# EE 622: Optimal Control Systems

## Homework-1

Prob.1 For the following function,

$$F(x_1, x_2, x_3) = 2x_1^4 + 3x_2^2 + 6x_3^2 - 3x_1x_2 - 6x_2x_3$$

(a) Find the minimum(s).

(b) Are there any other stationary points? If so, what are they?

Prob.2 This problem explores the steepest descent algorithm. For the following function,

$$F(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3 - 2x_1 - 8x_3$$

(a) Give an expression for the search direction $\mathbf{p}_k$ for the steepest descent method.

(b) Using $\mathbf{x}_0 = [1\ 1\ 1]^T$, write out the first iteration of the steepest descent algorithm. What is $\mathbf{x}_1$ in terms of $\alpha_0$? What is the optimal value for $\alpha_0$ that minimizes $F(\mathbf{x}_0 + \alpha_0\mathbf{p}_0)$?

(c) Write a MATLAB program to solve this problem using a steepest descent algorithm and an initial value of $\mathbf{x}_0 = [1\ 1\ 1]^T$. Using a tolerance of $10^{-6}$, how many iterations does it take to converge with $\alpha = 0.1$, $\alpha = 0.2$, $\alpha = 0.5$, $\alpha = 1$? Explain your results.

(d) Use MATLAB's `fminunc` function to solve for the minimum and compare its performance with your algorithm.

Prob.3 **Problem 2.** Find the rectangle of maximum perimeter that can be inscribed in an ellipse; i.e., maximize

$$P = 4(x + y)$$

with the constraint

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

Prob.4 **Problem 4.** *Quadratic performance index with linear constraints.* Show that the control vector $u$ that minimizes the nonnegative definite quadratic form

$$L = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu\,,$$

with the linear constraints

$$f(x,u) = x + Gu + c = 0\,,$$

is

$$u = -(R + G^TQG)^{-1}G^TQc\,.$$

Show, also, that the minimum value of $L$ is

$$J = L_{\min} = \tfrac{1}{2}c^T\,(Q - QG(R + G^TQG)^{-1}G^TQ)\,c$$

and that

$$\lambda = (Q - QG(R + G^TQG)^{-1}G^TQ)\,c$$
$$\equiv (Q^{-1} + GR^{-1}G^T)^{-1}c \qquad \text{if} \qquad Q^{-1} \text{ exists;}\dagger$$
$$x = -(I - G(R + G^TQG)^{-1}G^TQ)\,c\,.$$

Note, also, that

$$\lambda^T = \frac{\partial J}{\partial c}\,.$$

†This is known as the "matrix inversion lemma."

Prob.5

**Problem 5.** *Sail setting and heading for maximum upwind velocity.* A simplified model of a sailboat moving at constant velocity is shown in Figure 1.3.2.

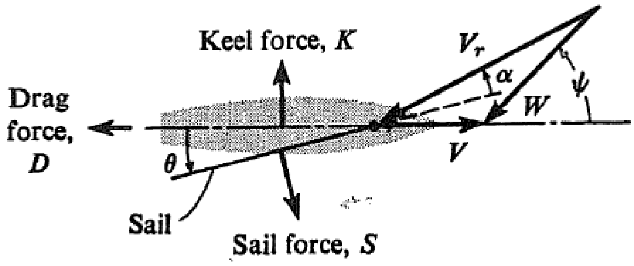


**Figure 1.3.2.** Force equilibrium of sailboat.

The sailboat's velocity relative to the water is $V$, at an angle $\psi$ to the wind, which is blowing with velocity $W$ relative to the water. The sail is set at an angle $\theta$ to the centerline of the boat, and the aerodynamic force $S$ is assumed to act normal to the sail. The hydrodynamic force on the hull is resolved into components perpendicular to the centerline $K$ and parallel to the centerline $D$. The magnitude of $S$ is assumed to vary with the square of the relative wind, $V_r$, and the sine of the sail angle of attack, $\alpha$:

$$S = C_1 V_r^2 \sin\alpha ,$$

where $C_1$ is a constant and $V_r$ and $\alpha$ are as defined in Figure 1.3.2. The drag is assumed to vary with the square of the boat velocity, $V$:

$$D = C_2 V^2 ,$$

where $C_2$ is a constant. For equilibrium of forces parallel to the centerline, we have

$$D = S \sin\theta .$$

*Show that*: (a) For given $\psi$, maximum $V$ is obtained when $\alpha = \theta$. (b) The maximum velocity for $\psi = 180°$ (running before the wind) is $W\mu/(1+\mu)$ and is obtained when $\theta = 90°$, where $\mu^2 = C_1/C_2$. (c) The maximum upwind velocity, $V\cos\psi$, is equal to $W\mu/4$ and is obtained when the sail setting and the heading are chosen to be

$$\theta \cong [(\mu+2)^2 + 4]^{-1/2} , \qquad \psi \cong 45° .$$

Assume for this part of the problem that $\alpha$ and $\theta$ are small angles so that $\sin\alpha \cong \alpha$, $\sin\theta \cong \theta$, $\cos\alpha \cong 1$, $\cos\theta \cong 1$.

Prob.6

**Problem 6.** *Angle of attack and bank angle for maximum lateral range glide.* A quasisteady approximation for gliding turns of a low-speed (subsonic) glider, made with constant angle of attack and constant bank angle, gives lateral gliding range, $y_f$, as

$$y_f = r(1 - \cos\beta_f) ,$$

where

$$r = \frac{\ell}{\alpha}\frac{\cos^2\gamma}{\sin\sigma} = \text{radius of the helix},$$

$$\beta_f = \frac{z_0}{\ell}\frac{\alpha\sin\sigma}{\sin\gamma\cos\gamma} = \text{final heading angle},$$

$$\gamma = \tan^{-1}\left[\left(\alpha + \frac{\delta^2}{4\alpha}\right)\sec\sigma\right] = \text{gliding helix angle},$$

and

$\alpha = \eta\bar{\alpha}$; $\bar{\alpha}$ = angle of attack,
$\sigma$ = bank angle
} (decision parameters),

$z_o$ = initial altitude,

$\ell = \dfrac{2m\eta}{\rho S C_{L_\alpha}}$ = characteristic length ($\cong$ 10 ft for typical sailplane),

$\delta = 2(\eta C_{D_0}/C_{L_\alpha})^{1/2}$ = minimum drag to lift ratio ($\cong \frac{1}{30}$ for typical sailplane),

$\eta$ = efficiency factor $(0 < \eta < 1)$.

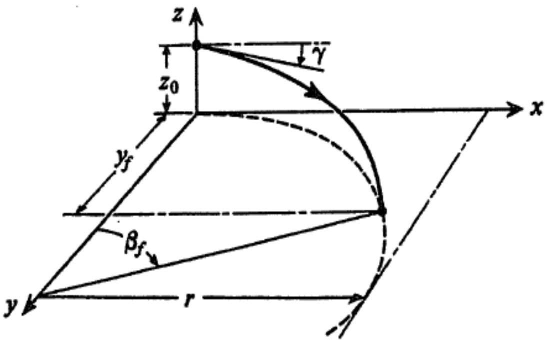


**Figure 1.3.3.** Geometry of flight path for lateral turn.

*Show that* the maximum value of $y_f$ for a given $z_o$ is obtained when we have

$$\tan\frac{\beta_f}{2} = \frac{\beta_f}{1 + (4\beta_f^2/\zeta^2)},$$

which may be regarded as a transcendental equation for $\beta_f$ as a function of $\zeta = z_o/\ell$. The corresponding values of $\sigma$, $\alpha$, and $\gamma$ are obtained from

$$\tan\sigma = \frac{2\beta_f}{\zeta}, \qquad \alpha = \frac{\delta}{2\sqrt{\cos 2\sigma}}, \qquad \gamma = 2\alpha\cos\sigma.$$

Assume that $\alpha$, $\gamma$, $\delta$ are $<<$ 1.

