Controller Design Technique

▷ **Passivity based control**: Consider the p-input-p-output systems

$$(*)\begin{cases} \dot{x} = f(x,u) \\ y = h(x) \end{cases}$$

$f \to$ locally Lipschitz in (x,u)
$h \to$ continuous in x $\forall x \in \mathbb{R}^n$
$f(0,0) = 0$; $h(0) = 0$.

Thm: If $(*)$ is (1) passive with a radially unbdd. positive definite storage $f^{\underline{n}}$ and (2) if no solution of $\dot{x} = f(x,0)$ can stay identically in the set $\{x : h(x) = 0\}$ other than the trivial sol$^{\underline{n}}$. $x(t) \equiv 0$; then the origin $x = 0$ can be globally stabilized by $u = -\phi(y)$ where ϕ is any locally Lipschitz $f^{\underline{n}}$ s.t. $\phi(0) = 0$ & $y^T\phi(y) > 0$ $\forall$ $y \neq 0$.

Proof: Since $(*)$ is passive, $\exists$ by (1), a p.d. C^1 $f^{\underline{n}}$ $V(x)$ s.t. $u^T y \geq \dot{V}$ $\forall x, u$

Use $V(x)$ for Lyap. $f^{\underline{n}}$ candidate for

$$\dot{x} = f(x, -\phi(y))$$

$$\dot{V} = \frac{\partial V}{\partial x} f(x, -\phi(y)) \leq -y^T\phi(y) \leq 0$$

Rest of the proof: Exercise.

Example:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1^3 + u \\ y &= x_2 \end{aligned}$$

Let $V(x) = \frac{x_1^4}{4} + \frac{x_2^2}{2}$

$$\dot V = x_1^3 x_2 - x_2 x_1^3 + x_2 u = x_2 u = yu.$$

Also $y(t) \equiv 0 \Rightarrow x(t) \equiv 0$

So using the thm: $u = -kx_2$ as $u = -\left(\frac{2k}{\pi}\right)\tan^{-1}(x_2)$ $k>0$

is globally stabilizing.

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2) Sliding Mode Control

Consider
$$(*)\begin{cases} \dot x_1 = x_2 \\ \dot x_2 = h(x) + g(x)u \end{cases} \qquad \frac{g(x) \geq g_0 > 0}{\forall x}$$

1) Let $s = a_1 x_1 + x_2 = 0$ (How to select a_1?)

2) Let $\left| \frac{a_1 x_2 + h(x)}{g(x)} \right| \leq \rho(x) \quad \forall x \in \mathbb{R}^2$

Consider the dynamics of s allowed by $(*)$.

$$\dot s = a_1 \dot x_1 + \dot x_2 = a_1 x_2 + h(x) + g(x) u$$

We want to design u s.t. every trajectory of $(*)$, reach s in finite time.

Consider the Lyapunov fn candidate

$$V = \frac{1}{2} s^2 \Rightarrow \dot V = s\dot s = s[a_1 x_2 + h(x)] + g(x) s u$$

Now assume

$u = -[\rho(x) + \beta_0]\,\mathrm{sgn}(s)$

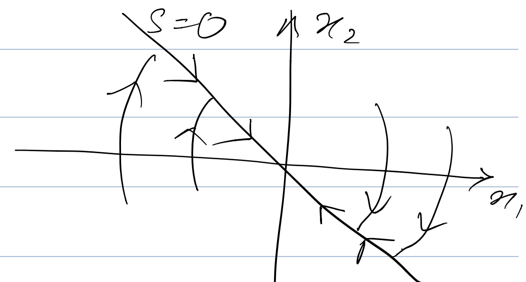
where $\beta_0 > 0$

& $\mathrm{sgn}(s) = \begin{cases} 1 & s>0 \\ 0 & s=0 \\ -1 & s<0 \end{cases}$

$$\leq g(x)|s|\rho(x) + g(x) s u$$
$$= g(x)|s|\rho(x) - g(x)[\rho(x) + \beta_0] s\,\mathrm{sgn}(s)$$
$$= g(x)\beta_0 |s| \leq -g_0 \beta_0 |s| \quad (1)$$

FACT (without proof) : (1) proves that the trajectory reaches $s=0$ in finite time.
(use Comparison Lemma if interested)

Once it reaches $s=0$, it cannot leave it since $\dot{V} \leq -g_0\beta_0|s|$.



