Uniqueness & Existence of Sol^{n}s

Banach Space:

Def^n: A normed linear space is an ordered pair $(X, \|\cdot\|)$ where X is a linear vector space & $\|\cdot\| : X \to \mathbb{R}$ is a real valued function defined on X s.t. the following axioms hold:

1) $\|x\| \geq 0$, $\forall x \in X$; $\|x\| = 0$ iff $x = 0_X$
2) $\|\alpha x\| = |\alpha| . \|x\|$ $\forall x \in X$, $\forall \alpha \in \mathbb{R}$ or $\mathbb{C}$
3) $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in X$

Examples: 1) $X = \mathbb{R}^n$, $\|\cdot\|_\infty : \mathbb{R}^n \to \mathbb{R}$ defined as $\|x\|_\infty = \max\limits_{1 \leq i \leq n} |x_i|$] X - finite dim

2) Let $C[a,b]$ be the set of all continuous functions mapping $[a,b]$ into $\mathbb{R}$.
Let $X = C[a,b]$ & $\|x\|_C = \max\limits_{t \in [a,b]} |x(t)|$

Def^n: A sequence $\{x_i\}$ in a normed linear space $(X, \|\cdot\|)$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, $\exists N = N(\varepsilon)$ s.t. $\|x_i - x_j\| < \varepsilon$ for $i, j > N$.

The set of rational nos (with Euclidean norm) is not complete. e.g. $(1, 1.4, 1.41, 1.414, \cdots)$ converges to $\sqrt{2}$, but does not converge in $\mathbb{Q}$.

Defn: A normed linear space $(X, \|\cdot\|)$ is said to be a complete normed linear space or a Banach space if every Cauchy seq. in $(X, \|\cdot\|)$ converges to an element in X.

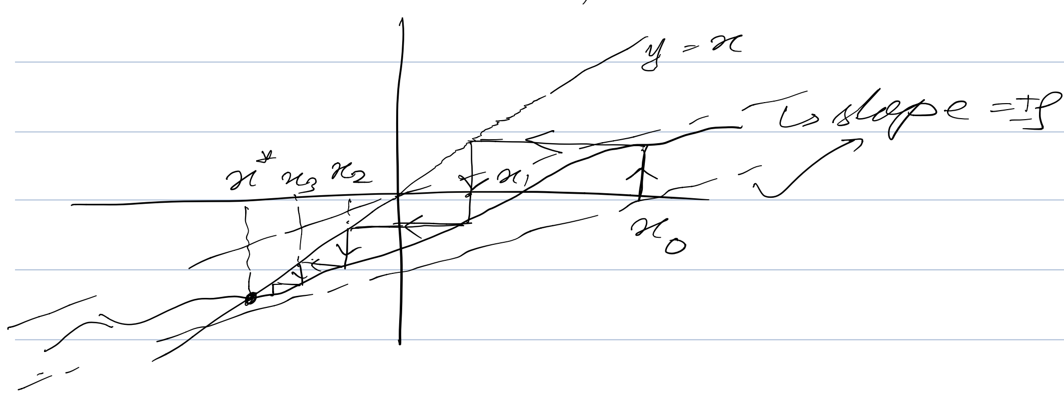
Ex: $C[a,b]$ with $\|x\|_\infty = \max_{t \in [a,b]} |x(t)|$ is a Banach space.

Defn: Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed linear spaces & let $f : X \to Y$. Then f is said to be continuous at $x_0 \in X$ if for every $\varepsilon > 0$ $\exists$ $\delta = \delta(\varepsilon, x_0)$ s.t. $\|f(x) - f(x_0)\|_Y < \varepsilon$ whenever $\|x - x_0\|_X < \delta$.

Contraction Mapping theorem

Thm: Let $(X, \|\cdot\|)$ be a Banach space and let $T: X \to X$. Let $\exists\, \rho < 1$ s.t.
$\|Tx - Ty\| \le \rho \|x - y\| \quad \forall x, y \in X$
Then $\exists$ exactly one $x^* \in X$ s.t. $Tx^* = x^*$
Moreover for each $x_0 \in X$, the seq. $\{x_n\}$ defined by $x_{n+1} = Tx_n$ converges to x^*.
($x^* \leftarrow$ fixed pt. of the map T.)

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be continuously diff. & let $\sup_{x \in \mathbb{R}} |f'(x)| := \rho < 1$



Local Existence & Uniqueness

① $\dot{x} = f(t, x(t))$, $t \ge 0$, $x(0) = x_0$; $x(t) \in \mathbb{R}^n$
$f: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$. If $x(t)$ is a sol^n over $[0, T]$
Clearly, ② $x(t) = x_0 + \int_0^t f(\tau, x(\tau))\, d\tau$, $t \in [0, T]$

① and ② are equivalent. i.e. every solution of ① is also a solⁿ of ② & vice versa.

<u>Thm</u>: Let the function f be continuous in t & x and satisfies: $\exists$ finite constants T, r, h and K such that

$$\|f(t,x) - f(t,y)\| \leq K\|x-y\| \quad \forall x,y \in B \quad \forall t \in [0,T]$$

$$\|f(t,x_0)\| \leq h \quad \forall t \in [0,T] \quad \text{where}$$

$$B = \{x \in \mathbb{R}^n : \|x - x_0\| \leq r\}$$

Then ① has exactly one solution over $[0,\delta]$ for δ sufficiently small.

