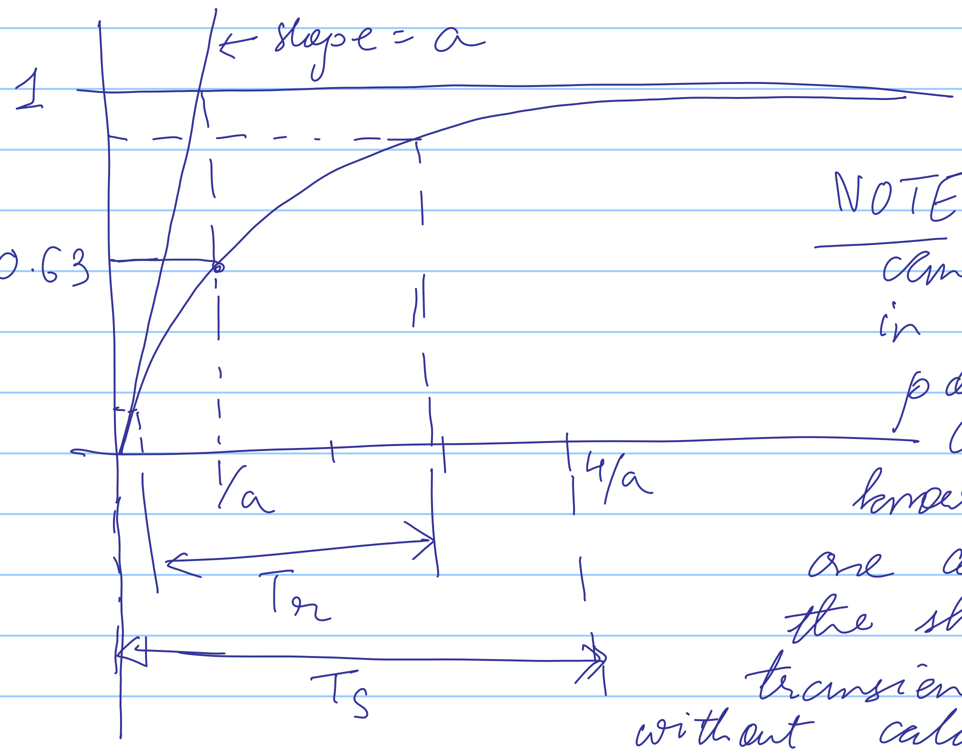
2) Alternatively one can use <u>Rise Time</u>: $T_r$

Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value

$$\left.\begin{array}{l} c(t_1) = 1 - e^{-at_1} = 0.1 \\ c(t_2) = 1 - e^{-at_2} = 0.9 \end{array}\right\} \quad T_r = t_1 - t_2 = \frac{2.2}{a}$$

3) <u>Settling Time</u>: $T_s$ : The time for the response to reach, and stay within, 2% of its final value. (2% settling time)

$$1 - e^{-aT_s} = 0.98 \Rightarrow T_s = \frac{4}{a}$$



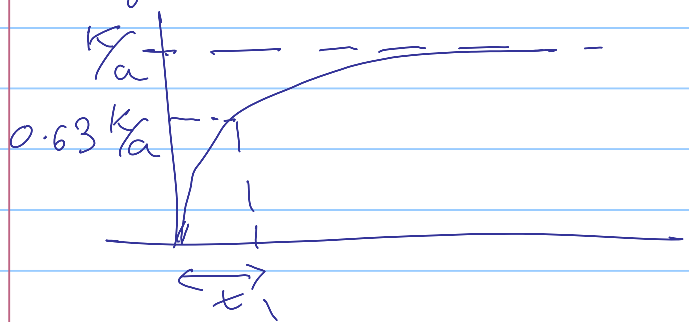

<u>NOTE</u>: All specs can be expressed in terms of the pole. Conversely, knowing the pole one can predict the shape of transient response without calculation.

# First Order System Identification

Q) Assume only the In and Output of a 1st order system are accessible. Calculate the t.f.

Assume $G(s) = \dfrac{K}{s+a}$

Apply a unit step input : $u(t)$ ← Observe the o/p




$$C(s) = \frac{K}{s(s+a)} = \frac{K/a}{s} - \frac{K/a}{s+a}$$

$$c(t) = K/a\left[1 - e^{-at}\right] \qquad c(\infty) = \frac{K}{a}$$

So final value $= \dfrac{K}{a}$. while $t_1 = \dfrac{1}{a}$

Hence we can solve for K and a.

# Second Order Systems:

Q) Can we predict the transient (step) response just by looking at the pole/zero map?

Let $G(s) = \dfrac{b}{s^2+as+b}$ [Additional scalar gains are neglected]

