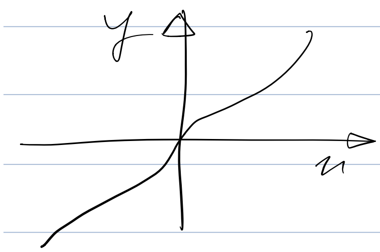
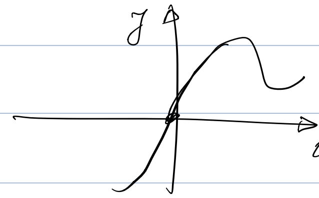
Passivity

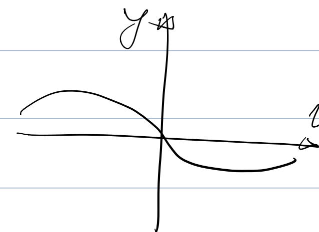
Static maps : $y = h(t, u)$ are said to be *passive* if $u^T y \geq 0 \quad \forall u$.



Passive



Passive



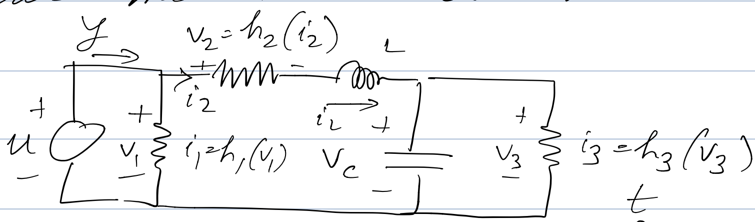
Not passive

Passivity for Dynamical System

Intuitively: Total energy supplied over a period of time = Increase in stored energy + energy dissipated.

i.e. Total energy supplied $\geq$ Increase in stored energy
$[0, t]$ $\qquad\qquad\qquad\qquad$ $[0, t]$

Consider the RLC network:



Energy supplied over $[0, t]$ = $\int_0^t u(s)\, y(s)\, ds$

$\underbrace{\text{voltage} \times \text{current}}_{\text{Power}}$

Stored energy : energy stored in L & C.

$=: V(x) := \frac{1}{2} L x_1^2 + \frac{1}{2} C x_2^2$. So according to the above def$^{\underline{n}}$ passivity implies

$$(1) \quad \int_0^t u(s) y(s)\, ds \geq V(x(t)) - \underbrace{V(x(0))}_{\text{initial energy in } L \,\&\, C.}$$

Since this eqn must hold for all $t \geq 0$,

$$(2) \quad u(t)\, y(t) \geq \dot{V}(x(t), u(t)) \quad \forall t \geq 0.$$

So instead of (1) we equivalently can use (2) as the def$^{\underline{n}}$ of passivity.

In this example, we can check:

$$V = \frac{1}{2} L x_1^2 + \frac{1}{2} C x_2^2$$

$$\dot{V} = L x_1 \dot{x}_1 + C x_2 \dot{x}_2 = \underbrace{\cdots\cdots}_{\text{Exercise}} = uy - u h_1(u) - x_1 h_2(x_1) - x_2 h_3(x_2)$$

$$\Leftrightarrow uy = \dot{V} + u h_1(u) + x_1 h_2(x_1) + x_2 h_3(x_2)$$

If h_1, h_2, h_3 are passive then $uy \geq \dot{V} \Rightarrow$ passive.

Consider: $(*) \begin{cases} \dot{x} = f(x, u) \\ y = h(x, u) \end{cases}$ $f \to$ locally Lipschitz, $h \to$ continuous, $f(0,0) = 0$, $h(0,0) = 0$

Important: u & y are of same dimension.

Def$^{\underline{n}}$: System $(*)$ is said to be passive if $\exists$ C^1 positive semi-definite $V(x)$ [called storage f$^{\underline{n}}$]

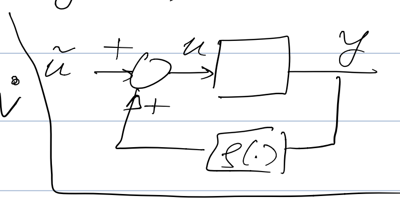
s.t. $\quad u^T y \geq \dot{V} \quad \forall t \geq 0.$

<u>Moreover:</u> 1) lossless if $u^T y = \dot{V}$

2) output feedback passive if $u^T y \geq \dot{V} + y^T \rho(y)$ for some function ρ.

3) output strictly passive if $u^T y \geq \dot{V} + y^T \rho(y)$ & $y^T \rho(y) > 0 \quad \forall y \neq 0.$

4) strictly passive if $u^T y > \dot{V}$



<u>Examples:</u> $\dot{x} = u,\ y = x$ is lossless. [Exercise]

<u>Lemma:</u> If (*) is output strictly passive with $u^T y \geq \dot{V} + \delta y^T y$ for some $\delta > 0$, then it is finite gain L_2-stable (gain $\leq \frac{1}{\delta}$)

<u>Proof:</u>
$$\dot{V} \leq u^T y - \delta y^T y = \underbrace{-\frac{1}{2\delta}\underbrace{(u-\delta y)^T(u-\delta y)}_{\text{square}} + \underbrace{\frac{1}{2\delta}u^T u - \frac{\delta}{2} y^T y}_{\text{Remaining term}}}_{\substack{\text{Completion of squares} \\ \text{(Useful trick in passivity theory)}}}$$
$$\leq \frac{1}{2\delta} u^T u - \frac{\delta}{2} y^T y$$

<u>Rest of proof: exercise</u>

<u>Lemma:</u> If (*) is passive with <u>p.d.</u> storage fn $V(x)$, then origin of $\dot{x} = f(x, 0)$ is stable.

