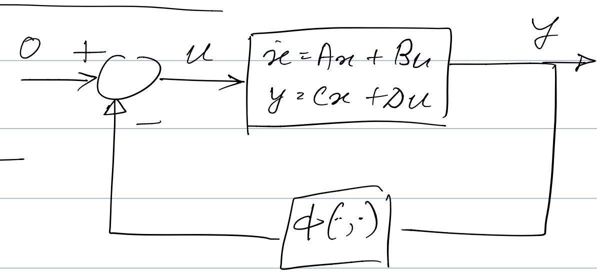
The Luré Problem :

$(*)$ —— 

$u(t), y(t) \in \mathbb{R}^m$

$m < n$.

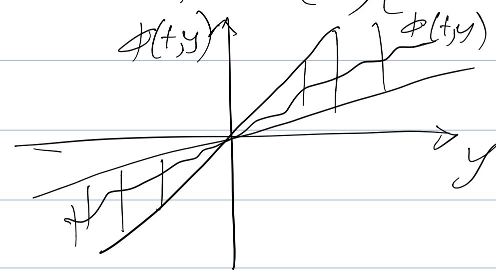
$H(s) = C(sI - A)^{-1}B + D$

1) $u(t) = -\phi(t, y(t))$. $\{ \dot{x} = Ax + Bu, \; y = Cx + Du \}$ —— (1)

2) $\{A, B\}$ controllable (3) $\{A, C\}$ observable

3) $\phi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ & $a, b \in \mathbb{R}$ with $a < b$. Then ϕ is said to belong to the sector $[a, b]$ if (i) $\phi(t, 0) = 0 \;\; \forall t \in \mathbb{R}_+$ & (ii) $[\phi(t,y) - ay]^T [by - \phi(t,y)] \geq 0$ $\forall t \in \mathbb{R}^+, \forall y \in \mathbb{R}^m$

For scalar :



For scalar :

$ay \leq \phi(t,y) \leq by$

Absolute Stability: Under the above assumptions, derive conditions involving only $H(\cdot)$ & (a, b) s.t. $x = 0$ is a globally uniformly asymptotically stable equilibrium of (1) for every f^{n} $\phi : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}^m$ belonging to sector $[a, b]$.

Kalman-Yakubovich-Popov (KYP) Lemma : Consider the system $(*)$ above, where (i) A is Hurwitz (ii) (A, B) is controllable (iii) (C, A) is observable, and

(iv) $\inf_{\omega \in \mathbb{R}} \lambda_{min} [H(j\omega) + H^*(j\omega)] > 0$ [H^* denotes conjugate transpose]

($\lambda_{min} \to$ smallest eigenvalue) $\boxed{\text{Q. Why is } \lambda_{min} \text{ real?}}$

Under these conditions, $\exists$ a $P = P^T > 0$, matrices $Q \in \mathbb{R}^{m \times n}$, $W \in \mathbb{R}^{m \times m}$, and an $\varepsilon > 0$ s.t.

(1) $A^T P + PA = -\varepsilon P - Q^T Q$

(2) $B^T P + W^T Q = C$

(3) $W^T W = D + D^T$

<u>Notes:</u> 1) $H(\cdot)$ satisfying (iv) is called <u>strictly positive real</u> (SPR)

2) For a scalar t.f $h(s)$, condition (iv) is equivalent to the Nyquist plot of $h(s)$ lying entirely in the <u>open</u> right half plane.

We skip the general proof, but try to provide some intuition below for SISO systems

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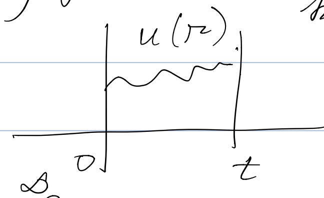
<u>Non-rigorous intuition</u>

$$\left[\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned}\right]$$

SPR $\Rightarrow$ Strictly passive

$$h(s) = c(sI - A)^{-1} b.$$

Consider an input



$$u(r) = 0 \begin{cases} r < 0 \\ r > t \end{cases}$$

$$\int_0^t y(r)u(r)\,dr = \int_{-\infty}^{\infty} y(r)u(r)\,dr \quad \left[\text{since } u(r) \text{ is truncated}\right]$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} y(j\omega)\, u^*(j\omega)\, d\omega \quad \left[\text{By Parseval's theorem}\right]$$

But $y(j\omega) = h(j\omega)\, u(j\omega)$. Hence

$$\int_0^t y(\tau)\, u(\tau)\, d\tau = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(j\omega) \left| u(j\omega) \right|^2 d\omega$$

Using $h(-j\omega) = [h(j\omega)]^*$ [F.T. of a real signal]

$$\int_0^t y(\tau)\, u(\tau)\, d\tau = \frac{1}{\pi} \int_0^{\infty} \mathrm{Re}[h(j\omega)] \left| u(j\omega) \right|^2 d\omega$$

Clearly if $\mathrm{Re}[h(j\omega)] > 0$, LHS $> 0 \Rightarrow$ passive.