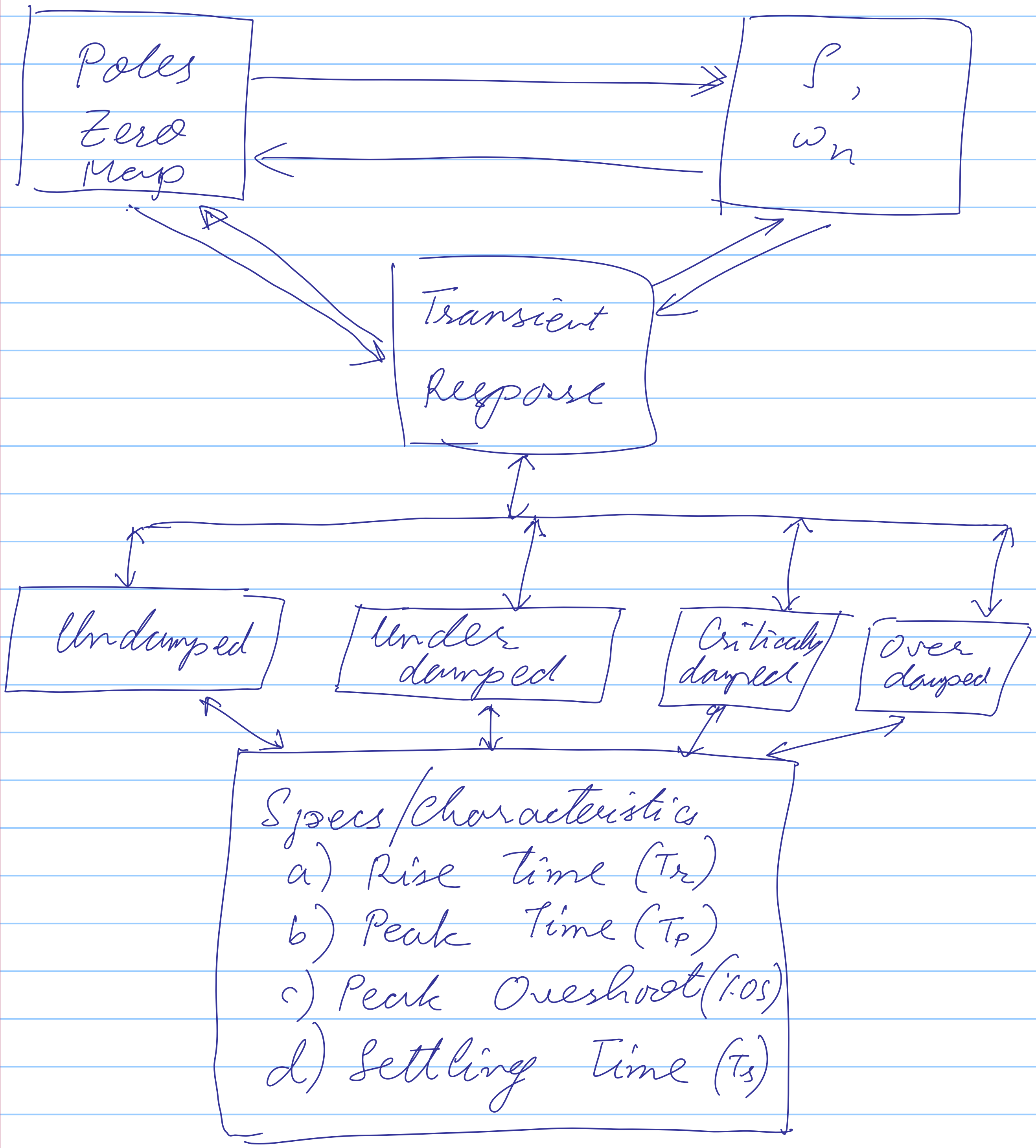
Poles : $\dfrac{-a \mp \sqrt{a^2-4b}}{2} = \left(\dfrac{a}{2\sqrt{b}}\right)\sqrt{b} \pm \sqrt{b}\sqrt{\left(\dfrac{a}{2\sqrt{b}}\right)^2 - 1}$

$$= \left(\frac{a}{2\sqrt{b}}\right)\sqrt{b} \pm \sqrt{b}\sqrt{\left(\frac{a}{2\sqrt{b}}\right)^2 - 1}$$

We name : $\zeta = \frac{a}{2\sqrt{b}}$ =: damping ratio

and $\omega_n = \sqrt{b}$ =: natural freq

We want the following interconnection



Hence $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

We can have 3 cases:

**Case I : Real Poles:** $\left\{ \dfrac{a}{2\sqrt{b}} > 1 \right.$ or $\left\{ \zeta > 1 \right.$

**Case Ia : Distinct Real Poles**

Let $\sigma_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$ $\quad \zeta > 1$

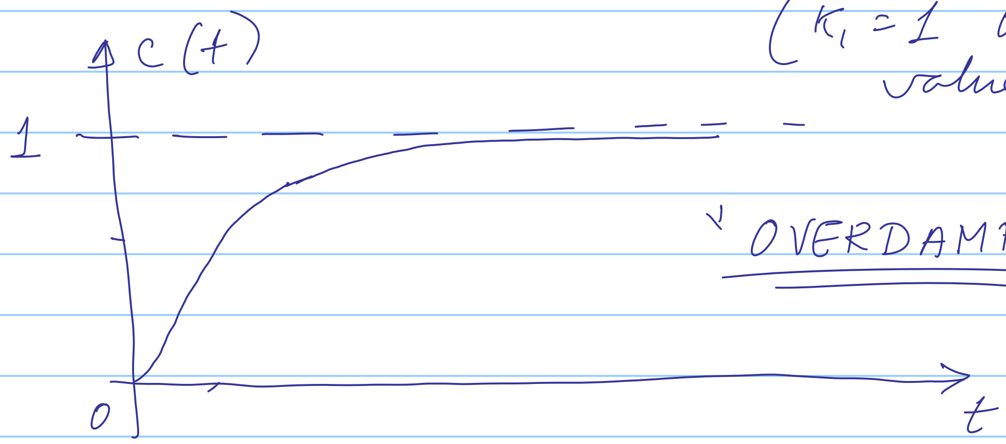
$\sigma_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$R(s) = \dfrac{1}{s} \Rightarrow C(s) = \dfrac{\omega_n^2}{s(s-\sigma_1)(s-\sigma_2)}$

$= \dfrac{K_1}{s} + \dfrac{K_2}{s-\sigma_1} + \dfrac{K_3}{s-\sigma_2}$

$c(t) = \left[1 + K_2 e^{-\sigma_1 t} + K_3 e^{-\sigma_2 t}\right] u(t)$

($K_1 = 1$ by final value thm)

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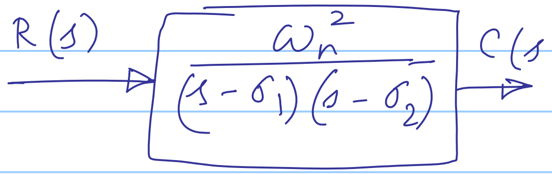
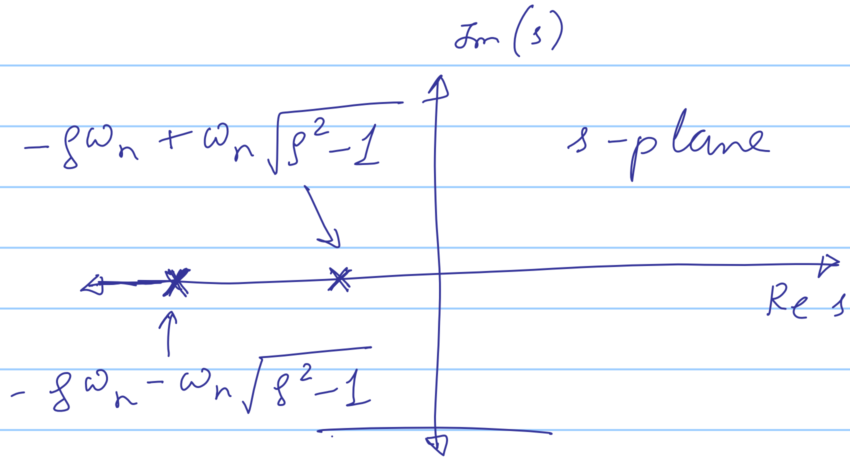





