Lyapunov Stability

Std assumptions: $\dot{x} = f(t, x(t)) \quad t \geq 0 \; ; \; x(t) \in \mathbb{R}^n$ (*)

1) $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous

2) Unique solⁿ corr. to each initial condition.

3) Notation: $s(t, t_0, x_0) \leftarrow$ solⁿ of (*) corr. to $x(t_0) = x_0$, evaluated at t.

   a) $s(t_0, t_0, x_0) = x_0$ $\quad \forall t \geq t_1 \geq t_0 \geq 0$

   b) $s(t, t_1, s(t_1, t_0, x_0)) = s(t, t_0, x_0)$

4) If x_0 is an eq. pt. then $s(t, t_0, x_0) = x_0 \quad \forall t \geq t_0 \geq 0$

5) '0' is the eq. pt. $\Leftrightarrow f(t, 0) = 0 \quad \forall t \geq 0$

   or $s(t, t_0, 0) = 0 \quad \forall t \geq t_0$

Defⁿ: The eq. '0' is stable if for each $\varepsilon > 0$ & each $t_0 \in \mathbb{R}_+$, $\exists$ a $\delta = \delta(\varepsilon, t_0)$ s.t.

$$\|x_0\| < \delta(\varepsilon, t_0) \Rightarrow \|s(t, t_0, x_0)\| < \varepsilon \quad \forall t \geq t_0$$

\# It is uniformly stable if for each $\varepsilon > 0$, $\exists \; \delta = \delta(\varepsilon)$ s.t. $\|x_0\| < \delta(\varepsilon)$, $t_0 \geq 0$

$$\Rightarrow \|s(t, t_0, x_0)\| < \varepsilon, \; \forall t \geq t_0$$

\# The eq. is unstable if it is not stable.

Stability $\equiv$ Continuity

Note:

1) Very close to global existence condition studied earlier. → continuity from $C^n[t_0, T] \to C^n[t_0, T]$ as long as $T < \infty$.

Instead consider $C^n[t_0, \infty)$ & let $BC^n[t_0, \infty) \subset C^n[t_0, \infty)$ denote the set of bdd. continuous fn.. Define $\|x(\cdot)\|_s = \sup_{t \in [t_0, \infty)} \|x(t)\|$

Then $BC^n[t_0, \infty)$ is a Banach space.

<u>Stability</u> $\Leftrightarrow$ For each $t_0 \geq 0$, $\exists\, d(t_0)$ s.t. $s(\cdot, t_0, x_0) \in BC^n[t_0, \infty)$ whenever $x_0 \in B_{d(t_0)}$ where $B_d = \{x \in \mathbb{R}^n : \|x\| < d\}$

$\Leftrightarrow$ The map $s(\cdot, t_0, x_0) : \underset{B_{d(t_0)}}{\underset{\cap}{x_0}} \longmapsto \underset{BC^n[t_0, \infty)}{\underset{\cap}{s(t, t_0, x_0)}}$

is continuous at $x_0 = 0$ $\forall\, t_0 \geq 0$.

Note: 1) Stability $\Leftrightarrow$ continuous dep. on initial condition over infinite interval.

2) For autonomous systems, stability $\Leftrightarrow$ uniform stability

Ex: $\dot{x}(t) = (6t\sin t - 2t)\,x(t) \qquad t \geq t_0$

Sol^n: $x(t) = x(t_0)\exp\{6\sin t - 6t\cos t - t^2$
$\qquad - 6\sin t_0 + 6t_0\cos t_0 + t_0^2\}$

$\left|\dfrac{x(t)}{x(t_0)}\right| = \exp\{- - - - - -\}$

Check: $c(t_0) = \sup\limits_{t \geq t_0} \exp\{- - - -\} < \infty$

Thus, $\|x(t_0)\| < \dfrac{\varepsilon}{c(t_0)} \Rightarrow \|x(t,t_0)\| < \varepsilon \quad \forall t > t_0$

So '0' is stable eq. pt.

But, $t_0 = 2n\pi$, $\Rightarrow x[(2n+1)\pi] = x(2n\pi)\exp\{(4n+1)(6-\pi)\pi\}$

Clearly $c(t_0)$ is unbdd in t_0. So δ cannot be ind. of t_0. So not U.S.