Necessity of this condition can also be argued.

For stable minimal $h(s) = c(sI - A)^{-1} b$,
for every $Q = Q^T > 0$, $\exists\, P = P^T > 0$ s.t.

$$A^T P + PA = -Q \quad ; \quad V(x) = \frac{1}{2} x^T P x$$

$$\dot{V}(x) = x^T P [Ax + Bu] = x^T PBu - \frac{1}{2} x^T Q x$$

If now $C = B^T P$, then $\dot{V} = y^T u - \frac{1}{2} x^T Q x$

$\Rightarrow \dot{V} < y^T u \rightarrow$ St. passive

<u>Note</u>: Above arguments don't really prove the KYP.

Lemma: The LTI minimal realization $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

$H(s) = C(sI - A)^{-1}B + D$ is strictly passive if $H(s)$ is strictly positive real

Proof: $V(x) = \frac{1}{2}x^T P x$ as storage f^{n}.

$$u^T y - \dot{V} = u^T y - \frac{\partial V}{\partial x}[Ax + Bu]$$

Using KYP equation

$$= u^T[Cx + Du] - x^T P(Ax + Bu)$$

$$= u^T Cx + \frac{1}{2}u^T(D + D^T)u - \frac{1}{2}x^T(PA + A^T P)x - x^T PBu$$

$$= u^T(B^T P + W^T Q)x + \frac{1}{2}u^T W^T W u + \frac{1}{2}x^T Q Q^T x + \frac{1}{2}\varepsilon x^T P x - x^T PBu$$

$$= \frac{1}{2}(Qx + Wu)^T(Qx + Wu) + \frac{1}{2}\varepsilon x^T P x$$

$$\geq \frac{1}{2}\varepsilon x^T P x > 0 \quad (\text{since } \varepsilon > 0)$$

Hence strictly passive.

So $\boxed{\text{Hurwitz + minimal + SPR}} \Rightarrow$ KYP equations $\Rightarrow$ Strict Passivity

Examples 1) $G(s) = \frac{1}{s}$. $Re(G(j\omega)) = Re\left[\frac{1}{j\omega}\right] = 0$ $\forall \omega \neq 0$

$\hookrightarrow$ Not Hurwitz.

Hence $G(s)$ is not SPR.

2) $G(s) = \frac{1}{s+a}$, $a > 0$. $Re(G(j\omega)) = \frac{a}{\omega^2 + a^2} > 0$ $\forall \omega$

Hence SPR.

3> $G(s) = \begin{bmatrix} \frac{s+2}{s+1} & \frac{1}{s+2} \\ \frac{-1}{s+2} & \frac{2}{s+1} \end{bmatrix}$ → Check → it is Hurwitz

↳ $A, B, C, D = ?$

$$G(j\omega) + G^T(-j\omega) = \begin{bmatrix} \frac{2(2+\omega^2)}{1+\omega^2} & -\frac{2j\omega}{4+\omega^2} \\ \frac{2j\omega}{4+\omega^2} & \frac{4}{1+\omega^2} \end{bmatrix} > 0 \quad \forall\omega$$

↳ P.D.

$G(s)$ is SPR.

