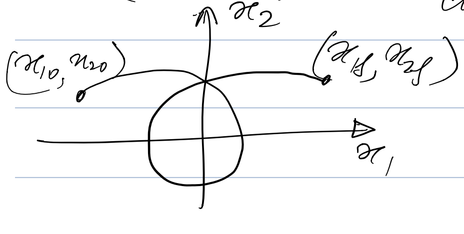
We will consider only autonomous systems

$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}\right\}$$

The locus in the $x_1 - x_2$ plane of $x(t)$ $\forall\, t \geq 0$ is a phase-plane plot / trajectory

$x_0 = (x_{10}, x_{20})^T$

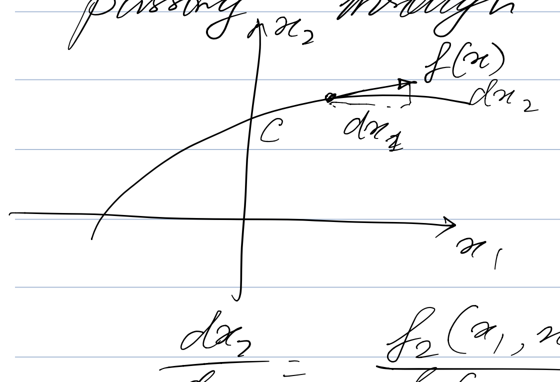
The family of all such curves is called a phase-portrait.



Defn: The vector field associated with $\begin{bmatrix}\dot{x}_1 = f_1(x_1, x_2)\\ \dot{x}_2 = f_2(x_1, x_2)\end{bmatrix}$

is defined as the continuous function $f := \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$. The direction of the vector field f at a point $x \in \mathbb{R}^2$ is denoted by $\theta_f(x) = \tan^{-1} \dfrac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$

$f(x)$ is tangent to the phase plane trajectory passing through x.



$$\frac{dx_1}{dt} = f_1(x_1, x_2)$$

$$\frac{dx_2}{dt} = f_2(x_1, x_2)$$

$$\frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$$

Q: Can the phase trajectories intersect?

Phase Portraits of Linear Systems

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} \qquad \begin{matrix} x_1(0) = x_{10} \\ x_2(0) = x_{20}\end{matrix} \;=\; x(0) = x_0$$

$$\dot{x} = A \, x$$

$z(t) = M^{-1}x(t) \rightarrow \dot{z} = M^{-1}AM\,z(t)$; $z(0) = M^{-1}x_0$

By choosing M appropriately, $M^{-1}AM$ can be made into one the following 3 forms:

1) $M^{-1}AM = \begin{bmatrix}\lambda_1 & 0 \\ 0 & \lambda_2\end{bmatrix}$ λ_1, λ_2 are real (not necessarily distinct)

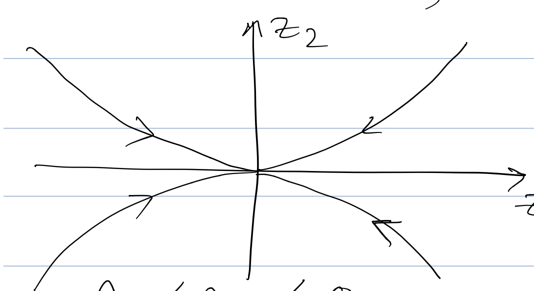
2) $M^{-1}AM = \begin{bmatrix}\lambda & 1 \\ 0 & \lambda\end{bmatrix}$ λ is repeated real

3) $M^{-1}AM = \begin{bmatrix}\alpha & \beta \\ -\beta & \alpha\end{bmatrix}$ $\alpha \pm j\beta$ are the eigenvalues.