Typical phase portrait

Dynamics on the surface

$$\dot{s} = a_1\dot{x}_1 + \dot{x}_2 = 0$$

$$a_1x_2 + \dot{x}_2 = 0 \quad \Big| \text{ since } \dot{x}_1 = x_2$$

$$\dot{x}_2 = -a_1x_2 \quad \text{--- (2)}$$

If $a_1 > 0$, (2) is asymp. stable (reaches origin in infinite time)

Also $s=0 \Rightarrow a_1x_1 + x_2 = 0$

$$\Leftrightarrow x_2 = -a_1x_1 \quad \text{--- (3)}$$

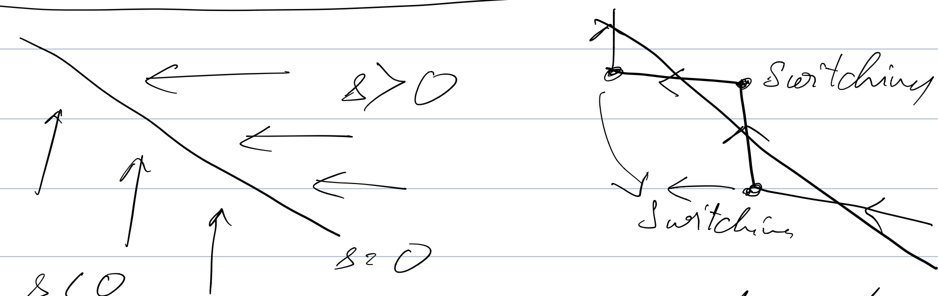
(2) + (3) $\Rightarrow$ $\dot{x}_1 = -a_1x_1 \Rightarrow x_1$ is asymp stable.

Note: $h(x)$ & $g(x)$ are not required to be exact: only upper bdd $\rho(x)$ is needed.

If $\left|\dfrac{a_1x_2 + h(x)}{g(x)}\right| \leq k_1$ for some k_1

Then $u = -k\,\mathrm{sgn}(s)$ $k > k_1$ is sufficient for reaching $s=0$.

Real implementation leads to chattering



Resulting trajectory

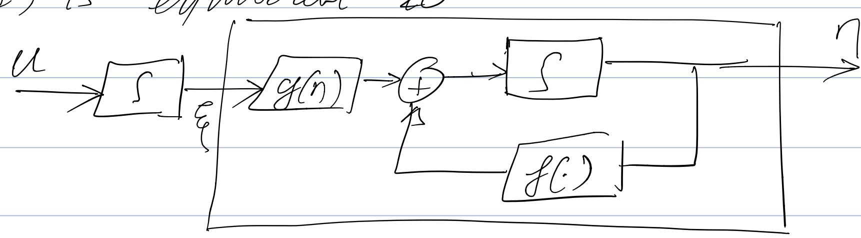
3) Backstepping Consider

$$(1)\begin{cases} \dot{\eta} = f(\eta) + g(\eta)\xi \\ \dot{\xi} = u \end{cases}$$

$\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}$, $u \in \mathbb{R}$ $f(0) = 0$, $0 \in D$ | $f : D \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$

Aim: Design a state feedback control s.t. $(\eta = 0, \xi = 0)$ is stabilized.

(1) is equivalent to



Assumption: We know a state feedback law $\xi = \phi(\eta)$ with $\phi(0) = 0$ s.t. the origin of $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$ is asymptotically stable.

(b) a Lyapunov $f^{\underline{n}}$ $V(\eta)$ which satisfies

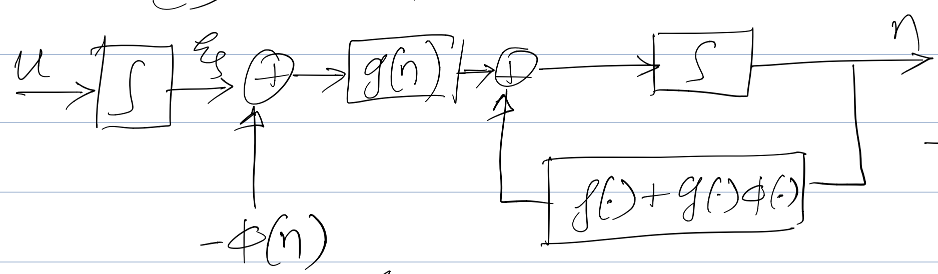
$$\frac{\partial V}{\partial \eta}\left[f(\eta)+g(\eta)\phi(\eta)\right] \leq -W(\eta)$$

$$W(0)=0, \quad W(\eta)>0, \quad \forall \eta \in D$$

Now consider (1) with $\xi \equiv -\phi(\eta)$:

$$(2)\left\{\begin{array}{l}\dot{\eta} = f(\eta)+g(\eta)\phi(\eta) \\ \dot{\xi} = u\end{array}\right\}$$