## Case Ib: Real Repeated Poles:

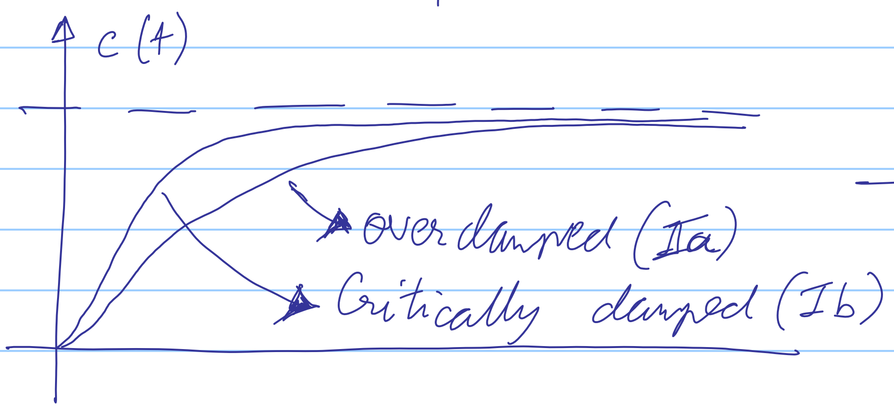
$$\frac{a}{2\sqrt{b}} = 1 \quad \text{or} \quad \zeta = 1$$

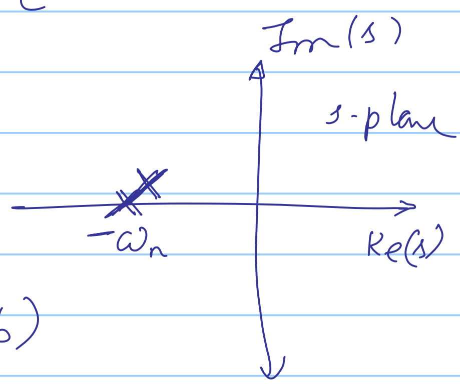
$$\sigma = -\omega_n$$

$$G(s) = \frac{\omega_n^2}{(s+\sigma)^2} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{1}{s} + \frac{K_1}{s+\omega_n} + \frac{K_2}{(s+\omega_n)^2}$$

$$= 1 + K_1 e^{-\omega_n t} + K_2 t e^{-\omega_n t}$$

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## Case II: Complex Poles:

$$0 < \frac{a}{2\sqrt{b}} < 1 \quad \text{or} \quad 0 < \zeta < 1$$

$$G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad R(s) = \frac{1}{s}$$

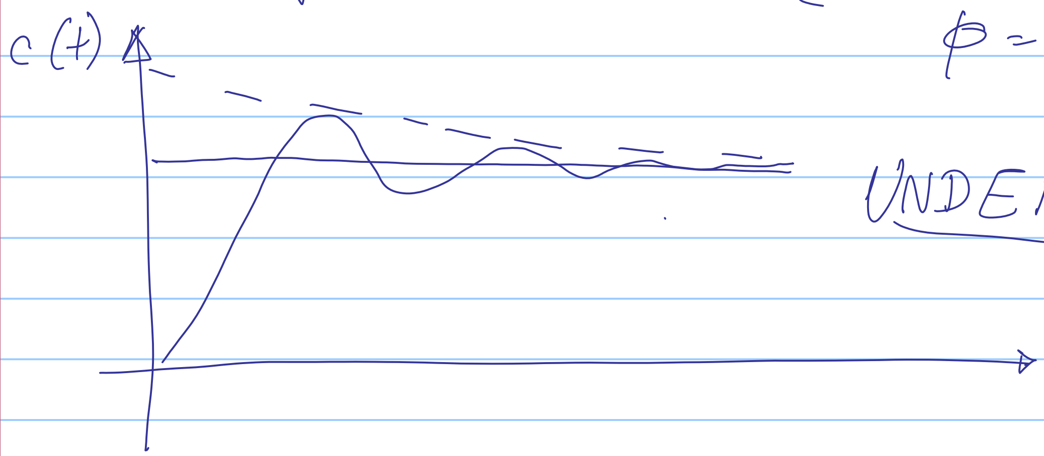
$$C(s) = \frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2+2\zeta\omega_n s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+\zeta\omega_n) + \frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\{(\omega_n\sqrt{1-\zeta^2})t\} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\{(\omega_n\sqrt{1-\zeta^2})t\}\right]$$

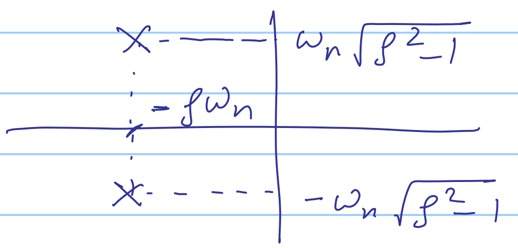
$$= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\{(\omega_n\sqrt{1-\zeta^2})t - \phi\}$$

$$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$




UNDERDAMPED

### Simulink Simulation ###




← Pole-zero plot

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Case III : Purely Imaginary Poles

