Input - Output Stability

Consider a system $y = Hu$ | $u : [0,\infty) \to \mathbb{R}^m$
↳ piecewise continuous

L_∞^m : $\|u\|_{L_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$ ↳ finite dim norm (all are equivalent)

L_2^m : $\|u\|_{L_2} = \sqrt{\int_0^\infty u^T(t) u(t)\, dt} < \infty$

L_p^m : $(1 \leq p < \infty)$: $\|u\|_{L_p} = \left(\int_0^\infty \|u(t)\|^p dt \right)^{1/p} < \infty$

Exercise:
Check: Are these norms? Are these normed linear spaces? Are they Banach spaces?

Fundamental problem: Output of unstable system with $u(t) \in L_\infty^m$ may not be in L_∞^m.

Extended spaces: $L_e^m = \{ u \mid u_\tau \in L^m, \forall \tau \in [0,\infty) \}$

with $u_\tau(t) = \begin{cases} u(t) & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases}$

Exam: $u(t) = t \Rightarrow u(t) \in L_{\infty e}$ but $u(t) \notin L_\infty$

$H : L_e^m \to L_e^q$ → this defn. takes care of possible 'unstable' systems.

Exercise: Is L_e^m a normed linear space?
Does $L_e^m \supset L^m$

Causality: A mapping $H : L_e^m \to L_e^q$ is said to be causal if the value of the output $(Hu)(t)$ at t depends on values of $u(t)$ upto t. $\Leftrightarrow (Hu)_\tau = (Hu_\tau)_\tau$

Causality is automatic in state space system $\rightarrow$ not so for I/O maps.

Def^{n}: A mapping $H: L_e^m \rightarrow L_e^q$ is L-stable if there exist a class-K f^n α on $[0,\infty)$ & $\beta \geq 0$ s.t. $\|(Hu)_\tau\|_L \leq \alpha(\|u_\tau\|_L) + \beta$ — (*) $\forall u \in L_e^m$ & $\tau \in [0,\infty)$. It is finite gain L-stable if $\exists$ $\gamma, \beta \geq 0$ s.t. $\|(Hu)_\tau\|_L \leq \gamma \|u_\tau\|_L + \beta$ $\forall u \in L_e^m$ & $\tau \in [0,\infty)$ — (**)

$\beta \rightarrow$ bias term - H.0 might not be zero.

Smallest γ (if exists) s.t. $\exists \beta$ satisfying (**) is called the gain of the system.

Example: $y(t) = \int_0^t h(t-\sigma)u(\sigma)d\sigma$ | $h(t) = 0 \ \forall t < 0$

Let $h(t) \in L_{1e}$. Now if $u \in L_{\infty e}$ then

$$|y(t)| \leq \int_0^t |h(t-\sigma)|\,|u(\sigma)|\,d\sigma$$

$$\leq \int_0^t |h(t-\sigma)|\,d\sigma \cdot \sup_{0\leq\sigma\leq t} u(\sigma) \leq \int_0^t |h(s)|\,ds \cdot \|u_\tau\|_{L_\infty}$$

(for any $\tau > t$)

i.e. $\|y_\tau\|_{L_\infty} \leq \|h_\tau\|_{L_1} \|u_\tau\|_{L_\infty} \quad \forall \tau \in [0,\infty)$

Q. Does this prove fg-L_∞ stable? No. - since $\|h_\tau\|_1$ may not be ind. of τ e.g $h(t) = e^t$. Then clearly $\|h_\tau\|_{L_1} = e^\tau - 1 \Rightarrow h \in L_{1e}$ but $h \notin L_1$.

If now $h \in L_1$ then $\|y_\tau\|_{L_\infty} \leq \|h\|_{L_1} \|u_\tau\|_{L_\infty}$
$\Leftrightarrow$ fg-L_∞ stable.

traditional condition of impulse response absolutely integrable $\Leftrightarrow$ BIBO stable.

<u>Def$^{\underline{n}}$</u>: A mapping $H: L_e^m \rightarrow L_e^q$ is small-signal L stable (small signal fg-L stable) if $\exists\, r > 0$ s.t. (*) (resp. (**)) is satisfied for $\forall\, u \in L_e^m$: $\sup_{0 \leq t \leq \tau} \|u(t)\| \leq r$. (Note: $\not\Rightarrow \|u\|_{L_p}$ is small)

<u>Ex</u>: $y = \tan u \rightarrow$ clearly not L_∞ stable since $y(t) \rightarrow \infty$ for $u \rightrightarrows \pi/2 \quad \forall t \geq 0$.
But if $|u(t)| \leq r < \pi/2$ then
$|y| \leq \left(\frac{\tan r}{r}\right)|u| \Rightarrow \|y\|_{L_p} \leq \left(\frac{\tan r}{r}\right)\|u\|_{L_p}$
$\hookrightarrow$ Exercise.