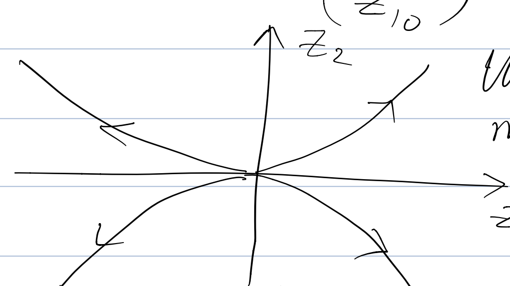
$\boxed{\text{For } \lambda_1 \neq 0, \lambda_2 \neq 0}$

Case 1: $z_1(t) = z_{10}e^{\lambda_1 t}$; $z_2(t) = z_{20}e^{\lambda_2 t}$

eliminate t, to get $z_2 = z_{20}\cdot\left(\dfrac{z_1}{z_{10}}\right)^{\lambda_2/\lambda_1}$

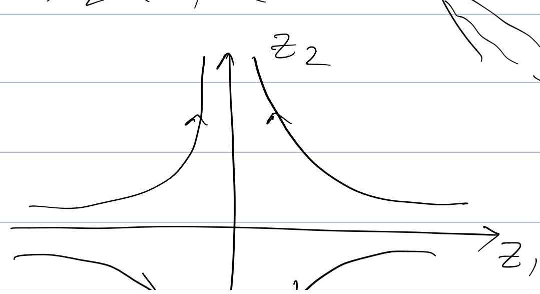


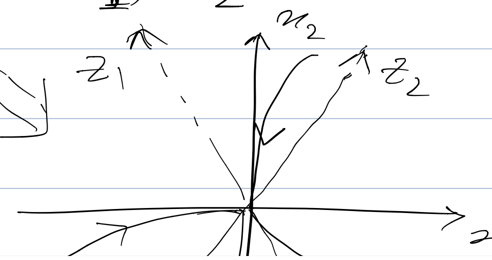
$\lambda_2 < \lambda_1 < 0$

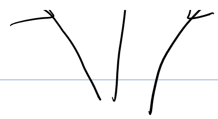


Unstable node

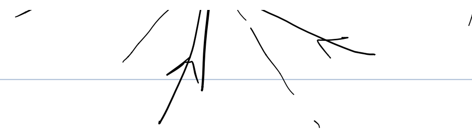
$\lambda_1 > \lambda_2 > 0$



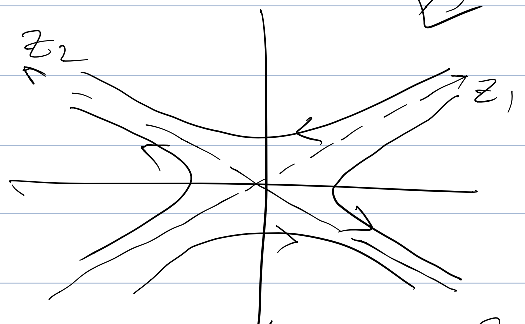




$\lambda_1 < 0 < \lambda_2$



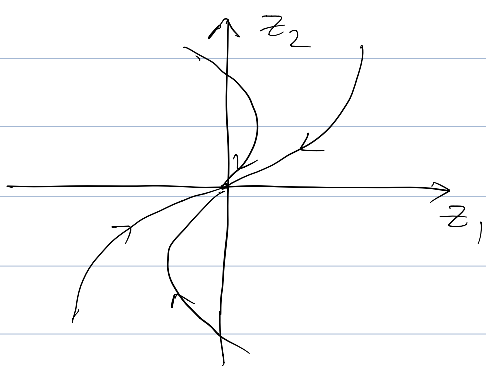
Stable node

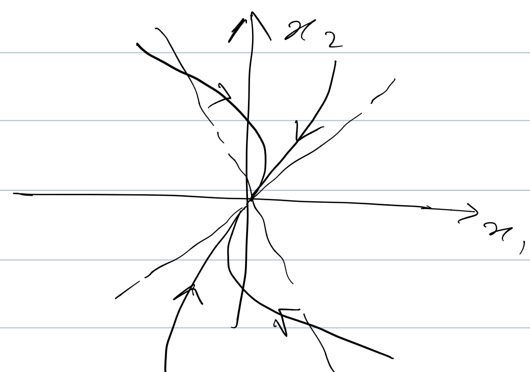


Saddle point?
Why?