Solution to the Luré Problem

Thm: $(*)\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ u = -\phi(y,t) \end{cases}$ Let (i) A be Hurwitz

(ii) (A, B) controllable

$H(s) = C(sI-A)^{-1}B + D$ (iii) (C, A) observable

(iv) $\inf\limits_{\omega\in\mathbb{R}} \lambda_{min}\left[H(j\omega) + H^*(j\omega)\right] > 0$

(v) ϕ belongs to the sector $[0,\infty) \equiv \begin{cases} (i)\ \phi(t,0) = 0 \quad \forall t \\ (ii)\ y^T\phi(t,y) \geq 0 \\ \quad \forall t \geq 0,\ \forall y \in \mathbb{R}^m \end{cases}$

Note

Then $(*)$ is globally exponentially stable. ($\Rightarrow$ global uniformly asymptotically stable)

Proof: $V(x) = x^TPx$. $\dot{V}(x) = \dot{x}^TPx + x^TP\dot{x}$

$= [Ax - B\phi]^T Px + x^TP[Ax - B\phi]$ (Using $u = -\phi(y,t)$)

$= x^T[A^TP + PA]x - \phi^TB^TPx - x^TPB\phi$

[Note that $\phi^T B^T P x = \phi^T C x - \phi^T W^T Q x = \phi^T (y + D\phi) - \phi^T W^T Q x$]

Hence, $\dot{V} = x^T [A^T P + PA] x - \phi^T (D + D^T) \phi - \phi^T W^T Q x - x^T Q^T W \phi$
$- \phi^T y - y^T \phi$

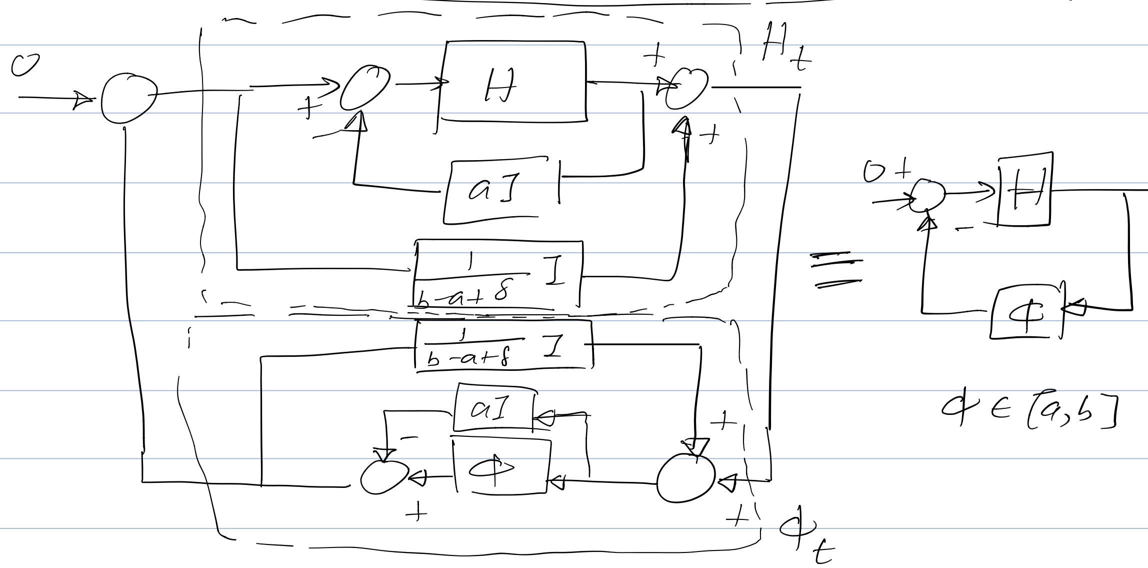
(Using KYP Lemma)

$= -\varepsilon x^T P x - [Qx + W\phi]^T [Qx + W\phi]$ [Exercise: Intermediate steps]

$\leq -\varepsilon x^T P x < 0$

$\Rightarrow$ Global uniform asymp. stability.

For general sector non-linearities $\phi \in [a, b]$



$\phi_t \in [0, \Delta)$

$\phi_t = (\phi - aI)\left[I - \frac{1}{b-a+\delta}(\phi - aI)\right]^{-1}$

ϕ_t belongs to sector $(0, \Delta)$

$H_t = H(s)[I + aH(s)]^{-1} + \frac{1}{b-a+\delta} I$

Exercise: Verify calculation from Fig.