<u>I/O vs Lyapunov Stability</u>

Linear Systems : $\dot{x} = Ax + Bu$; $y = Cx + Du$
$H(s) = C(sI - A)^{-1}B + D.$ } (1)

Thm 1: Let (1) be stabilizable & detectable. Then (1) is L_∞-stable iff $\dot{x} = Ax$ is (globally) asymp. stable. (Revise from Kailath)

Thm 2: If $\dot{x} = Ax$ is asymp. stable, then (1) is L_p-stable for each $p \in [1, \infty]$ (Exercise)

We want non-linear version of the above thms:

$$(*)\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \forall t \geq 0, \; f(t,0,0) = 0 \;\; \forall t \geq 0 \\ y(t) = g(t, x(t), u(t)) & g(t,0,0) = 0 \end{cases}$$

$\Rightarrow$ '0' is eq. pt. for unforced system.

First a non-linear version of Th 2:

Thm: Suppose '0' is an exponentially stable eq of $\dot{x} = f(t, x, 0)$, f is C^1 and f, g are Lipschitz continuous i.e

$$\left.\begin{aligned} \|f(t,x,u) - f(t,z,v)\| &\leq k_f\left[\|x - z\| + \|u - v\|\right] \\ \|g(t,x,u) - g(t,z,v)\| &\leq k_g\left[\|x - z\| + \|u - v\|\right] \end{aligned}\right\} \begin{aligned} &\forall t \geq 0 \\ &\forall x, z, u, v \end{aligned}$$

Then $(*)$ with $x_0 = 0$ is small signal L_p-stable with finite gain & zero bias $\forall p \in [1, \infty]$

If '0' is globally exp. stable eq, then $(*)$ is L_p-stable with finite gain & zero bias $\forall p \in [1, \infty]$

L_2 - Gain:

Thm: Consider $\left.\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}\right\}$ where A is Hurwitz.

Let $G(s) = C(sI - A)^{-1}B + D$. Then the L_2 gain of the system is $\sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2$

Proof: $x(0) = 0$ WLOG. $Y(j\omega) = \int_0^\infty y(t) e^{-j\omega t} dt$

$Y(j\omega) = G(j\omega)\, U(j\omega)$

Using Parseval's thm (for causal $y \in L_2$, $\int_0^\infty y^T(t) y(t) dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y^*(j\omega) Y(j\omega) d\omega$)

$$\|y\|_{L_2}^2 = \int_0^\infty y^T(t) y(t) dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} Y^*(j\omega)\, Y(j\omega) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^*(j\omega)\, G^T(-j\omega)\, G(j\omega)\, U(j\omega) d\omega$$

$$\leq \left[\sup_{\omega \in \mathbb{R}} \|G(j\omega)\|_2\right]^2 \frac{1}{2\pi}\int_{-\infty}^{+\infty} U^*(j\omega) U(j\omega) d\omega$$

$$= \|G(s)\|_{H_\infty}^2 \; \|u\|_{L_2}^2$$

$\|G(j\omega)\|_2$ = induced 2-norm

$= \sqrt{\lambda_{max}[G^T(-j\omega)\, G(j\omega)]}$

$= \sigma_{max}[G(j\omega)]$

$y = \underset{n\times n}{A} x$

$\|y\|_2 = \|Ax\|_2 \leq \|A\|_2 \|x\|_2$

$\hookrightarrow$ induced 2-norm $\|A\|_2 = \sup_{\|x\|=1} \frac{\|Ax\|}{\|x\|}$

Hence L_2 gain $\underset{=\,\checkmark}{\leq}$ $\|G(s)\|_{H_\infty}$ (Equality req a little more work)

Thm: Consider $\dot{x} = f(x,u)$; $y = h(x,u)$ where f be locally Lipschitz & h is continuous $\forall x \in \mathbb{R}^n$ & $u \in \mathbb{R}^m$. Let $V(x)$ be positive semidefinite s.t. $\dot{V} = \frac{\partial V}{\partial x} f(x,u) \leq a(\nu^2\|u\|^2 - \|y\|^2)$ $(a, \nu > 0)$

Then for each $x(0) \in \mathbb{R}^n$, the system is fg - L_2 stable & its L_2 gain is $\leq \nu$.