Case 2: Jordan Form :
$$\begin{aligned}\dot{z}_1 &= \lambda z_1 + z_2 \\ \dot{z}_2 &= \lambda z_2\end{aligned} \qquad \begin{aligned} z_1(0) &= z_{10} \\ z_2(0) &= z_{20}\end{aligned}$$

$$\begin{aligned} z_1(t) &= z_{10} e^{\lambda t} + z_{20} t e^{\lambda t} \\ z_2(t) &= z_{20} e^{\lambda t}\end{aligned}$$
Still possible to eliminate t, but messy.





Stable node if $\lambda < 0$
Unstable node if $\lambda > 0$.
Eq pt. $(0,0)$.

Case 3 : Complex Conjugate

$\dot{z}_1 = \alpha z_1 + \beta z_2 \qquad z_1(0) = z_{10}$

$$(*)\ \dot z_2 = -\beta z_1 + \alpha z_2 \qquad z_2(0) = z_{20}$$

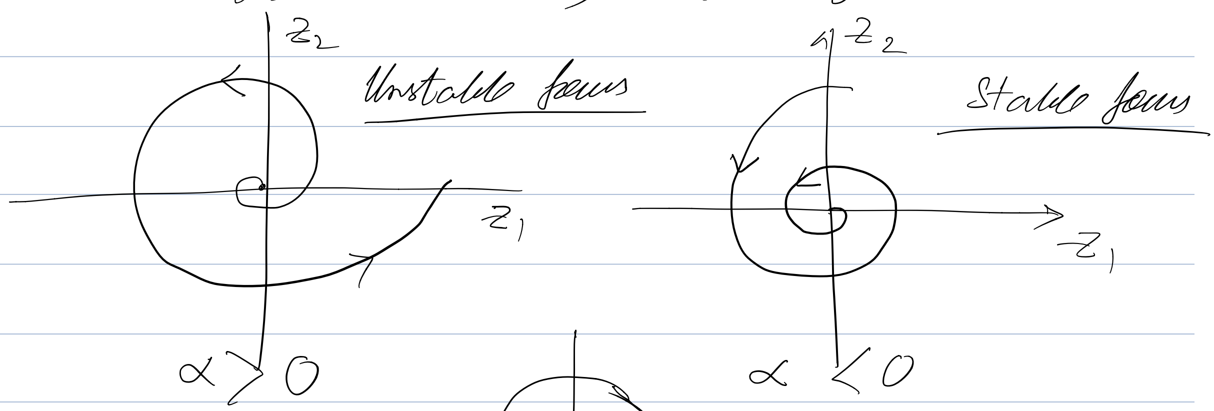
Introduce polar coordinates to simplify:

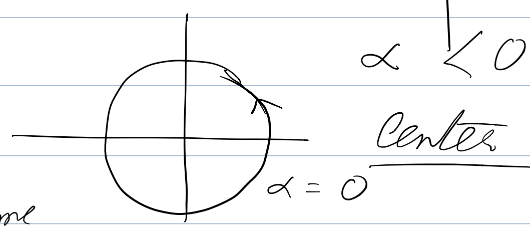
$$r = \sqrt{z_1^2 + z_2^2} \qquad \phi = \tan^{-1}\frac{z_2}{z_1}$$