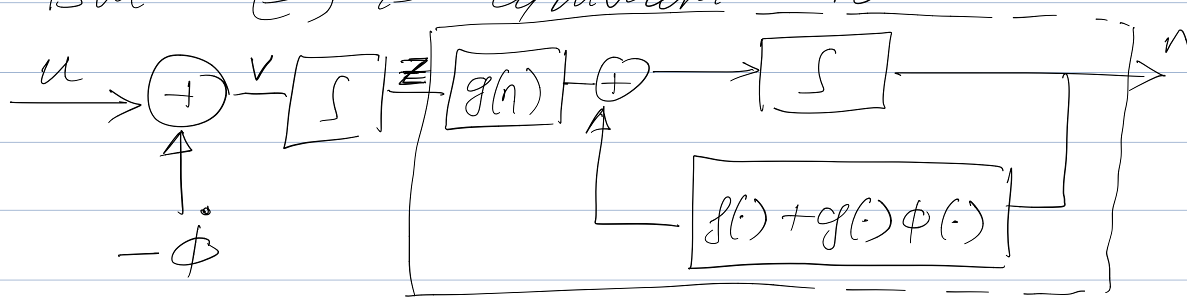
Now (2) is equivalent to



—— (3)

since $(2) \Leftrightarrow \left\{\begin{array}{l}\dot{\eta} = f(\eta)+g(\eta)\phi(\eta)+g(\eta)\underbrace{[\xi-\phi(\eta)]}_{=0} \\ \dot{\xi} = u\end{array}\right.$ —— (4)

But (3) is equivalent to:



—— (5)

with $z=\xi-\phi(\eta)$ and $v=u-\dot{\phi}$

Equations for (5), which are also eq. to eq. (4) are: (Using $\dot{\phi}=\frac{\partial \phi}{\partial \eta}\cdot\dot{\eta}$)

$$(6)\begin{cases}\dot{\eta} = \left[f(\eta) + g(\eta)\phi(\eta)\right] + g(\eta)z \\ \dot{z} = v\end{cases}$$

Note: (6) is of the form (1)

Q. Is (6) stabilizable with a state feedback?

Let V_c be a Lyap. fn candidate:

$$V_c = V(\eta) + \frac{1}{2}z^2$$

$$\dot{V}_c = \frac{\partial V}{\partial \eta}\left[f(\eta) + g(\eta)\phi(\eta)\right] + \frac{\partial V}{\partial \eta}g(\eta)z + zv$$

$$\leq -W(\eta) + \frac{\partial V}{\partial \eta}g(\eta)z + zv$$

Choose: $v = -\frac{\partial V}{\partial \eta}g(\eta) - kz,\ k > 0.$

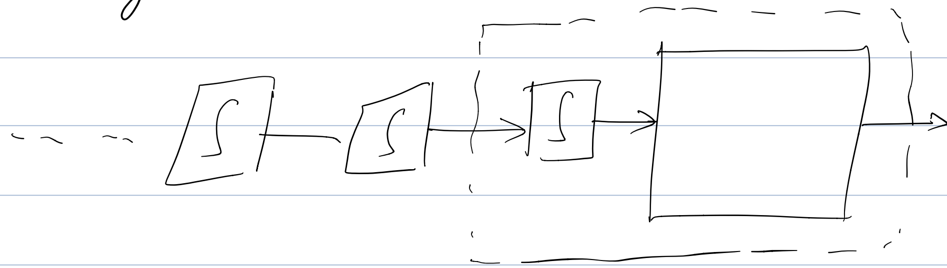
Then $\dot{V}_c \leq -W(\eta) - kz^2 \Rightarrow$ asymp. stable $(\eta = 0, z = 0)$.

Since $\phi(0) = 0 \Rightarrow (\eta = 0, \xi = 0)$ is asymp. stable.

The resulting control law:

$$u = \frac{\partial \phi}{\partial \eta}\left[f(\eta) + g(\eta)\xi\right] - \frac{\partial V}{\partial \eta}g(\eta) - k\left[\xi - \phi(\eta)\right]$$

This brings us to the state right after the "Assumptions" in the above derivation.
So we can repeat this process for any no. of integrators cascaded from the left.



Q. What if the outer system is not an integrator?

$$\dot{\eta} = f(\eta) + g(\eta)\xi_q$$
$$\dot{\xi}_q = f_a(\eta, \xi_q) + g_a(\eta, \xi_q)\, u$$

If $g_a(\eta, \xi_q) \neq 0$ over domain of interest,
Let $u = \dfrac{1}{g_a(\eta, \xi_q)}\left[u_a - f_a(\eta, \xi_q)\right]$

Then $\dot{\xi}_q = u_a$. → Design for u_a.
But apply u.