Corollary: Consider the Luré system. Let (i) (A, B) be controllable (ii) (C, A) be observable (iii) Φ belong to sector $[a, b]$. Define $H_a(s) = H(s)[I + aH(s)]^{-1}$. Let

(iv) $$\inf_{\omega \in \mathbb{R}} \lambda_{min}\left[H_a(j\omega) + H_a^*(j\omega)\right] + \frac{2}{b-a} > 0$$

and (v) All poles of $H_a(s)$ have -ve real parts. Under these conditions, the system is exponentially stable. ($\Rightarrow$ uniformly globally asymp. stable)

Proof: Exercise

Specialization to Scalar Case (Circle Criterion)

The above corollary has nice geometric interpretations for scalar $h(s)$. First note condition (iv) for scalar $h(s)$ simplifies to

$$\mathrm{Re}\left[\frac{h(j\omega)}{1 + a h(j\omega)}\right] + \frac{1}{b-a} > 0 \qquad (*)$$

Let $z = h(j\omega)$.

FACT: $(*)$ is true iff

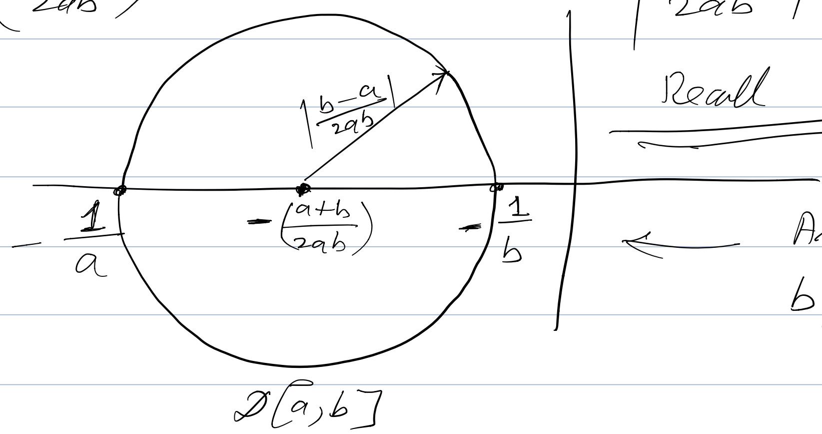
$$\left| z + \frac{b+a}{2ab} \right| > \left| \frac{b-a}{2ba} \right| \quad \text{if } ab > 0$$

and

$$\left| z + \frac{b+a}{2ab} \right| < \left| \frac{b-a}{2ab} \right| \quad \text{if } ab < 0$$

Proof: Exercise

Let $\mathcal{D}[a,b]$ denote the closed disk in the complex plane centered at $-\left(\frac{b+a}{2ab}\right)$ with radius $\left|\frac{b-a}{2ab}\right|$.



Recall $b > a$ always.

Assumption in pic. $b > a > 0$

Thm (Circle Criterion): Consider the Luré System with $m = 1$. Let (i) (A, b, c, d) be minimal realization of $h(s)$, (ii) ϕ belong to sector $[a,b]$. Then the feedback system is globally exponentially stable if one of the following conditions, as appropriate holds:

Case (i) $0 < a < b$: The Nyquist plot of $h(j\omega)$ lies outside & is bounded away from the disk $\mathcal{D}[a,b]$. Moreover, the plot encircles $\mathcal{D}[a,b]$ exactly ν times in the counterclockwise direction, where ν is the no. of eigenvalues of A with +ve real part.

Case ii) $0 = a < b$: A is Hurwitz; and
$$\inf_{\omega \in \mathbb{R}} \operatorname{Re} h(j\omega) + \frac{1}{b} > 0$$

Case (iii) $a < 0 < b$: A is Hurwitz; the plot of $h(j\omega)$ lies in the interior of $D[a,b]$ and is bdd. away from $D[a,b]$

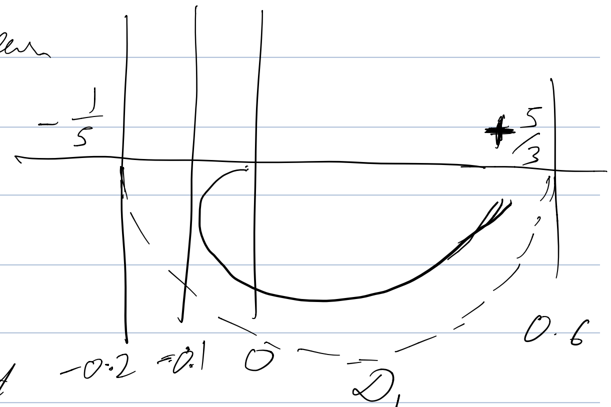
Case (iv): $a < b \leq 0$. Replace $h(\cdot)$ by $-h(\cdot)$, a by $-b$, b by $-a$ & apply (i) or (ii) as appropriate

Proof: Exercise

Q.. What happens if $(b-a) \to 0$?
Q.. Why does the encirclement statement appear in (i) & not in (iii) ?
Q. Which case is related to the small gain theorem? Which one is more general?