Then $(*)$ becomes $\dot r = \alpha r(t)$ & $\dot\phi = -\beta$

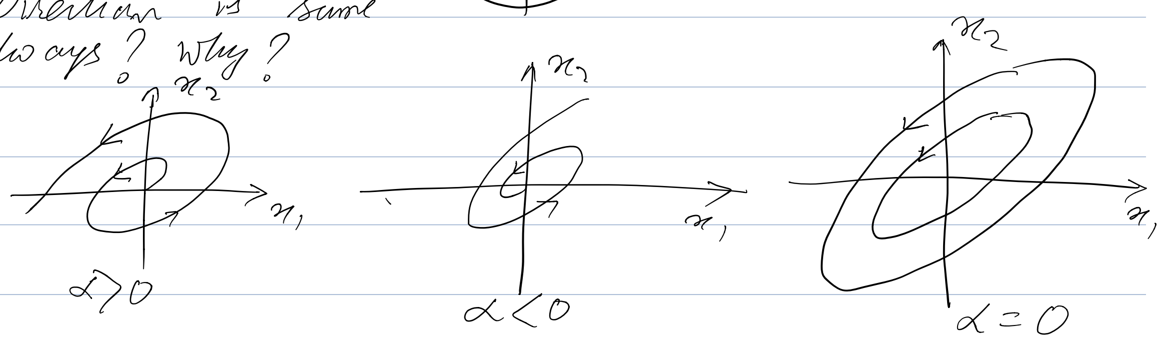
$$\left[\begin{aligned} \dot r &= \frac{\partial r}{\partial z_1}\dot z_1 + \frac{\partial r}{\partial z_2}\dot z_2 = \frac{1}{2}\left[\frac{2z_1}{r}\dot z_1 + \frac{2z_2}{r}\dot z_2\right] \\ &= \frac{1}{r}\left[\alpha z_1^2 + \beta z_1 z_2 - \beta z_1 z_2 + \alpha z_2^2\right] = \alpha r. \end{aligned}\right]$$

Solution: $r(t) = r_0 e^{\alpha t}$, $\phi(t) = \phi_0 - \beta t$





Q. Direction is same always? Why?



Exercise: Phase Portraits for one or both eigenvalues zero.

Non-linear systems

$$\left.\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2)\\ \dot{x}_2 &= f_2(x_1, x_2)\end{aligned}\right\} \dot{x} = f(x)$$

Linearization : Linearize f_1, f_2 in the neighbourhood of any of the eq. pts.
→ Determine the behaviour of non-linear system trajectories from the linearized system.

1) Assume $(0,0)$ is an eq. pt. (WLOG. → Why? $\tilde{x}_1 = x_1 - x_{10}$; $\tilde{x}_2 = x_2 - x_{20}$)
2) Assume f_1, f_2 are continuously diff. around $(0,0)$.
3) Define
$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix}\right|_{\substack{x_1 = 0\\ x_2 = 0}} =: \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}$$

$$f_1(x_1, x_2) = \underbrace{f_1(0,0)}_{=0} + a_{11}x_1 + a_{12}x_2 + r_1(x_1, x_2)$$
$$f_2(x_1, x_2) = \overbrace{f_2(0,0)}^{=0} + a_{21}x_1 + a_{22}x_2 + r_2(x_1, x_2)$$

4) Define the linear system:
$$\left.\begin{aligned}\dot{\xi}_1 &= a_{11}\xi_1 + a_{12}\xi_2\\ \dot{\xi}_2 &= a_{21}\xi_1 + a_{22}\xi_2\end{aligned}\right\} (*)$$

(*) is a linearization of $\dot{x} = f(x)$ around $(0,0)$.
(Will prove later) Trajectories of (*) and $\dot{x} = f(x)$ are qualitatively same in some suitably small neighbourhood of $(0,0)$.

	Linearized	Non-linear
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 < 0, \lambda_2 < 0$	Stable Node	Stable Node
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > 0, \lambda_2 > 0$	Unstable N.	Unstable N.
$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \lambda_2 < 0$	Saddle pt	Saddle pt.
$\lambda_1, \lambda_2 \in \mathbb{C}$, $Re\, \lambda_1 > 0$	Unstable focus	Unstable focus
$\lambda_1, \lambda_2 \in \mathbb{C}$, $Re\, \lambda_1 < 0$	Stable focus	Stable focus
λ_1, λ_2 pure imag	Center	?