Q. What are the types of instability?

Our fav. example:  Origin is not stable.

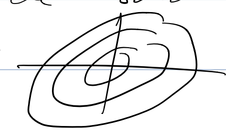
However this does not diverge to infinity.

Def^n: The eq. 0 is attractive if for each $t_0 \in \mathbb{R}_+$ $\exists\ \eta(t_0) > 0$ s.t. $\|x_0\| < \eta(t_0) \Rightarrow s(t_0+t, t_0, x_0) \to 0$ as $t \to \infty$.

The eq. 0 is uniformly attractive if $\exists\ \eta > 0$ s.t.

$\|x_0\| < \eta,\ t_0 \geq 0 \Rightarrow s(t_0+t, t_0, x_0) \to 0$ as $t \to \infty$
uniformly in x_0, t_0.

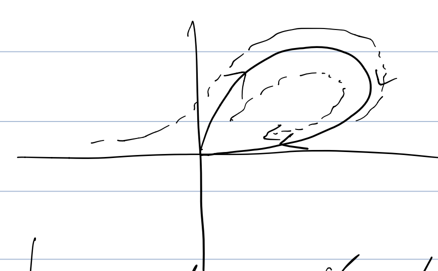
$\Leftrightarrow$ For each $\varepsilon > 0$, $\exists\ T = T(\varepsilon)$ s.t.
$\|x_0\| < \eta,\ t_0 \geq 0 \Rightarrow \|s(t_0+t, t_0, x_0)\| < \varepsilon\ \forall t \geq T(\varepsilon)$

\# For an eq. to be attractive it must be isolated.
\# Stable but not attractive : Center : 

Def^{n}: The eq. 0 is asymptotically stable if it is stable & attractive. It is "uniformly asymptotically stable" (u.a.s) if it is uniformly stable & uniformly attractive.

Q. Are these definitions req. → can there be an eq. which is attractive but not stable?

ANS: YES!



$$\dot{x}_1 = \frac{x_1^2(x_2 - x_1) + x_2^5}{(x_1^2 + x_2^2)\left[1 + (x_1^2 + x_2^2)\right]}$$

$$\dot{x}_2 = \frac{x_2^2(x_2 - 2x_1)}{(x_1^2 + x_2^2)\left[1 + (x_1^2 + x_2^2)\right]}$$

→ Not uniformly attractive
closer x_0 is to origin slower is the conv.

\# For autonomous system : stable + attractive $\Leftrightarrow$ uniformly attractive

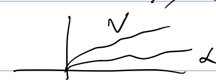
Is it possible to have uni. attr. but unstable?

→ Not known

Lyapunov's Direct Method / Second Method

Def^{n}: A function $V: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is said to be lpdf (locally positive definite f^{n}) if (i) it is continuous (ii) $V(t,0)=0 \; \forall t \geq 0$ (iii) $\exists r>0$ & a class K function α s.t.

$$\alpha(\|x\|) \leq V(t,x) \quad \forall t \geq 0, \; \forall x \in B_r$$



Def^{n}: A f^{n} $\phi: \mathbb{R}_+ \to \mathbb{R}_+$ is of class K if it continuous, strictly increasing & $\phi(0)=0$.

Easy to check conditions for lpdf: $W: \mathbb{R}^n \to \mathbb{R}$ (continuous) is lpdf iff (i) $W(0)=0$ AND (ii) $\exists r>0$ s.t. $W(x)>0 \quad \forall x \in B_r - \{0\}$

FACT:

$V: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ is lpdf iff (i) $V(t,0)=0 \; \forall t$ + (ii) $\exists$ lpdf $W: \mathbb{R}^n \to \mathbb{R}$ & $r>0$ s.t. $V(t,x) \geq W(x) \; \forall x \in B_r \; \forall t \geq 0$

Thm: The eq. 0 of $\dot{x}=f(t,x)$ is stable if $\exists$ a continuously diff (C^1) lpdf (Lyapunov f^{n}) $V: \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ & a constant $r>0$ s.t.

$$\dot{V}(t,x) \leq 0 \quad \forall t \geq t_0 \quad \forall x \in B_r$$

where $\dot{V}$ is evaluated along the trajectories of $\dot{x}=f(t,x)$.

