Lecture 2: Dynamic Programming

Note Title 11-06-2008

The Optimal Control Problem:

Find $u^*(t) \in \mathcal{U}$ which causes the system

$$\dot{x} = f(x(t), u(t), t) \quad \text{to}$$

follow a trajectory $x^*(t)$ that minimizes the performance measure

$$J = \phi[x(t_f), t_f] + \int_{t_0}^{t_f} L(x(t), u(t), t)\,dt$$

Some Typical J's

Minimum Time Problem: Transfer an arbitrary initial state $x(t_0)$ to a specified target set S in minimum time.

$$J = t_f - t_0 = \int_{t_0}^{t_f} dt \quad \Bigg| \quad \begin{array}{l} \phi = 0 \\ L = 1 \end{array}$$

e.g.s: interception, attack, slew mode of radar & gun system

Terminal Control: Minimize the deviation of the final state from a desired value $r(t_f)$.

Possible J's: $J = \sum_{i=1}^{n} \left[x_i(t_f) - r_i(t_f)\right]^2$

$$= [x(t_f) - r(t_f)]^T [x(t_f) - r(t_f)]$$

$$= \| x(t_f) - r(t_f) \|_2^2 \rightarrow \text{2-norm squared}$$

<u>More generally:</u>

$$J = [x(t_f) - r(t_f)]^T H [x(t_f) - r(t_f)]$$

$$= \| x(t_f) - r(t_f) \|_H^2 \qquad \text{with } H \geq 0$$

Q. What is the significance of h_{ii} (assume diagonal H)?

Q. Can the elements of H be used for normalization of various states?

<u>Minimum effort</u>: Transfer from arbitrary initial state $x(t_0)$ to a specified target set S with min. expenditure of control effort.

<u>Possible J's</u>: $J = \int_{t_0}^{t_f} \left[\sum \beta_i |u_i(t)| \right] dt$ ↳ magnitude

$$J = \int_{t_0}^{t_f} [u^T(t) R u(t)] dt \rightarrow \text{energy}$$

$$\boxed{R > 0}$$

<u>Tracking Problems</u>: Maintain $x(t)$ as close as possible to the desired

state $r(t)$ in $[t_0, t_f]$

$$J = \int_{t_0}^{t_f} \|x(t) - r(t)\|_Q^2 \, dt \quad , \quad Q \geq 0$$

This J is fine if $|u_i(t)| \leq \nu$, $\nu > 0$

Q. Otherwise what happens to the input?

A. It may become unbounded

<u>Modified J:</u>

$$J = \int_{t_0}^{t_f} \left[\|x(t) - r(t)\|_Q^2 + \|u(t)\|_R^2 \right] dt$$

$$Q \geq 0 \quad , \quad R > 0$$

<u>Another version</u> (with extra emphasis on terminal state)

$$J = \|x(t_f) - r(t_f)\|_H^2 + \int_{t_0}^{t_f} \left[\|x(t) - r(t)\|_Q^2 + \|u(t)\|_R^2 \right] dt$$

<u>$H, Q \geq 0 \quad , \quad R > 0$</u>

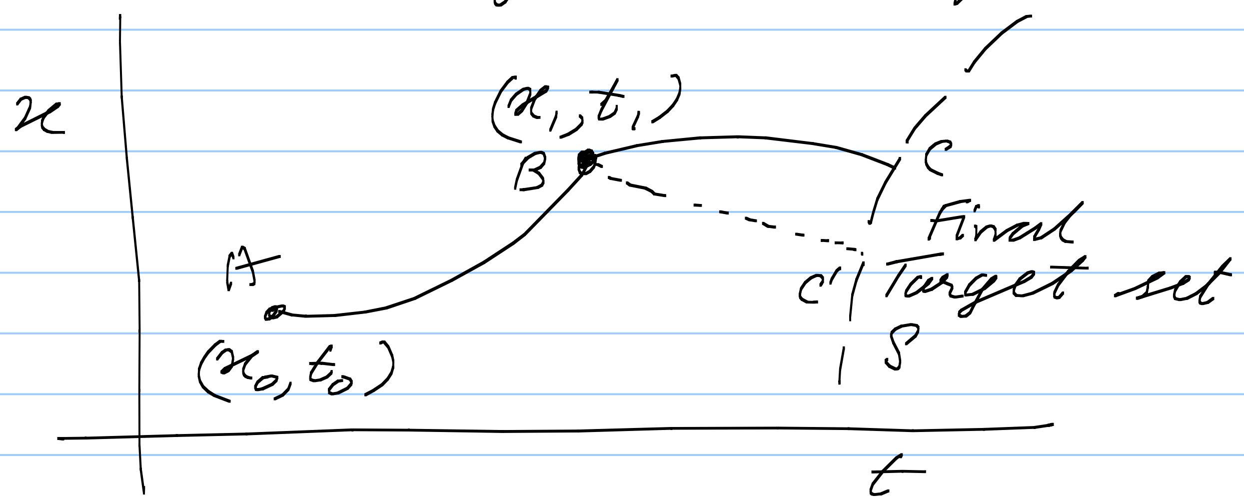
<u>Regulator Problem</u>: Special case where $r(t) = 0 \quad \forall t \in [t_0, t_f]$