[NOTE 1. Within this same approximation, the maximum value of $x_f$ for given $z_o$ is

$$x_f = z_o/\delta$$

and is obtained with

$$\alpha = (1/2)\delta, \qquad \sigma = 0 \Rightarrow \tan\gamma = \delta.]$$

[NOTE 2: Further definition of symbols:

$m$ = mass of glider, $V$ = velocity,
$\rho$ = density of the atmosphere (approximated as constant in this problem),
$C_{L_\alpha}$ = lift coefficient slope,
$C_{D_0}$ = zero-lift drag coefficient
$S$ = reference area for coefficients,

$$\text{Lift} = C_{L_\alpha}\bar{\alpha}\frac{\rho V^2}{2}S, \qquad \text{Drag} = (C_{D_0} + \eta C_{L_\alpha}\bar{\alpha}^2)\frac{\rho V^2}{2}S.]$$

Prob.7

**Problem 7.** ***Maximum steady rate of climb for an aircraft.*** For the problem stated in Example 2 of Section 1.2, find the maximum steady rate of climb at sea level and at altitudes of 10,000 ft, 20,000 ft, 30,000 ft, and 40,000 ft for an airplane with weight $mg = 34{,}000$ lbs and wing area $S = 530$ ft². The lift, drag, and thrust characteristics are given below:

$$L = C_{L_\alpha}\alpha\frac{\rho V^2}{2}S\,, \qquad D = (C_{D_0} + \eta C_{L_\alpha}\alpha^2)\frac{\rho V^2}{2}S\,.$$

Here $C_{L_\alpha}$, $C_{D_0}$, and $\eta$ are functions of Mach number $M \equiv V/c$, as shown in Figures 1.3.4 and 1.3.5; $c$ = speed of sound and $\rho$ = density of the air, both of which are functions of altitude, that is, $c = c(h)$, $\rho = \rho(h)$. These functions are given in Table 1.3.1. The thrust, $T$, at full throttle, is a function of Mach number and altitude, as shown in Figure 1.3.5. Use $\epsilon = 3°$.

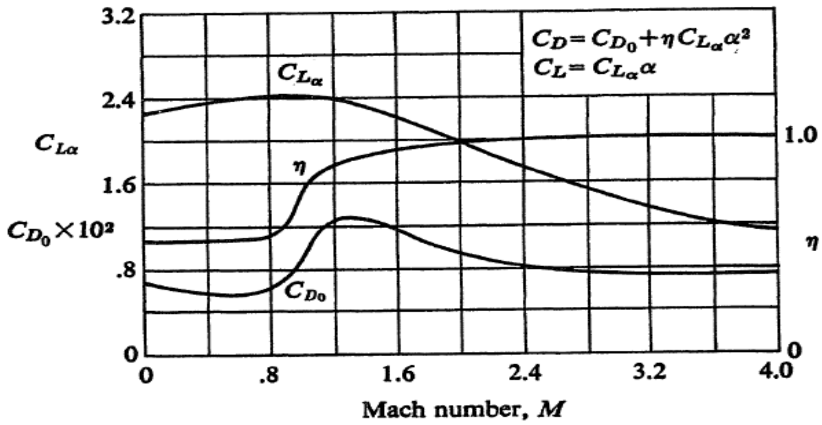


**Figure 1.3.4.** Drag and lift coefficients as function of Mach number.

Find, also, the altitude at which the maximum rate of climb is zero. This is called the "ceiling" of the airplane.