[——] [1]

Note: $\dot{V}(t,x) = \dfrac{\partial V(t,x)}{\partial t} + \left[\dfrac{\partial V}{\partial x}\right]^T \dot{x} = \dfrac{\partial V}{\partial t} + \dfrac{\partial V}{\partial x}^T f(t,x)$

Proof: Since V is lpdf, $\exists$ a class K fn α & a constant $s>0$ s.t.

$$\alpha(\|x\|) \leq V(t,x) \quad \forall t \geq 0, \forall x \in B_s$$

Let $\varepsilon > 0$, $t_0 \geq 0$. Let $\varepsilon_1 = \min\{\varepsilon, r, s\}$

& pick $\delta > 0$ s.t. $\sup\limits_{\|x\| \leq \delta} V(t_0, x) =: \beta(t_0, \delta) < \alpha(\varepsilon_1)$

[Q. Can such δ be found always?]

Then $\|x_0\| < \delta \Rightarrow V(t_0, x_0) \leq \beta(t_0, \delta) < \alpha(\varepsilon_1)$

But recall that $\dot{V}(t,x) \leq 0 \quad \forall t \geq t_0 \quad \forall \|x\| < \delta$

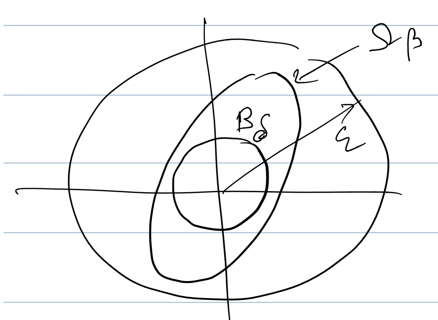
$\Rightarrow V(t, s(t,t_0,x_0)) \leq V(t_0, x_0) < \alpha(\varepsilon_1) \quad \forall t \geq t_0$ [Wrong] [check vidyasagar]

But $V[t, s(t,t_0,x_0)] \geq \alpha[\|s(t,t_0,x_0)\|]$

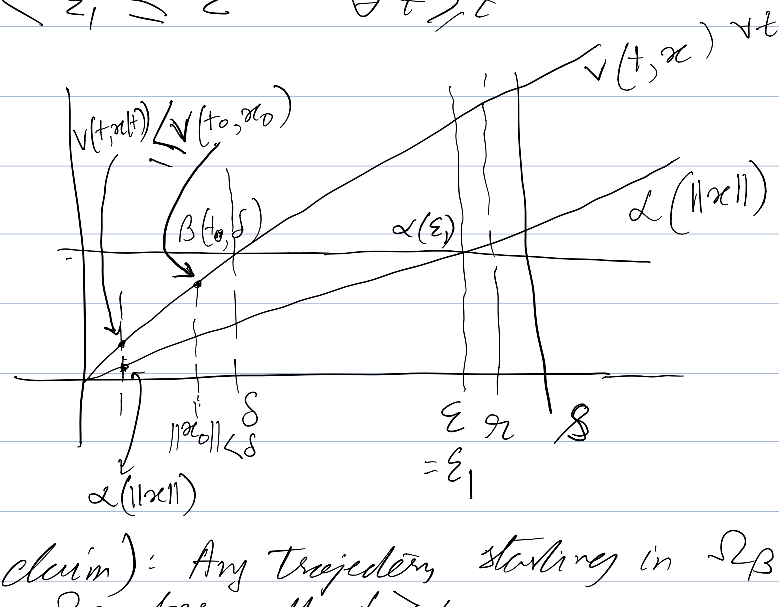
Hence $\alpha[\|s(t,t_0,x_0)\|] < \alpha(\varepsilon_1) \quad \forall t \geq t_0$

Since $\alpha(\cdot)$ is strictly increasing:

$$\|s(t,t_0,x_0)\| < \varepsilon_1 \leq \varepsilon \quad \forall t \geq t$$



$\Omega_\beta = \{x \mid V(x) \leq \beta\}$



Implicit (Unproven claim): Any trajectory starting in Ω_β at $t=t_0$, stays in Ω_β for all $t \geq t_0$.