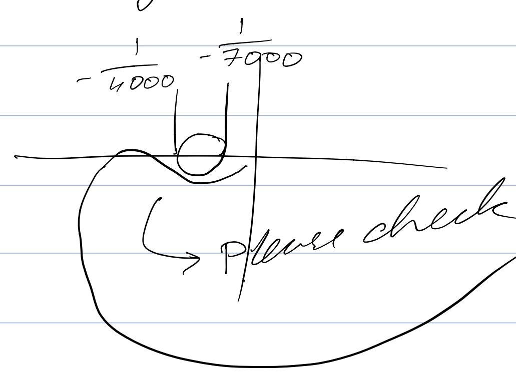
Example: $h(s) = \dfrac{(s+25)^2}{(s+1)(s+2)(s+3)(s+200)}$

A) $\phi_1 \in [-5/3, 5]$: D_1 shown
Case (iii) applies
$\Rightarrow$ exp. stable.

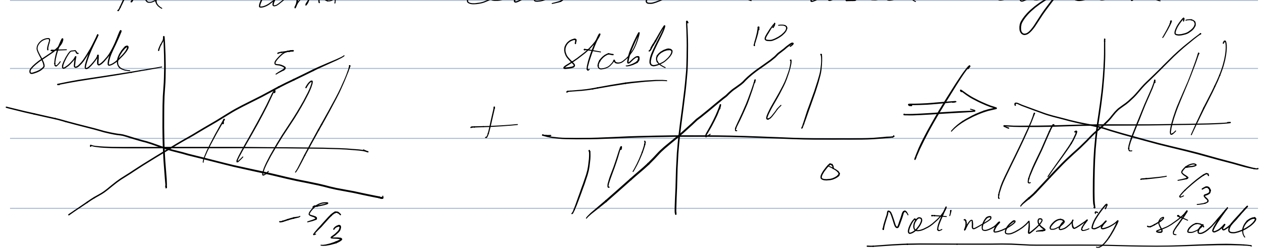


B)
$\phi_2 \in [0, 10]$. Case (ii) applies since Nyq. plot is strictly to the right of $-\frac{1}{10} = -0.1$
$\Rightarrow$ exp. stable.

C)
$\phi_3 \in [4000, 7000]$
Case (i) applies. No encirclements + No open loop unstable poles
$\Rightarrow$ exp. stable.

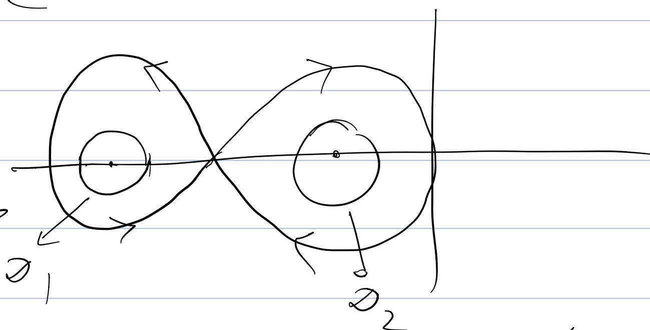


Note: A) & B) together does not imply a $\phi_4 \in [-\frac{5}{3}, 10]$ is exp. stable. The thm needs to be used again.



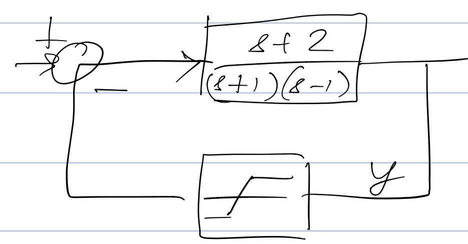
Example: $H(s) = \dfrac{4}{(s-1)\left(\frac{1}{2}s+1\right)\left(\frac{1}{3}s+1\right)}$

If some ϕ_1 corr. to D_1 then (i) applies ⇒ exp. stable



If sm ϕ_2 corr. to D_2, then (i) does not apply (encirclement is clockwise). So exp. stability cannot be guaranteed.

Exercise:



$u = -sat(y)$

$\phi \in [0, 1]$

Q. Can you use circle criterion?
Is this asymp. stable? Globally? Locally?