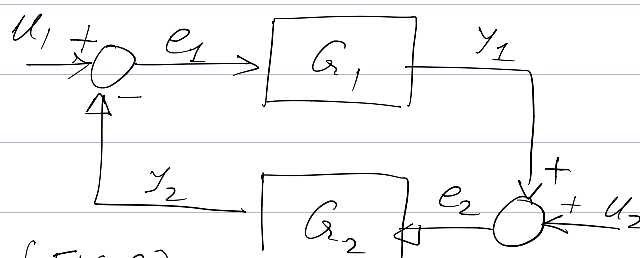
Proof: $V(x(\tilde{\tau})) - V(x(0)) \leq a\nu^2 \int_0^{\tilde{\tau}} \|u(t)\|^2 dt - a \int_0^{\tilde{\tau}} \|y(t)\|^2 dt$

$V(x) \geq 0$

$$\Rightarrow \int_0^{\tilde{\tau}} \|y(t)\|^2 dt \leq \nu^2 \int_0^{\tilde{\tau}} \|u(t)\|^2 dt + \frac{V(x(0))}{a}$$

$$\Leftrightarrow \|y_\tau\|_{L_2} \leq \nu \|u_\tau\|_{L_2} + \sqrt{\frac{V(x(0))}{a}}$$

Feedback Systems



$u_1, e_1, y_2 \in L_{pe}^{n_1}$

$u_2, e_2, y_1 \in L_{pe}^{n_2}$

Let $p \in [1, \infty]$

(FIG A)

$e_1 = u_1 - y_2$, $e_2 = u_2 + y_1$, $y_1 = G_1 e_1$, $y_2 = G_2 e_2$

Let $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ $e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$; $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $n = n_1 + n_2$

$G = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix}$ $F = \begin{bmatrix} 0 & I_{n_1} \\ -I_{n_2} & 0 \end{bmatrix}$ ← constant interconnection matrix

$$\left.\begin{aligned} e = u - Fy, \quad y = Ge \quad &\Rightarrow y = G(u - Fy) \\ &\Rightarrow (I + FG)e = u \end{aligned}\right\} (*)$$

Clearly, it is not necessary that each u will lead to a solution for y & e by (*).

So we assume the feedback connection is well-posed. $\Leftrightarrow$ $(I+FG_2)^{-1} : L_{pe}^n \to L_{pe}^n$ is a well-defined causal map.

Then $e = (I+FG_2)^{-1} u$ satisfies (*)

[Some more conditions are usually assumed (see Vidyasagar (1980))

$\Leftrightarrow$ All maps $u \to e$, $u \to y$, $y \to u$ are well defined

FACT: F.A.E (1) $u \to e$ is L_p stable (2) $u - y$ is L_p stable (3) Both $u \to e$ & $u \to y$ are L_p stable

So the statement "the feedback interconnection is L_p-stable" is unambiguous.

Small Gain thm: (leads to circle criterion)
Consider the feedback interconnection (Fig A), & let $p \in [1,\infty]$. Suppose G_1, G_2 are causal & L_p stable with finite gain & zero bias with gains $\gamma_{1p} = \gamma_p(G_1)$ $\gamma_{2p} = \gamma_p(G_2)$. Then the feedback interconnection is L_p stable if $\gamma_{1p}\gamma_{2p} < 1$.

Proof (Sketch) $\underbrace{\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{(\ell)}$; $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & G_2 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$

Since G_1, G_2 are causal + L_p stable $\overbrace{\text{with finite gain 0 bias}}^{\text{wb.}}$

$$\begin{bmatrix} \|y_1\|_{Tp} \\ \|y_2\|_{Tp} \end{bmatrix} \leq \begin{bmatrix} \gamma_{1p} & 0 \\ 0 & \gamma_{2p} \end{bmatrix} \begin{bmatrix} \|e_1\|_{Tp} \\ \|e_2\|_{Tp} \end{bmatrix} \quad \forall T \geq 0 \qquad \text{---}(1)$$

From (*)

$$\begin{bmatrix} \|e_1\|_{Tp} \\ \|e_2\|_{Tp} \end{bmatrix} \leq \begin{bmatrix} \|u_1\|_{Tp} \\ \|u_2\|_{Tp} \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \|y_1\|_{Tp} \\ \|y_2\|_{Tp} \end{bmatrix} \quad \forall T \geq 0 \qquad \text{---}(2)$$

From (1) & (2), $\forall T \geq 0$,

$$\begin{bmatrix} \|e_1\|_{Tp} \\ \|e_2\|_{Tp} \end{bmatrix} \leq \begin{bmatrix} \|u_1\|_{Tp} \\ \|u_2\|_{Tp} \end{bmatrix} + \begin{bmatrix} 0 & \gamma_{2p} \\ \gamma_{1p} & 0 \end{bmatrix} \begin{bmatrix} \|e_1\|_{Tp} \\ \|e_2\|_{Tp} \end{bmatrix}$$

or

$$\underbrace{\begin{bmatrix} 1 & -\gamma_{2p} \\ -\gamma_{1p} & 1 \end{bmatrix}}_{M} \begin{bmatrix} \|e_1\|_{Tp} \\ \|e_2\|_{Tp} \end{bmatrix} \leq \begin{bmatrix} \|u_1\|_{Tp} \\ \|u_2\|_{Tp} \end{bmatrix} \quad \forall T \geq 0$$