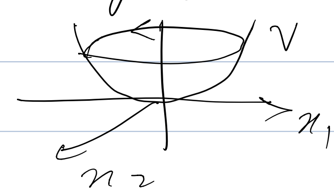
Thm: Consider $\dot{x} = f(x)$. Let $D \subset \mathbb{R}^n$ be a set s.t. $0 \in D$. Let $V : D \to \mathbb{R}$ be a C^1 f^{n} s.t. $V(0) = 0$ & $V(x) > 0$ in $D - \{0\}$

$$\dot{V}(x) \leq 0 \text{ in } D$$

Then x is stable. Moreover if $\dot{V}(x) < 0$ in $D - \{0\}$ the $x = 0$ is asymptotically stable.

\# For proving A.S. more work is needed. It is enough to show that $V(x(t)) \to 0$ as $t \to \infty$.

$\to$ As seen from figure above, this is obvious in 1-dimension $\to$ more complicated for higher dim.



Region of Attraction : $\Omega_c = \{x \mid V(x) \leq c\}$

Ex: $\ddot{\theta} + \sin\theta = 0 \to \dot{x}_1 = x_2 ; \dot{x}_2 = -\sin x_1$

Lyapunov f^{n} candidate : Total energy

$$V(x_1, x_2) = (1 - \cos x_1) + \frac{1}{2} x_2^2 \Rightarrow C^1, \text{ lpdf}$$

Easy to check : $V(x_1, x_2) > 0$; $V(0,0) = 0$
(lpdf)

$$\dot{V}(x_1, x_2) = \sin x_1 \cdot \dot{x}_1 + x_2 \dot{x}_2 = \sin x_1 (x_2) + x_2(-\sin x_1)$$
$$= 0 \Rightarrow V \text{ is a } \underline{\text{Lyapunov } f^{n}}.$$

Ex: $\dot{x}_1 = x_2$

$\dot{x}_2 = -p(t)\,x_2 - e^{-t}x_1$

Find Conditions on $p(t)$ to ensure stability.

Let $V(t, x_1, x_2) = x_1^2 + e^t x_2^2 \longrightarrow C^1$ Lyapunov fn candidate

$V(t, x_1, x_2) \geq W(x_1, x_2) := x_1^2 + x_2^2$

$\dot{V}(t, x_1, x_2) = e^t x_2^2 + 2x_1(x_2) + 2e^t x_2\left[-p(t)\,x_2 - e^{-t}x_1\right]$

$= e^t x_2^2\left[-2p(t) + 1\right]$

Here $\dot{V} \leq 0 \Rightarrow \underbrace{p(t) \geq 1/2}$ $\forall t \geq 0$.

very suff. — completely diff. condition possible for diff. Lyapunov fn.

Ex: $\dot{x}_1 = x_2$

$\dot{x}_2 = -a\sin x_1 - bx_2$

$V(x) = a(1-\cos x_1) + \frac{1}{2}(x_2^2)$

$\dot{V}(x) = a\dot{x}_1 \sin x_1 + x_2\dot{x}_2 = -bx_2^2$

Here $\dot{V}(x) \leq 0 \Rightarrow$ '0' is stable

But recall the phase portrait

So it looks A.S. (Lasalle's Inv Thm)

Try with diff. Lyapunov fn candidate

$V(x) = \frac{1}{2}x^T P x + a(1-\cos x_1)$, $P > 0$

$\dot{V}(x) = -\frac{1}{2}ab\,x_1 \sin x_1 - \frac{1}{2}bx_2^2$ $\quad P = \begin{bmatrix} bp_{12} & b/2 \\ b/2 & 1 \end{bmatrix}$

For $D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$, $V(x) > 0$ & $\dot{V}(x) < 0$ for $\forall x \in D$. $\Rightarrow$ '0' is asymptotically stable.

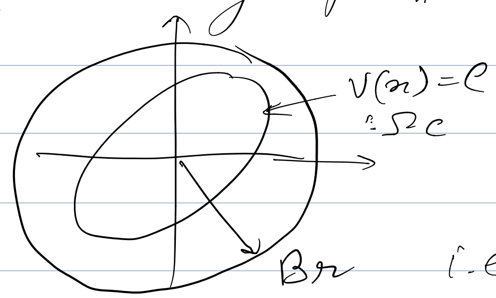
Def^{n}: An eq. pt. '0' of $\dot{x} = f(x)$ is globally asymptotically stable if $\forall x(0) \in \mathbb{R}^n$, $x(t, x_0) \to 0$ as $t \to \infty$.