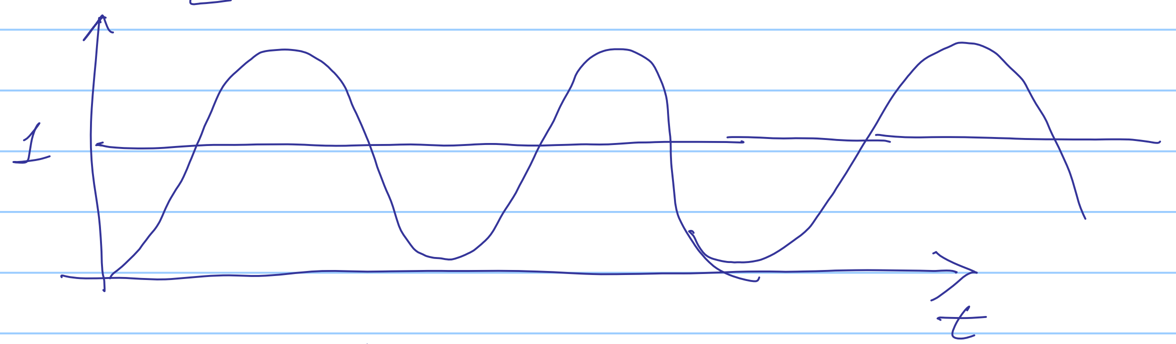
$a = 0$ or $\zeta = 0$

Poles are : $\sigma_{1,2} = \pm j\omega_n$

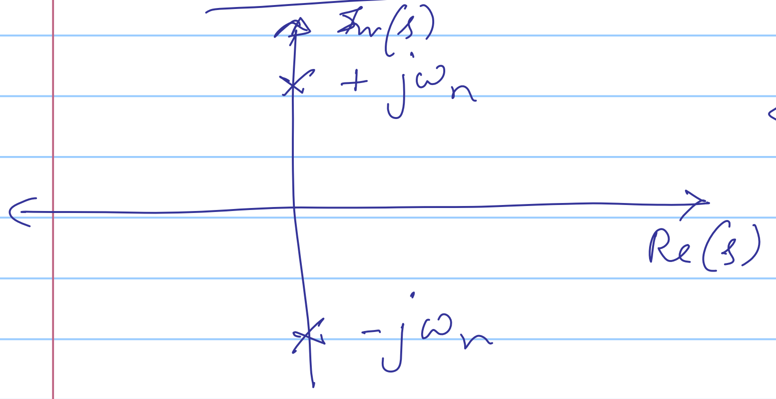
$$G(s) = \frac{\omega_n^2}{s^2 + \omega_n^2} \qquad R(s) = \frac{1}{s}$$

$$C(s) = \frac{\omega_n^2}{(s^2 + \omega_n^2)\,s} = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$$

$$c(t) = \left[1 - \cos(\omega_n t)\right] u(t)$$




UNDAMPED RESPONSE




Pole - Zero Plot

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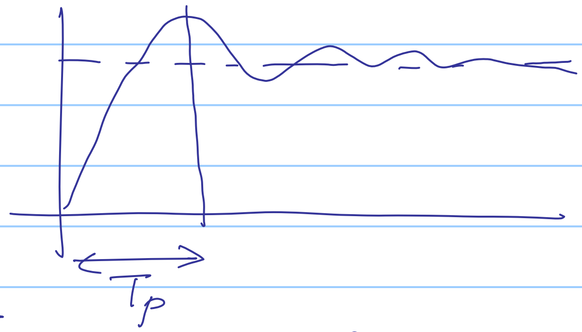
Transient Specifications in terms of $\zeta$ and $\omega_n$ :

Underdamped Case :

Peak Time : $T_p$ : the time required to reach first /

maximum peak.



$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}\right]$$

$$\frac{dc(t)}{dt} = -e^{-\zeta\omega_n t}\left[-\left(\omega_n\sqrt{1-\zeta^2}\right)\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\} + \frac{\zeta}{\cancel{\sqrt{1-\zeta^2}}}\,\omega_n\cancel{\sqrt{1-\zeta^2}}\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t\right]$$

$$-\left(-\zeta\omega_n\right)e^{-\zeta\omega_n t}\left[\cos\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}\right]$$

$$= \omega_n\sqrt{1-\zeta^2}\,e^{-\zeta\omega_n t}\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}$$

$$+ \frac{\zeta^2\omega_n}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}}\,e^{-\zeta\omega_n t}\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}$$

Putting $\frac{dc(t)}{dt} = 0$;

$$\left(\omega_n\sqrt{1-\zeta^2}\right)t = n\pi$$

$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$ , We are interested in the first maxima

Hence $\boxed{T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$ $(n=1)$

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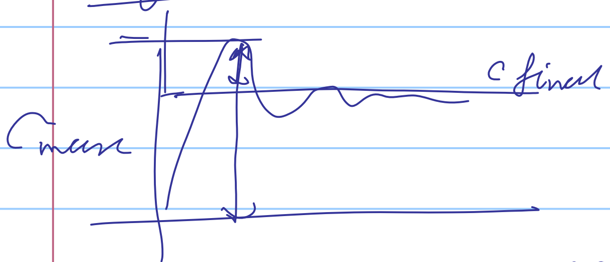
## % OS in terms of $\zeta, \omega_n$

**Def$^n$**: $\%OS = \frac{c_{max} - c_{final}}{c_{final}} \times 100$




$c_{max} = c(T_p)$

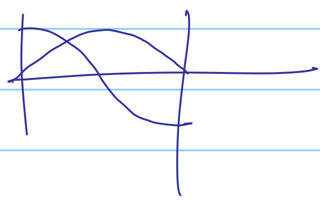
$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t\right\}\right]$$

$$c(T_p) = c\left(\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right) = 1 - e^{-\frac{\zeta\omega_n\pi}{\omega_n\sqrt{1-\zeta^2}}} *$$

$$\left[\cos\left\{\omega_n\sqrt{1-\zeta^2} * \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right\} + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left\{\omega_n\sqrt{1-\zeta^2}\,\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}\right\}\right]$$

$$= 1 - e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\left[\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right]$$

$$= 1 - e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$$



For unit step: $c_{final} = 1$

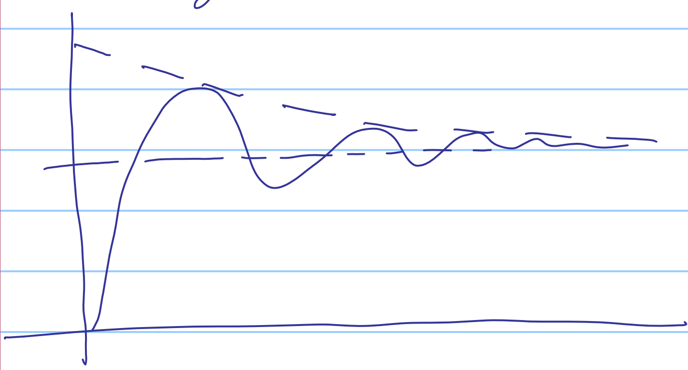
$$\boxed{\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100}$$

Conversely:

$$\boxed{\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}}$$

## Evaluation of $T_s$

Def$^{\underline{n}}$: Time in which $c(t)$ reaches and stays within 2% of the steady state value, $c_{final}$.