Table 1.3.1. Air density and speed of sound variation with altitude

| Altitude ~$h$, ft | Speed of sound ~$C$, ft/sec | Air density ~$\rho$, slugs/ft³ |
|---|---|---|
| 0 | 1,116 | 2,377 × $10^{-6}$ |
| 5,000 | 1,097 | 2,048 |
| 10,000 | 1,077 | 1,755 |
| 15,000 | 1,057 | 1,496 |
| 20,000 | 1,037 | 1,266 |
| 25,000 | 1,016 | 1,065 |
| 30,000 | 994.7 | 889.3 |
| 36,090 | 968.1 | 706.1 |
| 40,000 | 968.1 | 585.1 |
| 45,000 | 968.1 | 460.1 |
| 50,000 | 968.1 | 361.8 |
| 55,000 | 968.1 | 284.5 |
| 60,000 | 968.1 | 223.8 |
| 70,000 | 968.1 | 138.4 |
| 80,000 | 968.1 | 85.56 |
| 82,020 | 968.1 | 77.64 |
| 90,000 | 984.2 | 51.51 |
| 100,000 | 1,004 | 31.38 |

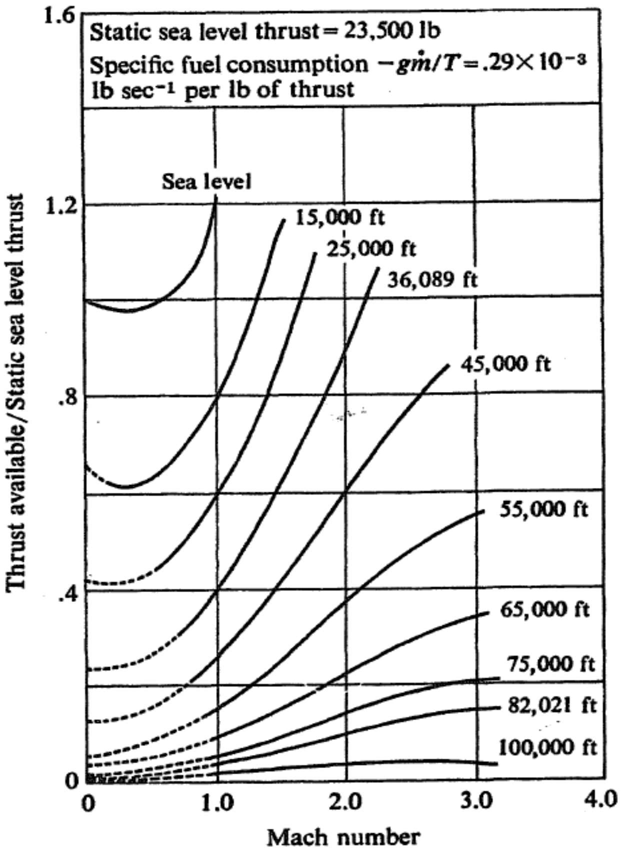


**Figure 1.3.5.** Thrust as function of Mach number and altitude at full throttle.

Prob.8 Problem 8. *Minimum fuel turn at constant altitude.* A steady turn ($\dot{V} = 0$ , $\dot{r} = 0$) at constant altitude is described by

$$(C_{D_0} + \eta C_{L_\alpha} \bar{\alpha}^2) \frac{\rho V^2}{2} S = T \qquad \text{(drag = thrust)},$$

$$mg = C_{L_\alpha} \bar{\alpha} \frac{\rho V^2}{2} S (\cos \sigma) \qquad \text{(weight = vertical component of lift)},$$

$$mV\dot{\beta} = C_{L_\alpha} \bar{\alpha} \frac{\rho V^2}{2} S (\sin \sigma) \qquad \text{(turn rate} \approx \text{horizontal component of lift)},$$

where

$\bar{\alpha}$ = angle-of-attack, $\sigma$ : bank angle } decision parameters

and the rest of the symbols are as defined in Problem 6.