<u>Sketch of Proof</u>: Notation:

$$\|x(t)\|_C = \max_{t\in[a,b]} \|x(t)\| \qquad x(t) \in C^n[0,\delta]$$

$\|x(t)\| \rightarrow$ normal Euclidean norm in $\mathbb{R}^n$

$$x_0(t) := x_0 \quad \forall t \in [a,b]$$

Let $S = \{x(\cdot) \in C^n[0,\delta] : \|x(\cdot) - x_0(\cdot)\|_C \leq r\}$

Let P be the map: $C^n[0,\delta] \rightarrow C^n[0,\delta]$

$$(Px)(t) = x_0 + \int_0^t f(\tau, x(\tau))\,d\tau \quad \forall t \in [0, \delta]$$

$x(t)$ is a sol^n of (1) iff $(Px)(\cdot) = x(\cdot)$
i.e. $x(\cdot)$ is a fixed pt. of P.

$$\|(Px)(t) - (Py)(t)\| = \left\|\int_0^t \{f(\tau, x(\tau)) - f(\tau, y(\tau))\}\,d\tau\right\|$$

$$\leq \int_0^t \|f(\tau, x(\tau)) - f(\tau, y(\tau))\|\,d\tau$$

$$\leq \int_0^t k\|x(\tau) - y(\tau)\|\,d\tau$$

$$\leq kt\|x(\cdot) - y(\cdot)\|_c \leq \rho\|x(\cdot) - y(\cdot)\|_c$$

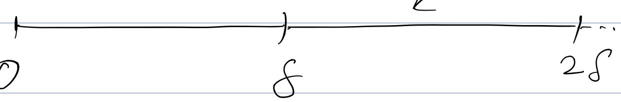
$$\Rightarrow \|(Px)(\cdot) - (Py)(\cdot)\|_c \leq \rho\|x(\cdot) - y(\cdot)\|_c$$
$\hookrightarrow$ why?

So contraction $\Longrightarrow$ $\exists$ unique fixed pt.

Many loose ends in the proof $\rightarrow$ will not be done!

Corollary: For $\dot{x} = f(t, x)$ (*), in some neighbourhood of $(0, x_0)$ the function $f(t, x)$ is continuously differentiable. Then (*) has exactly one solution over $[0, \delta]$ provided δ is suff. small.

It might be possible to extend local solutions indefinitely



could repeat same arguments starting at δ.

Corollary: Consider $\dot{x} = f(t,x)$ $(*)$, & let $f(t,x)$ be continuously differentiable everywhere. Then $\exists$ a unique no. $\delta_{max} = \delta_{max}(x_0)$, which could equal infinity, s.t. $(*)$ has a unique solution over $[0, \delta_{max})$ and over no larger interval. If δ_{max} is finite, then $\|x(t)\| \to \infty$ as $t \to \delta_{max}$.

IMP: [half open interval]

Ex: $\dot{x}(t) = 1 + x^2$ $\quad x(0) = 0$ $\longrightarrow \delta_{max} = \pi/2$

$x(t) = \tan t$

As $t \to \pi/2$, $x(t) \to \infty$. [Finite escape time]

Global Existence & Uniqueness (without proofs)

Th: Suppose for each $T \in [0, \infty)$ $\exists$ $k_T, h_T < \infty$ s.t.

$$\|f(t,x) - f(t,y)\| \leq k_T \|x - y\| \quad \forall x, y \in \mathbb{R}^n, \ \forall t \in [0,T]$$

$$\|f(t,x_0)\| \leq h_T \quad \forall t \in [0,T]$$

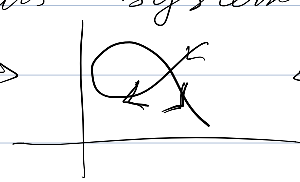
1) Then $(*)$ has exactly one solution over $[0, \infty)$

2) For each $z \in \mathbb{R}^n$ & $T \in [0,\infty)$ $\exists$ exactly one element $z_0 \in \mathbb{R}^n$ s.t. the unique sol$^{\underline{n}}$ over $[0,T]$ of $\dot{x} = f(t, x(t))$, $x(0) = z_0$ satisfies $x(T) = z$.

3) Let $T \in [0,\infty)$. For each $\varepsilon > 0$, $\exists\, \delta(\varepsilon, T) > 0$, s.t. $\|x_0 - y_0\| < \delta(\varepsilon, T) \Rightarrow \|x(\cdot) - y(\cdot)\|_C \leq \varepsilon$.
$[\dot{x}(t) = f(t,x),\ x(0) = x_0\ ;\ \dot{y}(t) = f(t,y),\ y(0) = y_0]$ — Cont. dep. on initial conditio[n]

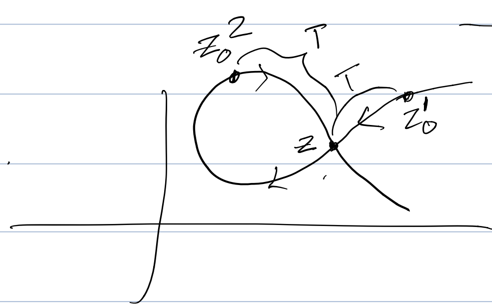
<u>To clarify</u>: For <u>autonomous</u> systems, a trajectory which passes through at least one pt. that is not an equilibrium pt. cannot cross itself unless it is a closed curve. In this case it is a periodic solution.

For non-autonomous system, trajectories can intersect →



← such curves are possible.

Q.



← Why does this fig. not contradict ⟨2⟩ of Thm above?

Important Disclaimers

\# The conditions are sufficient & not necessary

e.g. $\dot{x}(t) = -x^2 \quad x(0) = 1$

$x(t) = \frac{1}{t+1}$ ← unique sol^n over $[0, \infty)$

But $f(x) = -x^2$ is not globally Lipschitz cont.

\# Condition (3) does not guarantee continuous dependence on initial condition of the sol^n over $[0, \infty)$ ⟺ Otherwise Lyapunov theory would be redundant. !!