If $\gamma_{1p}\gamma_{2p} < 1$ then, M is non singular

$$\begin{bmatrix} \|e_1\|_{Tp} \\ \|e_2\|_{Tp} \end{bmatrix} \leq \frac{1}{1-\gamma_{1p}\gamma_{2p}} \begin{bmatrix} 1 & \gamma_{2p} \\ \gamma_{1p} & 1 \end{bmatrix} \begin{bmatrix} \|u_1\|_{Tp} \\ \|u_2\|_{Tp} \end{bmatrix} \quad \forall T \geq 0$$

$$\leq \underbrace{\frac{1}{1-\gamma_{1p}\gamma_{2p}} \begin{bmatrix} 1 & \gamma_{2p} \\ \gamma_{1p} & 1 \end{bmatrix} \begin{bmatrix} \|u_1\|_{p} \\ \|u_2\|_{p} \end{bmatrix}}_{\text{ind. of } T} \quad \text{if } u_1, u_2 \in L_p^{n_i}$$

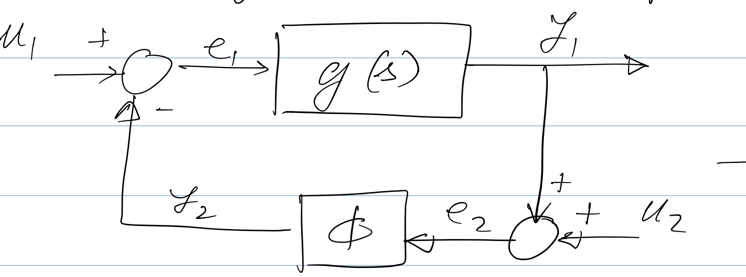
ind. of T. $\Rightarrow e_1, e_2 \in L_p^{n_i}$ — (3)

Using (1) & (3):

$$\begin{bmatrix} \|y_1\|_{p} \\ \|y_2\|_{p} \end{bmatrix} \leq \frac{1}{1-\gamma_{1p}\gamma_{2p}} \begin{bmatrix} \gamma_{1p} & \gamma_{1p}\gamma_{2p} \\ \gamma_{1p}\gamma_{2p} & \gamma_{2p} \end{bmatrix} \begin{bmatrix} \|u_1\|_{p} \\ \|u_2\|_{p} \end{bmatrix}$$

$\rightarrow$ proves $u \rightarrow y$ & $u \rightarrow e$ maps are L_p stable wb.

<u>Example:</u>



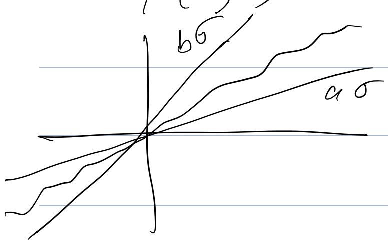
— (1)

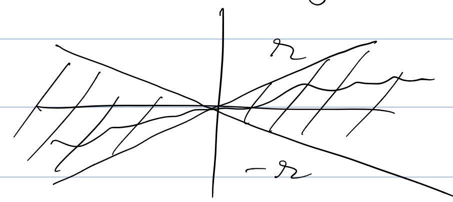
Let $g(s)$ be LTI and stable. ϕ is a memoryless possibly time varying non-linearity

$\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $\phi(t, x(t))$

We say "ϕ belongs to the sector $[a, b]$" if

$$\phi(t,0) = 0 \quad \& \quad a \leq \frac{\phi(t,\sigma)}{\sigma} \leq b \quad \forall \sigma \neq 0 \quad \forall t \geq 0 .$$





<u>Lemma</u>: Let $g(s)$ be BIBO stable & ϕ belong to the sector $[-r, r]$. Then the feedback configuration (1) is L_2-stable with finite gain & zero bias (wb) if

$$\sup_{\omega \in \mathbb{R}} |g(j\omega)| < \frac{1}{r}$$

<u>Proof</u>: We showed earlier $\gamma_2(g) = \sup_{\omega \in \mathbb{R}} |g(j\omega)|$

Clearly $\gamma_2(\phi) \leq r$.

Hence from small gain thm, L_2-stab (wb) of (1) follows if

$$\left\{ \sup_{\omega \in \mathbb{R}} |g(j\omega)| \right\} r < 1 .$$