Find $\alpha \triangleq \eta\bar{\alpha}$ and $\sigma$ to minimize the fuel in making a turn from $\beta = \beta_o$ to $\beta = \beta_f$ where fuel is proportional to

$$J \triangleq \int_0^{t_f} T\, dt \equiv \int_{\beta_0}^{\beta_f} \frac{T\, d\beta}{\dot{\beta}} \equiv \frac{T}{\dot{\beta}} (\beta_f - \beta_o);$$

that is, minimize

$$\frac{T}{\dot{\beta}} = \frac{(C_{D_0} + \eta C_{L_\alpha} \bar{\alpha}^2)\, mV}{C_{L_\alpha} \bar{\alpha} \sin \sigma}$$

subject to $mg = C_{L_\alpha} \bar{\alpha} \left(\frac{\rho V^2}{2}\right) S \cos \sigma$.

ANSWER. $\alpha = (\sqrt{3}/2)\delta$ , $\sigma = \cos^{-1}(1/\sqrt{3}) = 54.7°$ , where $\delta = 2\sqrt{\eta C_{D_0}/C_{L_\alpha}}$.

Note that this implies $V = \sqrt{2gl/\delta}$ , $\frac{L}{D} = (\sqrt{3}/2)(1/\delta)$ and $T = 2mg\,\delta$ , where $\ell = \frac{2m\eta}{C_{L_\alpha} \rho S}$.

Prob.9

**Problem 2.** *Aircraft cruise condition for minimum fuel consumption.* For the airplane described in Example 2, Section 1.2, and in Problem 7, Section 1.3, find the steady level-flight ($\gamma = 0$) condition for minimum fuel consumption per unit distance. Assume constant specific fuel consumption, $\sigma = .29 \times 10^{-3}$ lb sec$^{-1}$ per lb of thrust, so that fuel consumption per unit distance is given by

$$J = \frac{\sigma T}{V},$$

where

$$T \le T_{max}(V,h)$$

and $T_{max}(V,h)$ is as given graphically in Problem 7, Section 1.3.

The constraint equations are

$$L - mg + T \sin(\alpha + \epsilon) = 0, \qquad D - T\cos(\alpha + \epsilon) = 0,$$

where $L = L(V,h,\alpha)$, $D = D(V,h,\alpha)$ are as given in Problem 7, Section 1.3.

Prob.10

**Problem 3.** Write out a mathematical proof of the geometrical argument of Figure 1.7.2. In particular, show why $\lambda \ge 0$.

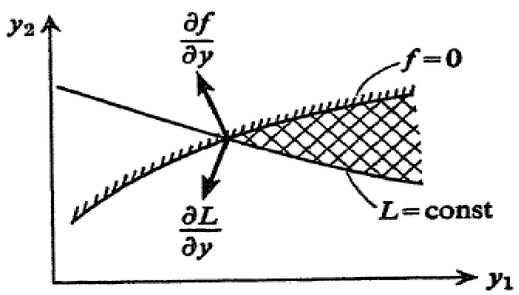


**Figure 1.7.2.** Two-dimensional illustration showing the necessity of Equation (1.7.10).

$$\frac{\partial L}{\partial y} \text{ parallel to } \frac{\partial f}{\partial y} \text{ and pointing in opposite directions.} \qquad (1.7.10)$$

Prob.11

For the following cost function, $F = x^2 + y^2 - 6xy - 4x - 5y$

(a) Show analytically how to minimize the cost subject to the constraints,

$$f_1 : -2x + y + 1 \ge 0$$

$$f_2 : x + y - 4 \le 0$$

$$f_3 : x \ge -1$$

(b) How is the optimal cost affected if the constraint $f_1$ is changed to,

$$f_1' = -2x + y + 1.1 \ge 0$$

Estimate this difference and explain your answer.