= Time taken for the decaying sinusoid to reach 0.02

$$c(t) = 1 - \underbrace{\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}} \cos(\omega_n\sqrt{1-\zeta^2}\,t - \phi)$$

We want $1.02 \leq c(t) \leq 0.98$

or $-0.02 \leq \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(*) \leq 0.02$

Now, $-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \leq \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(*) \leq +\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}$

So let us find a time (say $t_1$) for which

$$\frac{e^{-\zeta\omega_n t_1}}{\sqrt{1-\zeta^2}} \leq 0.02$$

$$t_1 = -\frac{\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

Clearly $T_s \leq t_1$, but we will use $t_1$ as a conservative estimate of $T_s$

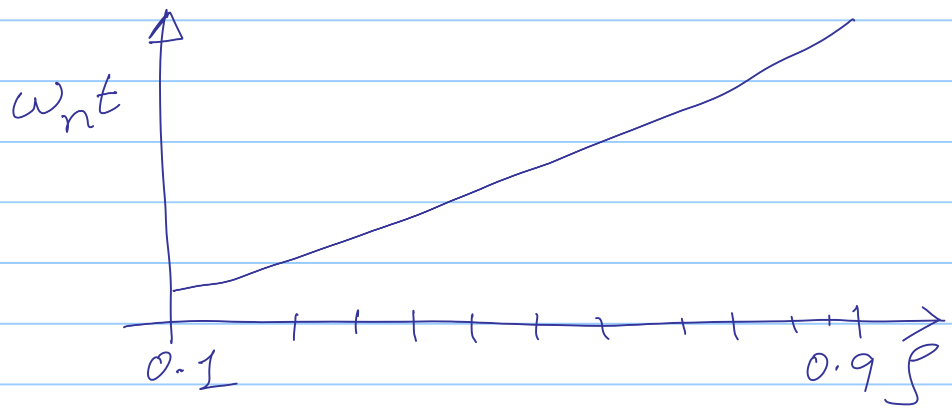
$$T_s = -\frac{\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

3.91 to 4.74 as $\zeta \sim 0$ to $0.9$

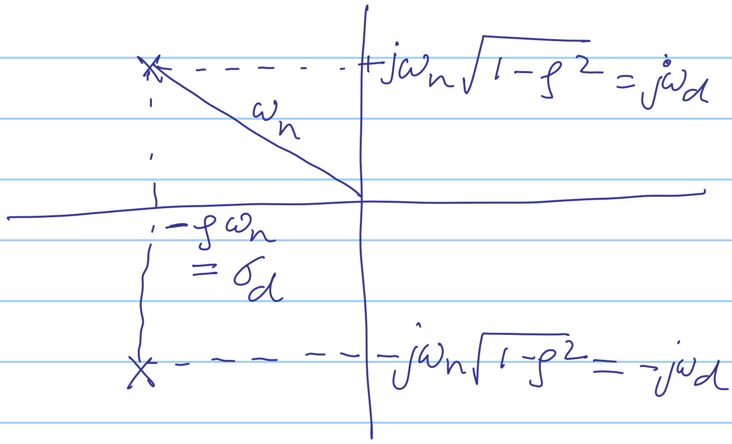
$$\boxed{T_s \approx \frac{4}{\zeta\omega_n}}$$

Rise Time : $T_r$ analytical expression cannot be found.

It is read out from the following graph:



## $T_p$, $T_s$ and %OS in terms of pole location:

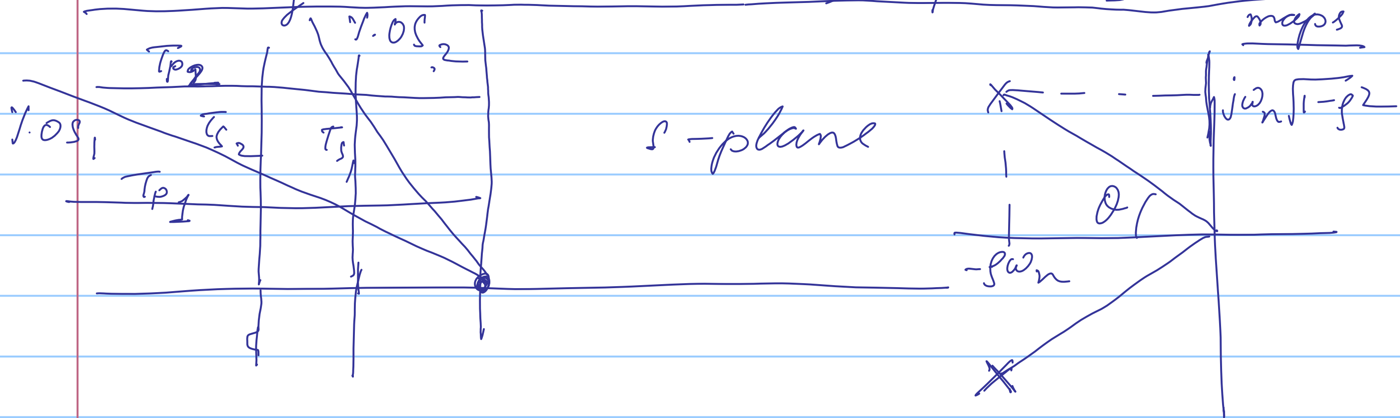



$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

$$\%OS = e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100 \leftarrow \text{function of only } \zeta$$

## Lines of Constant %OS, $T_p$ & $T_s$ in P-Z maps



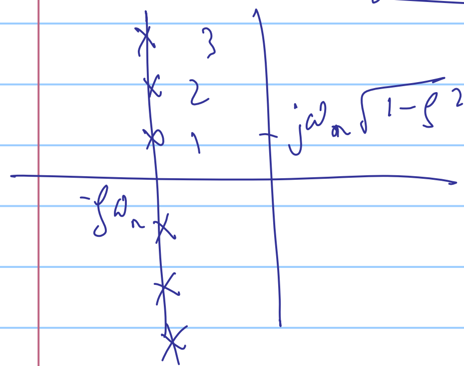

* For poles with constant $\theta$, $\zeta$ remains the same
  $\tan\theta = \left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

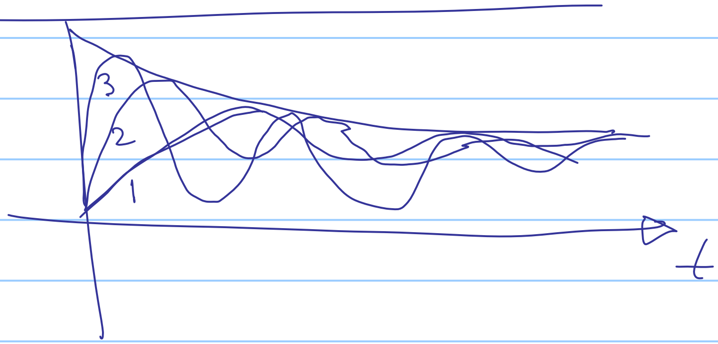
  But %OS is a function of $\zeta$ only.
  So %OS remains same on any line passing through the origin

* $T_{s_2} < T_{s_1}$: Settling time decreases as the poles move further left.