Proof (hint): Take V as Lyap. fn candidate.

Note: 1) V was assumed to be p.d. to be the Lyap. fn. candidate.

2) $\dot{V} \leq 0 \Rightarrow$ Asymp. stab. could not be proved.

Lemma: The origin of $\dot{x} = f(x,0)$ is asymp. stable if (∗) is strictly passive. Further if $V(x)$ is radially unbdd, origin is globally asymp. stable.

Proof: Exercise. (Hint: First show that $V(x) > 0$ $\forall x \neq 0$.)

Example:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_1 x_1^3 - k x_2 + u \\ y &= x_2 \end{aligned}\right\} \quad \begin{aligned} & a, k > 0 \\ & V(x) = \tfrac{1}{4} a x_1^4 + \tfrac{1}{2} x_2^2 \\ & \dot{V} = \cdots\cdots = -ky^2 + yu \end{aligned}$$

Hence output strictly passive with $f(y) = ky$.

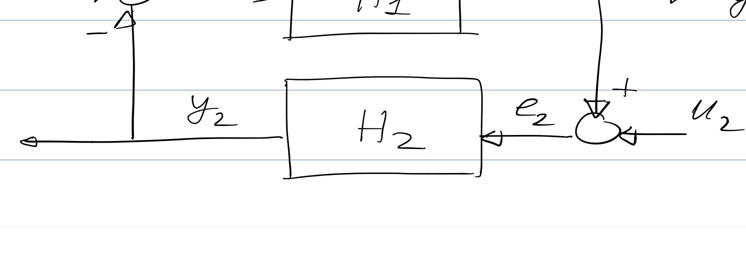
$\Rightarrow$ finite gain L_2 stable (gain $\leq \frac{1}{k}$)

For $u = 0$, origin of unforced system is stable

Q. Is it globally asymp. stable?

Passivity of Feedback Systems

(F1) ——



$$\dot{x}_i = f_i(x_i, e_i)$$
$$y_i = h_i(x_i, e_i)$$

Assumption: $u_1, y_1, u_2, y_2 \rightarrow$ same dimension.

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \qquad \left.\begin{array}{l} \dot{x} = f(x,u) \\ y = h(x,u) \end{array}\right\}$$

Thm: The feedback connection of two passive systems is passive.

Proof: $e_i^T y_i \geq \dot{V}_i \quad i=1,2$. Let $V = V_1 + V_2$

$$e_1^T y_1 + e_2^T y_2 = (u_1 - u_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2$$
$$= u^T y \geq \dot{V}$$

Thm: For (F1) above, let $e_i^T y_i \geq \dot{V}_i + \varepsilon_i e_i^T e_i + \delta_i y_i^T y_i$ for some storage f^n $V_i(x_i)$. Then $i=1,2$. the closed loop ($u \rightarrow y$ map) is finite gain L_2 stable if $\varepsilon_1 + \delta_2 > 0$ & $\varepsilon_2 + \delta_1 > 0$

Proof: $\underbrace{\dot{V} \leq -y^T L y - u^T M u + u^T N y}_{(1)}$ [Prove : Exercise]

where:

$$L = \begin{bmatrix} (\varepsilon_2 + \delta_1) I & 0 \\ 0 & (\varepsilon_1 + \delta_2) I \end{bmatrix} \quad M = \begin{bmatrix} \varepsilon_1 I & 0 \\ 0 & \varepsilon_2 I \end{bmatrix} \quad N = \begin{bmatrix} I & 2\varepsilon_1 I \\ -2\varepsilon_2 I & I \end{bmatrix}$$

$V(x) = V_1 + V_2$. Let $a = \min\{\varepsilon_2 + \delta_1, \varepsilon_1 + \delta_2\}$

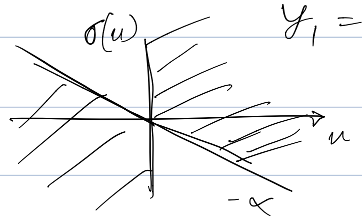
$b = \|N\|_2 \geq 0$ & $c = \|M\|_2 \geq 0$. Then from (1)

$$\dot{V} \leq -a\|y\|_2^2 + b\|u\|_2 \|y\|_2 + c\|u\|_2^2$$
$$= -\frac{1}{2a}(b\|u\|_2 - a\|y\|_2)^2 + \frac{b^2}{2a}\|u\|_2^2 - \frac{a}{2}\|y\|_2^2 + c\|u\|_2^2$$

$$\le \frac{k^2}{2a}\|u\|_2^2 - \frac{a}{2}\|y\|_2^2 \qquad \left(k^2 = b^2 + 2ac\right)$$

$$\Rightarrow \|y_\tau\|_{L_2} \le \frac{k}{a}\|u_\tau\|_{L_2} + \sqrt{\frac{2V(x(0))}{a}}$$

<u>Example:</u> $H_1 = \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a x_1^3 - \sigma(x_2) + e_1 \\ y_1 = x_2 \end{cases}$ $\qquad H_2 : y_2 = k e_2$



$\sigma \in [-\alpha, \infty]$ $\qquad a > 0$, $\alpha > 0$, $k > 0$

$$V(x) = \frac{a}{4} x_1^4 + \frac{1}{2} x_2^2$$

<u>Exercise:</u> Show closed loop is finite-gain L_2 stable.