Nonlinear Dynamical Systems

Consider n^{th}-order ordinary diff. eqns of the form:

$$\frac{d^n y(t)}{dt^n} = h\left[t, y(t), \dot{y}(t), \cdots, \frac{d^{n-1} y(t)}{dt^{n-1}}, u(t)\right] \quad t \geq 0$$

One can define:

$$x_1(t) = y(t) \qquad \dot{x}_1(t) = x_2(t)$$
$$x_2(t) = \dot{y}(t) \qquad \dot{x}_2(t) = x_3(t)$$
$$\vdots \qquad\qquad \vdots$$
$$x_n(t) = \frac{d^{n-1} y(t)}{dt^{n-1}} \qquad \dot{x}_n(t) = h[t, x_1(t), x_2(t), \cdots, x_n(t), u(t)]$$

Define: $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad f = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ h \end{bmatrix}$

Then $\dot{x}(t) = f(t, x(t), u(t)) \quad t \geq 0.$

In this course we will study

$$\dot{x}_1 = f_1(t, x_1, \cdots, x_n, u_1, \cdots, u_p)$$
$$\dot{x}_2 = f_2(t, x_1 \text{ ——— } - \; - \cdot)$$
$$\vdots$$
$$\dot{x}_n = f_n(t, \quad - - - \quad --)$$

denoted by

$$\dot{x} = f(t, x, u) \quad \text{———} \quad (1)$$

Sometimes output eqns : $y(t) = h(t, x, u)$

Non-linear : Anywhere superposition does not work is defined as non-linear.

Why study such systems : One can always "linearize"!

1> Many new phenomena :
a> Finite Escape time
b> Multiple isolated equilibria
c> Limit Cycles.
d> Almost-periodic sol^n / Bifurcation & chaos (Not here)
2> Non-existence & or non-uniqueness of solutions over $[0, \infty)$ - not such a practical issue

3> Better controllers:
a> Valid over large ranges
b> Hard non-linearities (Saturation, dead-zones
c> Controller design may be simpler (on-off controller)

Def^n : The system $\dot{x} = f(t, x)$ (*) is said to be autonomous if $f(t, x)$ is independent

[illegible]
of t and is said to be <u>non-autonomous</u> otherwise.

Q. what happened to $u(t)$?

<u>Def^n</u>: A vector $x_0 \in \mathbb{R}^n$ is said to be an equilibrium point at time $t_0 \in \mathbb{R}_+$ of (1) if $f(t, x_0) = 0 \quad \forall\, t \geq t_0$.

$\Rightarrow$ (1) If x_0 is eq. pt. of (*) at t_0, then x_0 is eq. pt. of (*) at all $t \geq t_0$.

$\Rightarrow$ For autonomous systems, x_0 is an eq. pt. at some time $\Leftrightarrow$ it is an eq. pt. at all times.

$\Rightarrow$ If $f(t_0, x_0) = 0$ then $\dot{x}(t) = f(t, x(t))$ $t \geq t_0$, $x(t_0) = x_0$ has the unique sol^n: $x(t) = x_0 \quad \forall\, t \geq t_0$.

<u>Def^n</u>: An eq. pt. x_0 at time t_0 of (*) is said to be isolated if $\exists$ a neighbourhood N of x_0 in $\mathbb{R}^n$ s.t. N contains no eq. pt. at time t_0 of (1) other than x_0.

Examples: 1) Multiple isolated eq.

Pendulum: $\ddot{\theta} + \frac{g}{l}\sin\theta(t) = 0$

Let $x_1(t) = \theta(t)$, $x_2(t) = \dot{\theta}(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{g}{l}\sin x_1(t) \end{bmatrix}$$

$$\text{Eq. pt} \iff \left.\begin{matrix} x_{20} = 0 \\ \sin x_{10} = 0 \end{matrix}\right\} \rightarrow \begin{bmatrix} 0 \\ n\pi \end{bmatrix}$$

$n = 0, \pm 1, \pm 2 \ldots$

$$\therefore \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \ \& \ \begin{bmatrix} 0 \\ \pi \end{bmatrix}$$

Q. What about $\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x$?

2) No solution; Non-unique sol^n

a) $\dot{x} = -\,\text{sign}[x(t)] \quad t \geq 0, \ x(0) = 0$

$$\text{sign}(x) := \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$$

No continuously diff. sol^n.

b) $\dot{x} = \frac{1}{2x(t)} \quad t \geq 0, \ x(0) = 0.$

2 sol^{ns}: $x_1(t) = t^{1/2}$, $x_2(t) = -t^{1/2}$

c) $\dot{x}(t) = 1 + x^2(t) \quad t \geq 0, \ x(0) = 0$

Unique soln : $x(t) = \tan t$ but valid only over $[0, 1)$.

No continuously diff. soln over all of $[0, \infty)$.

<u>Finite Escape time</u>: $\dot{x} = -x^2$, $x(0) = -1$

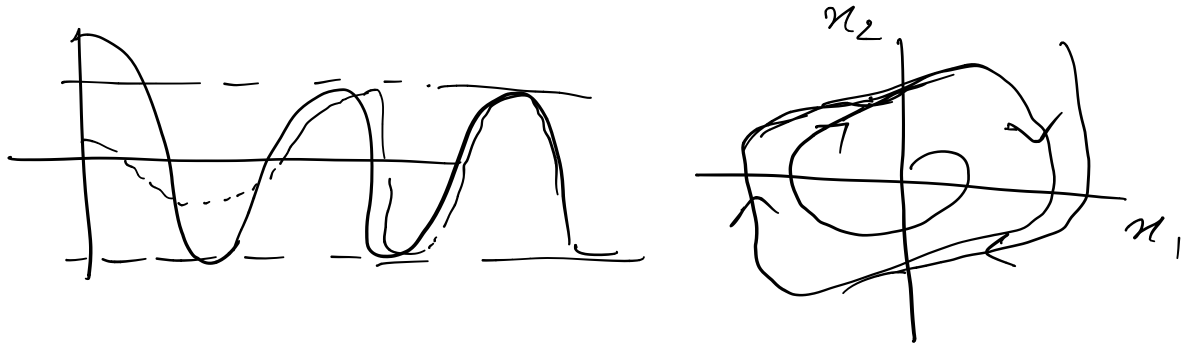
Unique soln: $x(t) = \frac{1}{t-1}$ on $[0, 1)$

As $t \to 1$, $x(t)$ leaves any bdd. set.

<u>Limit Cycles</u>: $m\ddot{x} + 2c(x^2-1)\dot{x} + kx = 0$

(Van der Pol eqn)

RLC ckt with non-linear resistor.



1) Amplitude does not depend initial condition

2) Oscillations are robust to system parameters.

Examples of Bifurcation & Chaos: Refer. Slotine Pa - 10 - 11.