Thm: Let $x=0$ be an eq. pt. of $\dot{x} = f(x)$. Let $V: \mathbb{R}^n \to \mathbb{R}$ be a C^1 f^{n} s.t.

1) $V(0) = 0$ & $V(x) > 0 \quad \forall x \neq 0$
2) $\|x\| \to \infty \Rightarrow V(x) \to \infty$ (Radially unbdd)
3) $\dot{V}(x) < 0, \quad \forall x \neq 0$

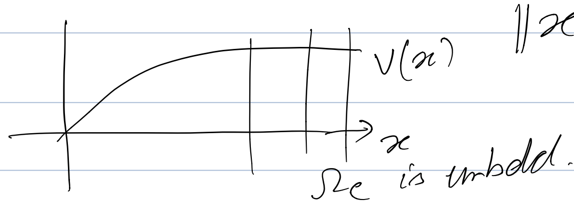
Then $x = 0$ is globally asymptotically stable.

Sketch: For any $p \in \mathbb{R}^n$, let $c = V(p)$.



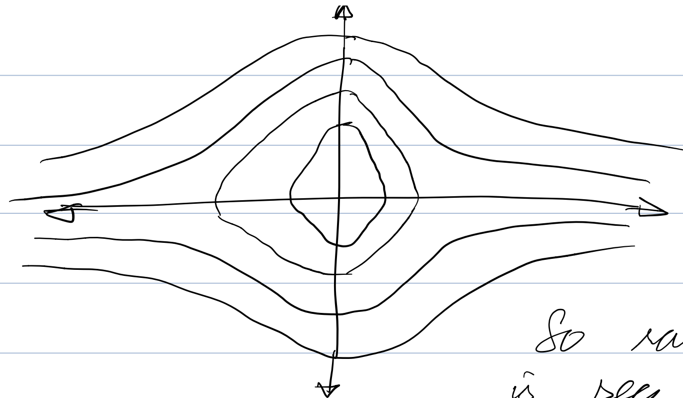
Condition (2) implies we can draw B_r s.t. $\Omega_c \subset B_r$

i.e. $\forall c$, $\exists r > 0$ s.t. $\|x\| > r \Rightarrow V(x) > c$
$\Rightarrow \Omega_c$ is bdd.



Q. Why is it required that Ω_c is bdd?

Consider $V(x) = \dfrac{x_1^2}{1+x_1^2} + x_2^2$. Then $\Omega_c = \{x \mid V(x) \leq c\}$ is bdd. for $c < 1$.



It is possible to have $\dot{V}(x) < 0$ but $\|x\| \not\to 0$.

So radially unbdd. condition is req.

<u>Lasalle's Invariance Thm:</u> (Krasovskii–Lasalle's Thm)

<u>Thm:</u> Let $\Omega \subset D$ be a compact set that is positively invariant w.r.t. $\dot{x} = f(x)$. Let $V: D \to \mathbb{R}$ be a C^1 fn s.t. $\dot{V}(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.

<u>Defn:</u> A set M is (+vely) invariant set w.r.t. $\dot{x} = f(x)$ if $x(0) \in M \Rightarrow x(t) \in M \quad \forall t \in \mathbb{R}$

$(\forall t \in \mathbb{R}_+)$

<u>Note:</u> 1) By this thm Regions of attraction are not restricted to $\Omega_c : \{x \mid V(x) \leq c\} \Rightarrow$ any compact +vely inv. set

2) Can be used for eq. sets rather than

isolated eq. pt.

3) $V(x)$ need not be positive definite.

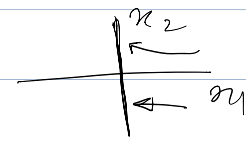
Ex: $\dot{y} = ay + u$, $u = ky$, $\dot{k} = \gamma y^2$ $\gamma > 0$

Taking $x_1 = y$, $x_2 = k \Rightarrow \dot{x}_1 = -(x_2 - a)x_1$
$\dot{x}_2 = \gamma x_1^2$

The line $x_1 = 0$ is an eq. set.

Consider $V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2 \quad (b > a)$

$\dot{V}(x) = -x_1^2(b-a) \leq 0.$



Since $V(x)$ is radially unbdd,
$\Omega_c = \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$ is compact
& truly invariant..

$E = \{x \in \Omega_c \mid x_1 = 0\}$ ← E is invariant.