Example: $\dot{x}_1 = -x_2 - \mu x_1 (x_1^2 + x_2^2)$
$\dot{x}_2 = x_1 - \mu x_2 (x_1^2 + x_2^2)$

Linearized eq: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -3\mu x_1^2 - \mu x_2^2 & -1 - 2\mu x_1 x_2 \\ 1 - 2\mu x_1 x_2 & \end{bmatrix}$

Eig $= \pm j$

$\longleftarrow = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad x_1 = 0$, $x_2 = 0$

But non-linear system: $x_1 = r\cos\theta$, $x_2 = r\sin\theta$

$\dot{r} = -\mu r^3 \qquad \dot{\theta} = 1$

$\rightarrow$ will resemble stable focus $\rightarrow \mu > 0$.
unstable focus $\rightarrow \mu < 0$.

Periodic Solution & Limit Cycles

Def: For $\dot{x} = f(x)$, $x(t)$ is a periodic solution if $\exists\, T$ s.t.
$x(t+T) = x(t)$. $\forall\, t$.

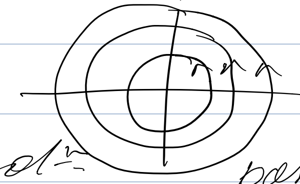
$\dot{x}_1 = x_2$; $\dot{x}_2 = -x_1$ | $x_1(0) = x_{10}$; $x_2(0) = x_{20}$

FACT.: Periodic sol^n $\Leftrightarrow$ a sol^n is a closed curve in $\mathbb{R}^2$

Sol^n: $x_1(t) = r_0 \cos(-t + \phi_0)$ $\quad\Big|\quad$ $r_0 = \sqrt{x_{10}^2 + x_{20}^2}$

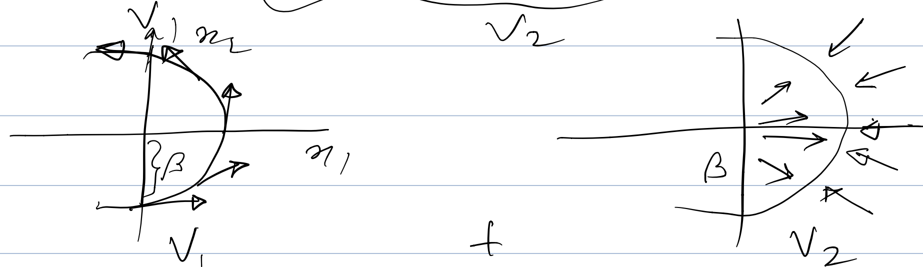
$x_2(t) = r_0 \sin(-t + \phi_0)$ $\quad\Big|\quad$ $\phi_0 = \tan^{-1} \dfrac{x_{20}}{x_{10}}$

Entire state space is covered with periodic solution.



Nonlinear Case: Isolated periodic sol^n possible

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + \alpha x_1(\beta^2 - x_1^2 - x_2^2) \\ \dot{x}_2 &= \underbrace{-x_1}_{V_1} + \underbrace{\alpha x_2(\beta^2 - x_1^2 - x_2^2)}_{V_2} \end{aligned}\right\} (*)$$



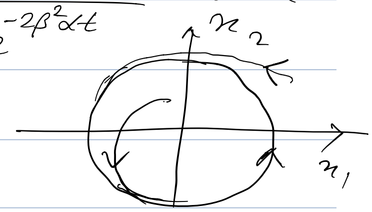
$V = V_1 + V_2$

Introduce polar coordinates: $r = \sqrt{x_1^2 + x_2^2}$; $\phi = \tan^{-1} \dfrac{x_2}{x_1}$

the $(*)$ becomes $\left.\begin{aligned} \dot{r} &= \alpha r(\beta^2 - r^2) \\ \dot{\phi} &= -1 \end{aligned}\right\}$