(c) Write a Matlab script to confirm your results in parts (a) and (b)

Example 2. *Maximum steady rate of climb for aircraft.* The net force on an aircraft maintaining a steady rate of climb must be zero. If we choose force components parallel and perpendicular to the flight path (see Figure 1.2.2), this requires that

$$f_1(V,\gamma,\alpha) = T\cos(\alpha+\epsilon) - D - mg\sin\gamma = 0\,,$$

$$f_2(V,\gamma,\alpha) = T\sin(\alpha+\epsilon) + L - mg\cos\gamma = 0\,,$$

where

$V$ = velocity,
$\gamma$ = flight path angle to horizontal,
$\alpha$ = angle-of-attack,
$m$ = mass of aircraft,
$g$ = gravitational force per unit mass,
$\epsilon$ = angle between thrust axis and zero-lift axis,

and, at a given altitude,

$L = L(V,\alpha)$ = lift force,
$D = D(V,\alpha)$ = drag force,
$T = T(V)$ = thrust of engine.

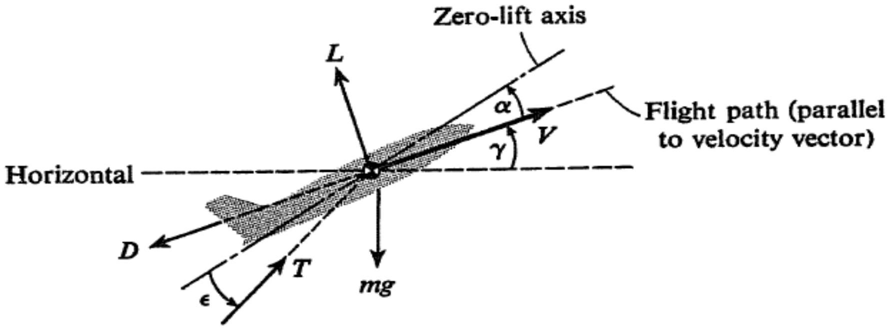


**Figure 1.2.2.** Force equilibrium of climbing aircraft.

The rate of climb is simply

$$V\sin\gamma\,.$$

We choose $V$ and $\gamma$ as state parameters and $\alpha$ as the control parameter since, at a given altitude, a choice of $\alpha$ determines $V$, $\gamma$ from the two force equilibrium relations.

The $H$ function is

$$H = V\sin\gamma + \lambda_1(T\cos(\alpha+\epsilon) - D - mg\sin\gamma) + \lambda_2(T\sin(\alpha+\epsilon) + L - mg\cos\gamma)\,.$$

Hence, the necessary conditions for a stationary value of rate of climb are:

$$f_1 = T(V)\cos(\alpha+\epsilon) - D(V,\alpha) - mg\sin\gamma = 0\,,$$

$$f_2 = T(V)\sin(\alpha+\epsilon) - L(V,\alpha) - mg\cos\gamma = 0\,,$$

$$\frac{\partial H}{\partial V} = \sin\gamma + \lambda_1\left[\frac{\partial T}{\partial V}\cos(\alpha+\epsilon) - \frac{\partial D}{\partial V}\right] + \lambda_2\left[\frac{\partial T}{\partial V}\sin(\alpha+\epsilon) + \frac{\partial L}{\partial V}\right] = 0\,,$$

$$\frac{\partial H}{\partial \gamma} = V\cos\gamma - \lambda_1\, mg\cos\gamma + \lambda_2\, mg\sin\gamma = 0\,,$$

$$\frac{\partial H}{\partial \alpha} = \lambda_1\left[-T\sin(\alpha+\epsilon) - \frac{\partial D}{\partial \alpha}\right] + \lambda_2\left[T\cos(\alpha+\epsilon) + \frac{\partial L}{\partial \alpha}\right] = 0\,.$$

These five equations for the five unknowns, $V$, $\gamma$, $\alpha$, $\lambda_1$, and $\lambda_2$, will, in general, have to be solved numerically for realistic lift, drag, and thrust functions.