Take $M \equiv E$. Hence by Lasalle's Thm,
every trajectory starting in Ω_c approaches
E as $t \to \infty$. i.e. $x_1(t) \to 0$ as $t \to \infty$

Also this conclusion is global since
c can be arbitrarily large.

<u>Linear Systems</u>: (Autonomous) $\dot{x} = Ax$; $x(t) = e^{At}x_0$

<u>Thm</u>: The eq. pt. $x = 0$ of $\dot{x} = Ax$ is stable <u>iff</u>
all eig. of A satisfy $\mathrm{Re}\,\lambda_i \leq 0$ & for

each eig. with $Re\,\lambda_i = 0$ & alg. mult $q_i \geq 2$
$\text{rank}(A - \lambda_i I) = n - q_i$.

The eq. pt. $x=0$ is Globally asymp. stable iff all eig. of A satisfy $Re\,\lambda_i < 0$.

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \rightarrow \text{unstable.}$$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \text{stable}$$

Proof: Exercise.

<u>Thm:</u> Given $A \in \mathbb{R}^{n \times n}$, the T.F.A.E:

(1) A is Hurwitz ($Re\,\lambda_i < 0 \;\; \forall i = 1, \ldots, n$)

(2) $\exists$ some p.d. matrix $Q \in \mathbb{R}^{n\times n}$ s.t.
$A^T P + PA = -Q$ has a ~~corr.~~ unique p.d. solution for P.

(3) For every p.d. matrix $Q \in \mathbb{R}^{n \times n}$, $A^T P + PA = -Q$ has a unique p.d. solution for P.

<u>Proof:</u> (3) $\Rightarrow$ (2) obvious

(2) $\Rightarrow$ (1). Let (2) be true for some Q

The consider Lyap. f^n candidate $V(x) = x^T P x$

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T A^T P x + x^T P A x$$

$$= x^T (A^T P + PA) x = - x^T Q x < 0$$

So by Lyap. thm $\Rightarrow \dot{x} = Ax$ is GAS & by the Thm above $\Rightarrow A$ is Hurwitz.

(1) $\Rightarrow$ (3) Let A be Hurwitz & let $Q \in \mathbb{R}^{n \times n}$ p.d.

Define $M = \int_0^\infty e^{A^T t} Q e^{At} dt$

$$A^T M + MA = \int_0^\infty \left[A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A \right] dt$$

$$= \int_0^\infty \frac{d}{dt} \left[e^{A^T t} Q e^{At} \right] dt = \left[e^{A^T t} Q e^{At} \right]_0^\infty$$

$$= -Q.$$

So some solution M exists. Remaining questions (1) Is $M > 0$? (2) Is M unique

Claim: $\underline{M > 0}$.

Let $M \leq 0$, then $\exists\, x \neq 0$ s.t. $x^T M x = 0$

But $x^T M x = \int_0^\infty x^T e^{A^T t} Q e^{At} x \, dt = 0$

$\Rightarrow x^T e^{A^T t} = 0 \quad \forall t \Rightarrow x = 0$ (Contradiction)

Claim: $\underline{M = P \text{ (ie } P \text{ is unique)}}$

Let $\exists\ M \neq P$ then

$$\left.\begin{aligned} MA + A^T M &= -Q \\ PA + A^T P &= -Q \end{aligned}\right\} \quad (M-P)A + A^T(M-P) = 0$$

Pre & post mult. by $e^{A^T t}$ & e^{At}:

$$e^{A^T t} \left[(M-P)A + A^T (M-P) \right] e^{At} = 0$$

$$\frac{d}{dt}\left[e^{A^T t}(M-P)e^{At}\right] = 0 \Rightarrow e^{A^T t}[M-P]\,e^{At} = \text{constant } \forall t.$$

Hence for $t=0$ & for $t=\infty$ $\Big]$ $M-P = \lim_{t\to\infty} e^{A^T t}[M-P]\,e^{At} = 0$

$\Rightarrow M=P.$

Note: 1) Lyap. Eqn. can be used to check whether A is Hurwitz $\to$ Choose $Q=I$ & then solve Lyap Eqn. $\to$ Check unique $P>0$ or not. (No comp. adv.)

2) Lyap. Eqn is of the form $Mx=y$.