* $T_{p_2} < T_{p_1}$ : Peak time decreases as the magnitude of imaginary part increases (Peak occurs earlier)

# Pictures from book + Simulation

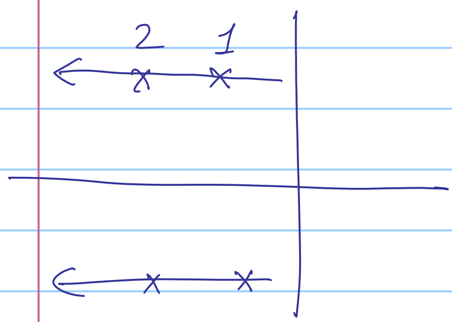


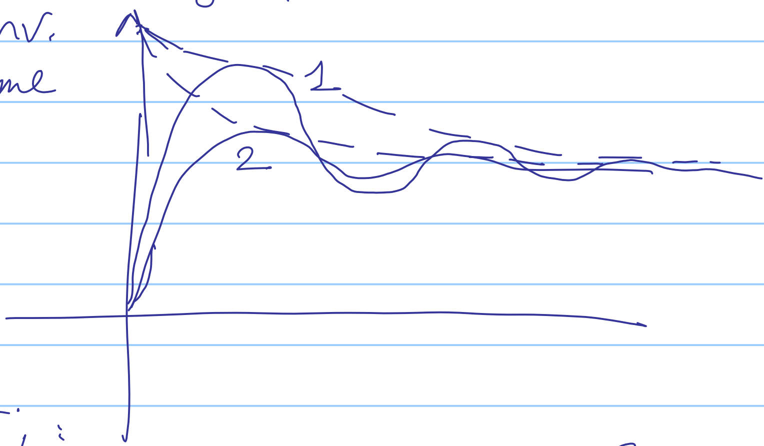




$$c(t) = 1 - \underbrace{\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}}_{\text{same}}\cos\{\underbrace{(\omega_n\sqrt{1-\zeta^2})}_{\text{increases}}t - \phi\}$$

same ⇒ exp. env. remains same

increases ⇒ freq increases



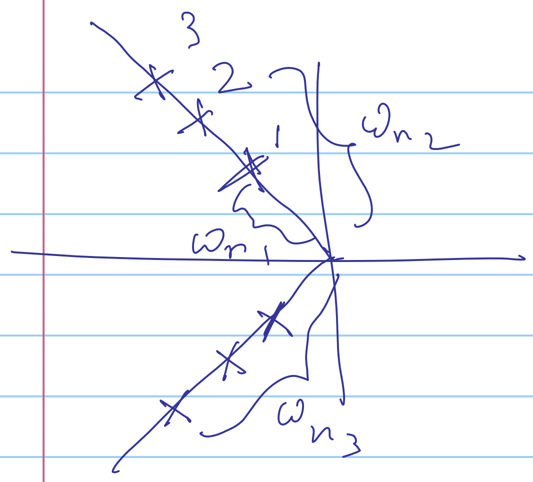



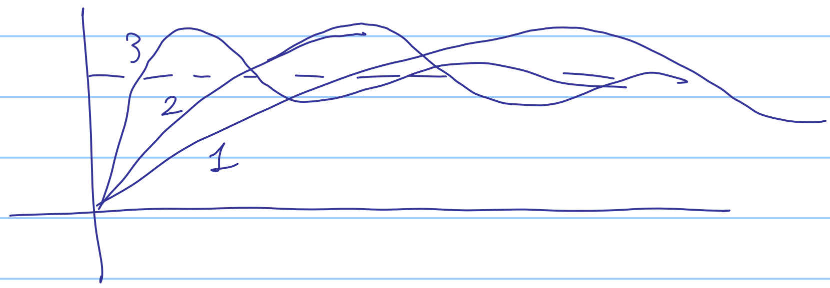

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\cos\{\underbrace{(\omega_n\sqrt{1-\zeta^2})}_{\text{same}}t - \phi\}$$

same ⇒ same freq.

more -ve

⇒ the exp. envelope decays faster





%OS remains same

$\zeta$ remains same

$\omega_{n_3} > \omega_{n_2} > \omega_{n_1}$ remains same

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos\left\{\left(\omega_n\sqrt{1-\zeta^2}\right)t - \phi\right\}$$

increases increase same

* Responses are of identical shape except for speed.
  * Speed is more as poles move left.

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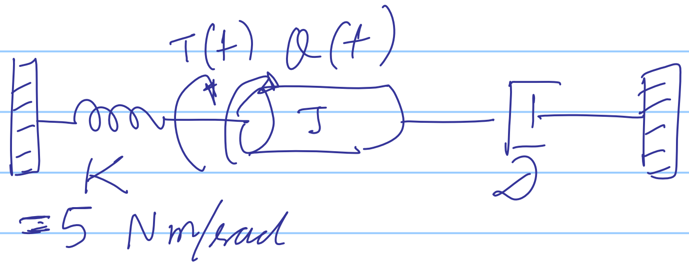
Example :



Find $J$ & $D$ to yield 20% overshoot and a settling time of 2 sec. for a step input of torque $T(t)$.

$$J\frac{d^2\theta}{dt^2} + D\frac{d\theta}{dt} + K\theta(t) = T(t)$$

$$\Rightarrow \quad Js^2\theta(s) + Ds\,\theta(s) + K\theta(s) = T(s)$$

$$\Rightarrow \quad \frac{\theta(s)}{T(s)} = \frac{1}{Js^2 + Ds + K}$$

$$= \frac{1/J}{s^2 + \frac{D}{J}s + \frac{K}{J}}$$

$\omega_n = \sqrt{\frac{K}{J}}$ $\quad 2\zeta\omega_n = \frac{D}{J}$

<u>Specs</u> : $T_s = 2$ $\quad \%OS = 20$

$$\frac{4}{\zeta\omega_n} = 2 \quad \text{— (1)} \qquad \zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2\left(\frac{\%OS}{100}\right)}} \quad \text{— (2)}$$