Solution (Exercise: Please verify): $r(t) = \dfrac{\beta}{\sqrt{1 + c_0 e^{-2\beta^2 \alpha t}}}$ and

$\phi(t) = \phi_0 - t$ $\qquad \left(c_0 = \dfrac{\beta^2}{r_0^2} - 1\right)$



Only one periodic sol^n: $r(t) = \beta$ / $x_1^2 + x_2^2 = \beta^2$

Also if $r_0 \neq 0$, every sol^n approach this periodic sol^n.

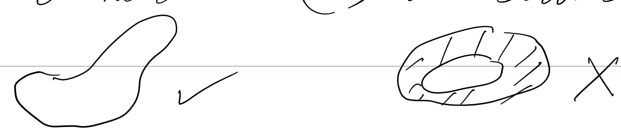
Def$^{\underline{n}}$: A *limit cycle* of $\begin{Bmatrix} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{Bmatrix}$ is a periodic solution. (sometimes "isolated" is insisted)

By convention, eq. pt. is not considered a limit cycle / periodic sol$^{\underline{n}}$).

Bendixson's Thm:

Def$^{\underline{n}}$: A connected region $\in \mathbb{R}^2$ is one which every two points in the set can be connected by a curve lying entirely within the set.

Def$^{\underline{n}}$: A set is simply connected if (1) it is connected (2) its boundary is connected.



Connected but not simply connected.

Thm: Let D be a simply connected set in $\mathbb{R}^2$ such that the quantity

$$\nabla f(x) := \frac{\partial f_1}{\partial x_1}(x_1, x_2) + \frac{\partial f_2}{\partial x_2}(x_1, x_2) \text{ is}$$

1) not identically zero over any subregion of D

2) does not change sign in D.

Then D contains no closed trajectories of $\begin{Bmatrix} \dot{x}_1 = f_1 \\ \dot{x}_2 = f_2 \end{Bmatrix}$.

$\longrightarrow$ This is a suff. condition for non-existence of a periodic sol$^{\underline{n}}$.

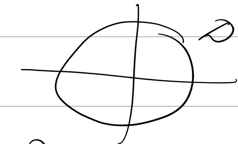
→ sufficient condition for non-existence of a periodic solution.

Ex: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) = 0$

Periodic solⁿ ⇔ Purely imag. eigenvalues

⇔ (1) $a_{11} + a_{12} = 0$, $a_{11}a_{22} - a_{12}a_{21} > 0$

$\nabla f = a_{11} + a_{12} \quad \forall x \in \mathbb{R}^2$



Here $a_{11} + a_{12} \neq 0$ over any

⇒ no periodic solⁿ in D.

Ex: $\dot{x}_1 = x_2 + x_1 x_2^2$

$\dot{x}_2 = -x_1 + x_1^2 x_2$

Eq. pt $(0,0)$ → Linearization

$\dot{x}_1 = x_2$; $\dot{x}_2 = -x_1$

(Remember warning about center!)

$\nabla f = x_1^2 + x_2^2 > 0$ for all $(x_1, x_2) \neq (0,0)$

⇒ No periodic solⁿ anywhere in $\mathbb{R}^2$ (except for $(0,0)$ ← eq. pt)

Ex: (D simply connected important):

$\dot{x}_1 = x_2 + \alpha x_1 (\beta^2 - x_1^2 - x_2^2)$

$\dot{x}_2 = -x_1 + \alpha x_2 (\beta^2 - x_1^2 - x_2^2)$

Let $D = \left\{ (x_1, x_2) : \frac{2\beta^2}{3} < x_1^2 + x_2^2 < 2\beta^2 \right\}$

$\nabla f = 2\alpha\beta^2 - 4\alpha(x_1^2 + x_2^2)$

Check that $\nabla f \geq 0$ everywhere on D.