<u>General form where backstepping can be applied</u>: (Strict feedback Systems)

$$\dot{x} = f_0(x) + g_0(x) z_1 \quad \text{——} (1)$$
$$\dot{z}_1 = f_1(x, z_1) + g_1(x, z_1) z_2 \quad \text{——} (2)$$
$$\dot{z}_2 = f_1(x, z_1, z_2) + g_2(x, z_1, z_2) z_3 \quad \text{——} (3)$$
$$\vdots$$
$$\dot{z}_k = f_k(x, z_1, \cdots z_k) + g_k(x, z_1, \cdots, z_k) u$$

$x \in \mathbb{R}^n$, $z_1, \cdots z_k$ are scalars; $f_i(0, \cdots 0) = 0$
$g_i(x, z_1, \cdots, z_i) \neq 0 \quad \forall\, 1 \leq i \leq k.$

Start with (1) + (2)] stabilize, create V_c etc.
Then consider $\{(1)+(2)\}$ and (3) $\cdots$ & iterate.

4) Feedback Linearization

$$\left.\begin{aligned} \dot{x} &= f(x) + G(x) u \\ y &= h(x) \end{aligned}\right] (1)$$

\# Find $u = \alpha(x) + \beta(x) v$ & a transformation $z = T(x)$ s.t. (1) becomes an equivalent linear system.

Example:
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\left[\sin(x_1 + \delta) - \sin\delta\right] - b x_2 + c u$$

If $u = \frac{a}{c}\left[\sin(x_1 + \delta) - \sin\delta\right] + \frac{v}{c}$, then

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -bx_2 + v\end{aligned}$$ equivalent linear system which if controllable,

can be stabilized by:

$$v = -k x_1 - k_2 x_2$$

$$\Leftrightarrow \quad \begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 x_1 - (k_2 + b) x_2\end{aligned}$$ k_1, k_2 can be chosen to place poles arbitrarily.

Overall feedback law:

$$u = \frac{a}{c}\left[\sin(x_1 + \delta) - \sin\delta\right] - \frac{1}{c}\left[k_1 x_1 + k_2 x_2\right]$$

In general: for this idea to work the non-linear system must be of the form: $\dot{x} = Ax + B\gamma(x)\left[u - \alpha(x)\right]$ — (1)

Then $u = \alpha(x) + \gamma^{-1}(x) v$ transforms (1) into $\dot{x} = Ax + Bv$. Then one can use $v = -kx$ to stabilize.

Overall control: $\boxed{u = \alpha(x) - \beta(x) kx}$

What if the system is not of form (1)?

Ex: $$\begin{aligned}\dot{x}_1 &= a\sin x_2 \\ \dot{x}_2 &= -x_1^2 + u\end{aligned}$$ Not of form (1)

We can try a non-linear transformation

of the state variables: $z = T(x)$ to try & bring them to form (1).

$$\left[\begin{array}{l} z_1 = x_1 \\ z_2 = a\sin x_2 = \dot{x}_1 \end{array}\right] \longrightarrow T$$

Then, $\left[\begin{array}{l} \dot{z}_1 = z_2 \\ \dot{z}_2 = a\cos x_2 \left(-x_1^2 + u\right) \end{array}\right]$ is in form (1).

or $\left[\begin{array}{l} \dot{z}_1 = z_2 \\ \dot{z}_2 = a\cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)\left(-z_1^2 + u\right) \end{array}\right]$ is in form (1)

Let $u = x_1^2 + \frac{1}{a\cos x_2} v \quad \left[-\pi/2 < x_2 < \pi/2\right]$

Then $\left[\begin{array}{l} \dot{z}_1 = z_2 \\ \dot{z}_2 = v \end{array}\right] \rightarrow$ Linearized System

Q. How to get back to original coordinates?

$$\left[\begin{array}{l} x_1 = z_1 \\ x_2 = \sin^{-1}\left(\frac{z_2}{a}\right) \end{array}\right] \leftarrow T^{-1} \Big\} \text{ defined over } -a < z_2 < a$$

Hence both T & T^{-1} needs to be well defined and C^1. Such maps are called "diffeomorphism"s

Problem with output linearization

Ex: $\dot{x}_1 = a \sin x_2$, $\dot{x}_2 = -x_1^2 + u$] $y = x_2$

The above transformation & input, results in:

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = v$$
$$y = \sin^{-1}\left(\frac{z_2}{a}\right)$$

← Non-linear; may pose difficulties in tracking / regulation.

Instead if one uses $u = x_1^2 + v$:

$$\dot{x}_1 = a \sin x_2$$ ← Non-linear

$$\left.\begin{aligned} \dot{x}_2 &= v \\ y &= x_2 \end{aligned}\right\}$$ $v \to y$ map is linear

This is called input-output linearization.
~~H~~However x_1 becomes unobservable from output y. So we have to separately ensure that x_1 is well-behaved.

THANKS FOR ATTENDING