Ex: $A=\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}$ $\quad Q=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ $\quad P=\begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}$

$$\left.\begin{matrix}A^TP+PA\\ =Q\end{matrix}\right] \equiv \begin{bmatrix}0 & 2 & 0\\ -1 & -1 & 1\\ 0 & -2 & -2\end{bmatrix}\begin{bmatrix}P_{11}\\ P_{12}\\ P_{22}\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ -1\end{bmatrix}$$

$$P=\begin{bmatrix}1.5 & -0.5\\ -0.5 & 1\end{bmatrix} > 0. \Rightarrow A \text{ is Hurwitz.}$$

Lyapunov's Linearization Method / Indirect Method

Thm: Let $x=0$ be the eq. pt. for $\dot{x}=f(x)$. & $0\in D\in\mathbb{R}^n$. Let $A=\left.\frac{\partial f}{\partial x}\right|_{x=0}$. 1) Then the origin is Asymp. stable if $\operatorname{Re}\lambda_i<0$

for all eigenvalues of A

2) The origin is unstable is $\text{Re}\,\lambda_i > 0$ for one or more eigenvalues of A.

Sketch of 1) Let $f(x) = \underbrace{f(0)}_{=0} + \underbrace{\left.\frac{\partial f}{\partial x}\right|_{x=0}}_{A} x + g(x)$

Exercise: Prove that $\frac{\|g(x)\|_2}{\|x\|_2} \to 0$ as $\|x\|_2 \to 0$

Let A be Hurwitz. Then for any $Q > 0$, solution of Lyap. eq. is $P > 0$. Let $V(x) = x^T P x$ be a Lyap. fn. candidate for $\dot{x} = f(x)$.

$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T P f(x) + f^T(x) P x$

$= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)] P x$

$= x^T (PA + A^T P) x + 2x^T P g(x) = \underbrace{-x^T Q x}_{-ve} + \underbrace{2x^T P g(x)}_{\text{indefinite in general}}$

$=$

But $\dot{V}(x) < 0$ is possible to enforce by choosing $\|x\|_2$ small enough $\to$ Exercise.

Proof of 2: skipped.

Ex: $\dot{x} = ax^3$. $A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^2 = 0$

So eig. on imag. axis. Thm. does not apply

Actually, stable for $a<0$, $a>0 \rightarrow$ unstable
$a=0 \rightarrow$ stable.

Ex: $\dot{x}_1 = x_2$
$\dot{x}_2 = -a\sin x_1 - bx_2$

Eq. pts
$x_1 = 0,\ x_2 = 0$ — ①
$x_1 = \pi,\ x_2 = 0$ — ②

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a\cos x_1 & -b \end{bmatrix}$$

$$\left.\frac{\partial f}{\partial x}\right|_{x=0} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \rightarrow \lambda_{1,2} = -\frac{1}{2}b \pm \frac{1}{2}\sqrt{b^2-4a}$$

For all $a,b>0$, $Re[\lambda_{1,2}]<0$

$\longrightarrow$ asymp. stable.

If $b=0$, $\lambda_{1,2}$ are on imag. axis. Thus we cannot determine stability through linearization. (Actually stable as seen earlier)

$$\left.\frac{\partial f}{\partial x}\right|_{(\pi,0)} = \begin{bmatrix} 0 & 1 \\ a & -b \end{bmatrix} \ ; \ \lambda_{1,2} = -\frac{1}{2}b \pm \sqrt{b^2+4a}$$

For $a>0,\ b\geq 0 \rightarrow$ one eig on RHP.
$\Longrightarrow$ unstable.

Ex: Feedback Stabilization of Non-linear Systems

$\dot{x} = f(x,u)$ find $u = g(x)$ s.t. the eq. pt.

'0' of the closed loop $\dot{x} = f(x, g(x))$ is asymptotically stable. Let $f(0,0) = 0$

Let $A = \left[\frac{\partial f}{\partial x}\right]_{x=0, u=0}$, $B = \left[\frac{\partial f}{\partial u}\right]_{x=0, u=0}$

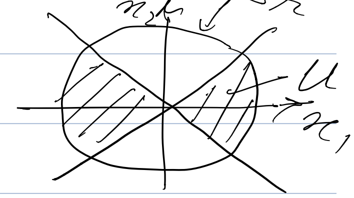
Let (A, B) be controllable $\Rightarrow$ $\exists K$ s.t. $(A-BK)$ have all eig. on LHP.
Let $u(t) = -Kx(t)$. Then $\dot{x} = f(x, -Kx(t))$ has '0' as asym. stable eq. pt.