But we know that D contains periodic solⁿ.

Poincare - Bendixson Thm

Def^n: Let $x(t)$ be a sol^n of $\dot{x} = f(x)$. A pt. $z \in \mathbb{R}^2$ is said to be a limit point of this trajectory if $\exists$ a sequence $\{t_n\}_1^\infty$ in $\mathbb{R}_+$ s.t. $t_n \to \infty$ as $n \to \infty$ and $x(t_n) \to z$ as $n \to \infty$. The set of all limit points of a trajectory $x(t)$ is called the limit set of the trajectory & is denoted by L.

As time progresses, trajectory passes arbitrarily close to z infinitely many times.

Thm: Let $S = \{x(t), t \geq 0\}$ denote a trajectory in $\mathbb{R}^2$ of $\dot{x} = f(x)$, & let L be its limit set. If L is contained in a closed bdd. region $M \subset \mathbb{R}^2$ & if M contains no. eq. pt of $\dot{x} = f(x)$, then either

(i) S is a periodic sol^n of $\dot{x} = f(x)$.

(ii) L is a " " " " .

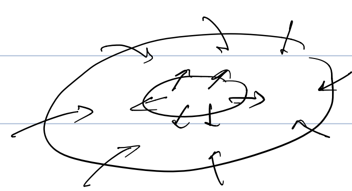
M has to contain **all** the limit pts.

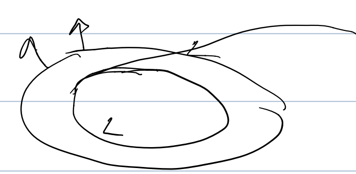
Q. How to check? — But M is closed.
So if $\exists\, t_0 < \infty$ s.t. $x(t) \in M \;\forall\, t \geq t_0$ then L is contained in M.

How to use P-B thm?

→ Find a region M s.t. (a) M contains no eq. pts
(b) some trajectory is eventually confined in M.

<u>Sufficient condition</u>: The vector field should point inward into M along the entire boundary of M.
→ Then any trajectory originating within M must remain in M.



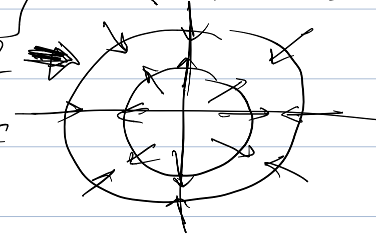


Not necessary

<u>Ex:</u>

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + x_1(1 - x_1^2 - x_2^2) \\ \dot{x}_2 &= -x_1 + x_2(1 - x_1^2 - x_2^2) \end{aligned}\right\} \quad M = \{(x_1, x_2) : 0.9 \leq x_1^2 + x_2^2 \leq 1\}$$

M contains no eq. pt. & ⇒

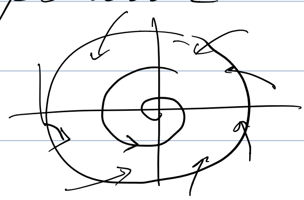


Hence M contains a periodic solution

Ex: (Eq. pt. exclusion is imp.):

$\dot{x}_1 = -x_1 + x_2$
$\dot{x}_2 = -x_1 - x_2$

Take M to be unit disk. Vector field pt. inward. So all trajectorie starting from M remain in M. But we know the origin is a stable focus. $\Rightarrow$ No periodic sol$^{\underline{n}}$.



An alternative statement of P-B thm

Consider $\dot{x} = f(x)$ in $\mathbb{R}^2$. Assume S is bdd & its limit set L contains no eq. pts. Then L is a periodic solution. If $\underline{S \neq L}$ the periodic orbit is called a limit cycle.

In some books limit cycles must have at least one more trajectory converging onto it.