Using $\%OS = 20$ in (2)
$\Rightarrow \zeta = 0.456$

From (1), $2\zeta\omega_n = \boxed{\frac{D}{J} = 4}$

and $\zeta = \frac{2}{\omega_n} = 2\sqrt{\frac{J}{K}} = 0.456$

$\boxed{\frac{J}{K} = 0.052}$ , (Given $K = 5$)

$J = 0.26$ kg-m$^2$ , $D = 1.04$ N-m-s/rad

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## <u>System Response with Additional Poles</u>

* All of calculations above is valid only for 2nd order systems
* Higher order system can sometimes be approximated by <u>dominant poles</u>

Consider a 3rd order system (extra real pole at $-\alpha_R$)

$$G(s) = \frac{\alpha_r \omega_n^2}{(s+\alpha_r)(s^2+2\zeta\omega_n s+\omega_n^2)}$$

Poles: $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$ , $-\alpha_r$

$$C(s) = \frac{A}{s} + \frac{B(s+\zeta\omega_n) + C\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + \frac{D}{s+\alpha_r}$$

$$c(t) = Au(t) + e^{-\zeta\omega_n t}(B\cos\omega_d t + C\sin\omega_d t) + De^{-\alpha_r t}$$

If $-\alpha_r << -\zeta\omega_n$ then it has no effect on $T_p$, $\%OS$, $T_s$ etc

Thus we can approx the response using a 2nd order approx.

Thumb rule: If $\alpha_r > 5\zeta\omega_n$ then dominant pole approx. is valid.

What about residue?

$$C(s) = \frac{bc}{s(s^2+as+b)(s+c)} = \frac{A}{s} + \frac{Bs+C}{s^2+as+b} + \frac{D}{s+c}$$

$$A = 1 \qquad B = \frac{ca-c^2}{c^2+b-ca}$$

$$C = \frac{ca^2-c^2a-bc}{c^2+b-ca} \qquad D = \frac{-b}{c^2+b-ca}$$

As $c \to \infty$, $A = 1$, $B = -1$, $C = -a$, $D = 0$

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## System Response with Zeros

Zeros affect the residue/amplitude.

Assumption: We add real axis zero to a two pole system

Case I : Add zero in LHP
Case II : Add zero in RHP
Case III : Pole zero cancellation

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Case I : Zero in LHP (Arguments valid only for REAL zeros)

* If zero is close to the Dominant pole: response changes
* If zero is far from Dominant poles response is unchanged.

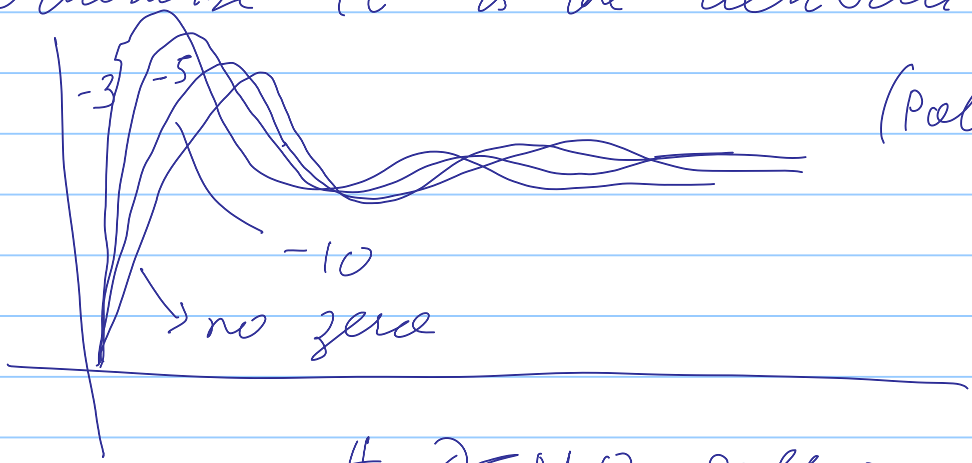
$$T(s) = \frac{(s+a)}{(s+b)(s+c)} = \frac{A}{s+b} + \frac{B}{s+c}$$

$$= \frac{\left(\frac{a-b}{c-b}\right)}{s+b} + \frac{\left(\frac{a-c}{b-c}\right)}{s+c}$$

$$\approx a\left[\frac{\frac{1}{c-b}}{(s+b)} + \frac{\frac{1}{b-c}}{(s+c)}\right] \quad \left(\text{Since } a \gg b \gg c\right)$$

$$\approx \frac{a}{(s+b)(s+c)}$$

Otherwise it is the derivative




(Poles $-1 \pm j\,2.8$)

# DEMO software

Case II : Real zero on RHP.

Let $C(s)$ be the response without zero.

$$(s-a)\,C(s) = \underbrace{s\,C(s)}_{\text{Derivative of response}} - \underbrace{a\,C(s)}_{\text{Scaled response}}$$

Derivative (+ve) is opposite sign of the scaled response

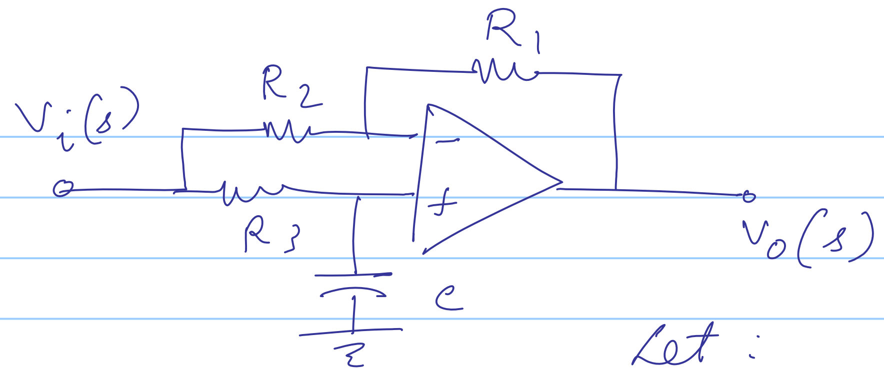

Final value th

$$\lim_{s \to 0} (s-a)\,\underbrace{G(s)\frac{1}{s}}_{C(s)} \cdot s = -a\,G(0) = -a$$

But initial derivative of $c(t)$ is +ve

Non-minimum Phase systems

Example :




$$\frac{V_o(s)}{V_i(s)} = \frac{R_2 - R_1 R_3 C s}{R_2 R_3 C s + R_2}$$

Let :
$R_1 = R_2$
$R_3 C = \frac{1}{10}$

$$= -\left(\frac{s-10}{s+10}\right)$$

$$= -\frac{s}{s+10} + 10\,\frac{1}{s+10}$$

$$C(s) = s\underbrace{\left(-\frac{1}{s+10}\cdot\frac{1}{s}\right)}_{C_0(s)} - 10\underbrace{\left\{\frac{-1}{(s+10)\,s}\right\}}_{C_0(s)}$$

Exercise: Calculate, plot & interpret $C_0(t)$, $C(t)$ (non-min phase behaviour)

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Case III : Pole-Zero Cancellation & 2nd order approx

$$T(s) = \frac{K(s+z)}{(s+p_3)(s^2+as+b)}$$

\# If $z$ is "close" to $p_3$ then a 2nd order approx is valid.