Proof: Let $h = f(x, -Kx) \Rightarrow \dot{x} = h(x) \leftarrow$ C.L.
$\left[\frac{\partial H}{\partial x}\right]_{x=0} = A - BK \leftarrow$ Hurwitz

Instability Thm (Chetaev's Thm)

1) Let $V : D \to \mathbb{R}$ be C^1 & $0 \in D \subset \mathbb{R}^n$.
$V(0) = 0$ & $\exists x_0$ with arbitrarily small $\|x_0\|$ s.t. $V(x_0) > 0$.

2) Let $r > 0$ be s.t. $B_r \subset D$ & let
$U = \{x \in B_r \mid V(x) > 0\}$
By assumption, $U \subset B_r$
U non-empty; $0 \in \partial U$



Ex: U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$

Thm: Let $x=0$ be eq. pt. for $\dot{x} = f(x)$. Let $V: D \to \mathbb{R}$ be a C^1 fn s.t. $V(0)=0$ & $V(x_0) > 0$ for some x_0 with arb. small $\|x_0\|$. If $\dot{V}(x) > 0$ in U, then x_0 is unstable.

Proof: Let $x_0 \in \text{int}(U)$ & $V(x_0) = a > 0$

Claim: $x(t)$ must leave U.

Since $\dot{V}(x) > 0$ in U, $V(x(t)) \geq a$.

Let $\gamma = \min\{\dot{V}(x) \mid x \in U \text{ \& } V(x) \geq a\}$

$$V(x(t)) = V(x_0) + \int_0^t \dot{V}(x(s))\,ds$$

$$\geq a + \int_0^t \gamma\,ds = a + \gamma t$$

Hence $x(t)$ must leave U since $V(x)$ is bdd. in U. (Why?)

$x(t)$ cannot leave U through the surface $V(x) = 0$ since $V(x(t)) \geq a$. Hence it must leave through $\|x\| = r$.

Converse Thm

Defn: The eq. pt. $x=0$ of $\dot{x} = f(t,x)$ is exponentially stable if $\exists\, c, k, \& \lambda$ s.t.

(*) $\|x(t)\| \leq k\|x(t_0)\| e^{-\lambda(t-t_0)}, \forall \|x(t_0)\| < c$

& globally exp. stable if (*) is satisfied for any initial state $x(t_0)$.

Clearly exp. stable $\Rightarrow$ asymp. stable.

Thm: Let $x=0$ be an eq. pt. of $\dot{x}=f(x)$, where $f(x)$ is C^1 in some neighborhood of $x=0$. Let $A = \frac{\partial f}{\partial x}\Big|_{x=0}$. Then $x=0$ is exponentially stable eq. pt. for $\dot{x}=f(x)$ iff A is Hurwitz.

Ex: $\dot{x} = -x^3$ $\quad \frac{\partial f}{\partial x}\Big|_{x=0} = 0 \rightarrow$ By thm, '0' is not exponentially stable.

But asymp. stable $\rightarrow V(x) = x^4$

$\dot{V}(x) = 4x^3\dot{x} = -4x^6 < 0 \Rightarrow$ asymp. stable.

More non-local thm (over all region of attraction)

Thm: Let $x=0$ be an asymp. stable eq. pt. for $\dot{x}=f(x)$. where $f: D \rightarrow \mathbb{R}^n$ is locally Lipschitz & $0 \in D \subset \mathbb{R}^n$. Let

$R_A \subset D$ be the region of attraction of $x=0$. Then there is a smooth p.d. f$^{\underline{n}}$ $V(x)$ & a continuous p.d. f$^{\underline{n}}$ $W(x)$, both defined over all $x \in R_A$, s.t.

$$V(x) \to \infty \quad \text{as} \quad x \to \partial R_A$$

$$\frac{\partial V}{\partial x} f(x) \leq -W(x), \quad \forall x \in R_A$$

& for any $c>0$, $\{x \mid V(x) \leq c\}$ is a compact subset of R_A. When $R_A = \mathbb{R}^n$, $V(x)$ is radially unbdd.