\# Only way to check is to calculate residues

$$C_1(s) = \frac{26.25(s+4)}{s(s+3.5)(s+5)(s+6)}$$ a) NOT VALID

2nd order A.

$$C_2(s) = \frac{26.25(s+4)}{s(s+4.01)(s+5)(s+6)}$$ b) VALID

Exercise: Justify conclusion (a) & (b) by calculating residues.

## Time Response of a Realization

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0^-) = x_0$$
$$y(t) = Cx(t)$$

Problem: Solve for $x(t)$.

Case I: The homogenous system OR $u(t) = 0$ for all $t$.

Then: $\dot{x}(t) = Ax(t) \qquad x(0^-) = x_0$
$y(t) = Cx(t)$

Using Laplace transform,

$$sX(s) - X(0^-) = AX(s)$$

or $X(s) = (sI - A)^{-1} x_0$

So $x(t) = \mathcal{L}^{-1}\{X(s)\} = \underbrace{\left\{\mathcal{L}^{-1}(sI-A)^{-1}\right\}}_{1} x_0$

Consider:

$(sI-A)^{-1}$ in the formula above.

Digression: Geometric Series

$$(1-\alpha)^{-1} = 1 + \alpha + \alpha^2 + \alpha^3 + \cdots\cdots$$

In this form the geometric series expansion is valid for a $n \times n$ matrix.

$$(sI-A)^{-1} = \frac{1}{s}\left(I - \frac{1}{s}\cdot A\right)^{-1}$$

$$= \frac{1}{s}\left[I + \frac{1}{s}A + \frac{1}{s^2}A^2 + \cdots\cdots\right]$$

$$= \frac{1}{s}I + \frac{1}{s^2}A + \frac{1}{s^3}A^2 + \cdots\cdots$$

Now apply the inverse Laplace Transform

$$\mathcal{L}^{-1}(sI-A)^{-1} = \mathcal{L}^{-1}\left(\frac{1}{s}\right)I + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right)A + \mathcal{L}^{-1}\left(\frac{1}{s^3}\right)A^2 + \cdots\cdots$$

$$\left[\text{Recall that } \mathcal{L}^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}, \; t \geq 0\right]$$

$$= I + tA + \frac{t^2}{2!}A^2 + \frac{t^3}{3!}A^3 + \cdots\cdots$$

Now where have we seen a similar form? Recall the series expansion of $e^{\alpha}$ for a scalar $\alpha$.

$$e^{\alpha} = 1 + \alpha + \frac{\alpha^2}{2!} + \frac{\alpha^3}{3!} + \cdots .$$

This looks so similar to the above formula, that we DEFINE the matrix exponential:

$$e^{At} := I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

$$= \sum_{i=0}^{\infty} \frac{t^i}{i!} A^i$$

From the above formula, for a $n \times n$ matrix $A$, $e^{At}$ is also a $n \times n$ matrix.

$$\boxed{\mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = e^{At}} \quad t \geq 0 \quad (*)$$

Hence, using $(*)$ in $(1)$,

$$x(t) = e^{At} x_0$$

Exercise: The matrix $e^{At}$ is sometimes called the state-transition matrix. Why?

Solving the general equation

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad x(0^-) = x_0$$
$$y(t) = Cx(t)$$

Take L.T. :

$$X(s) = (sI-A)^{-1}x_0 + (sI-A)^{-1}BU(s)$$

Now, $\mathcal{L}^{-1}\left\{(sI-A)^{-1}BU(s)\right\}$

$$= \mathcal{L}^{-1}\left\{(sI-A)^{-1}B\right\} * \mathcal{L}^{-1}\left\{U(s)\right\}$$

$$= e^{At}B * u(t)$$

Convolution

So, $x(t) = e^{At}x_0 + e^{At}B * u(t)$

$$= e^{At}x_0 + \int_{0^-}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$$

OR

$$\boxed{\begin{aligned} x(t) &= e^{At}\left[x_0 + \int_{0^-}^{t} e^{-A\tau}Bu(\tau)\,d\tau\right] \\ y(t) &= Cx(t) \end{aligned}}$$

<u>Example</u> : $\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$

$$x(0^-) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Suppose $u(t) = e^{t}$. Find $x(t)$.

Solⁿ. First calculate $e^{At}$

$$(sI-A)^{-1} = \begin{bmatrix} s-1 & -1 \\ 0 & s-1 \end{bmatrix}^{-1}$$

$\det(sI-A)$

$= (s-1)^2$

$\operatorname{adj}(sI-A)$

$$= \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}^T$$

$$= \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 1 \\ 0 & s-1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{s-1} & \frac{1}{(s-1)^2} \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

Hence $e^{At} = \mathcal{L}^{-1}\left\{(sI-A)^{-1}\right\} = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}$

According to ⊛ above,

$$x(t) = \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_{0^-}^{t} \begin{bmatrix} e^{-\tau} & -\tau e^{-\tau} \\ 0 & e^{-\tau} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{\tau} d\tau\right\}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_{0^-}^{t} \begin{pmatrix} -\tau e^{-\tau} \\ e^{-\tau} \end{pmatrix} e^{\tau} d\tau\right\}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \int_0^t \begin{pmatrix} -\tau \\ 1 \end{pmatrix} d\tau \right\}$$

$$= \begin{bmatrix} e^t & te^t \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 - t^2/2 \\ 1 + t \end{bmatrix}$$

$$= \begin{bmatrix} e^t - \frac{t^2}{2} e^t + te^t + t^2 e^t \\ e^t + te^t \end{bmatrix} = \begin{bmatrix} e^t + te^t + \frac{t^2}{2} e^t \\ e^t + te^t \end{bmatrix}$$

Exercise: Solve this problem using equation (**) instead of (*).