$$J = \int_{t_0}^{t_f} \left[x^T(t) Q x(t) + u^T(t) R u(t) \right] dt$$

$$Q \geq 0, \ R > 0$$

Dynamic Programming

Principle of Optimality: If the optimal solution for a problem passes through some intermediate point (x_1, t_1), then the optimal solution of the same problem starting at (x_1, t_1) must be the continuation of the same path



Proof: Let J_{AC} be the min. perf. measure from (x_0, t_0).

$$J_{BC'} < J_{BC}$$

But $J_{AC} = J_{AB} + J_{BC}$

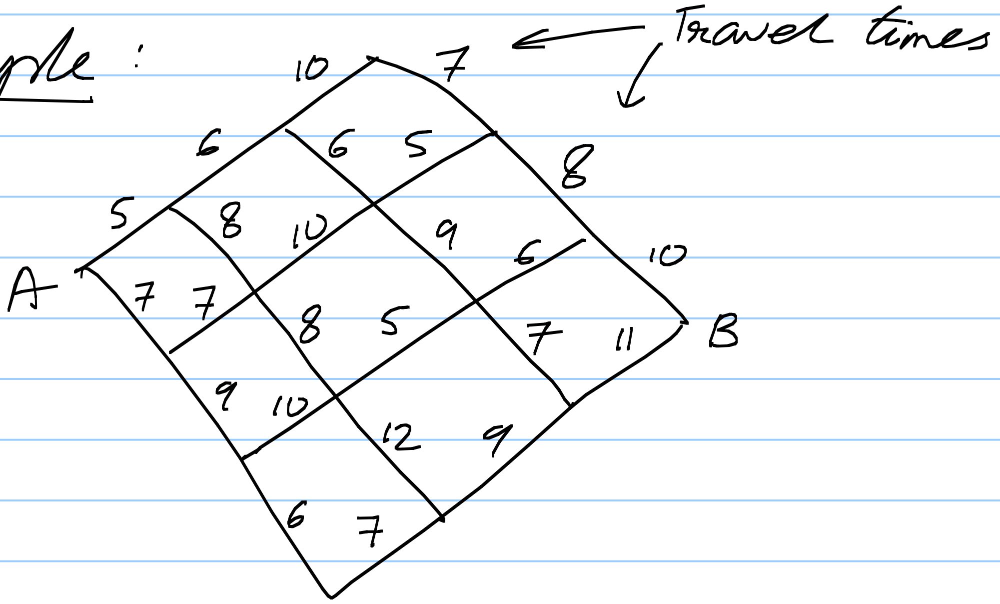
$$J_{AC'} = J_{AB} + J_{BC'} < J_{AC}$$

↳ contradiction since J_{AC} is min.

Uses:

1) Numerical solⁿ procedure
2) Theoretical structure of control law

Example :

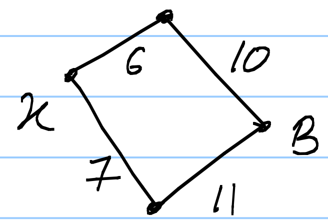


Travel from A to B in the min. time possible.

Option 1 : Check all 20 options.

Option 2 : Start at B and work backwards

Step 1 :



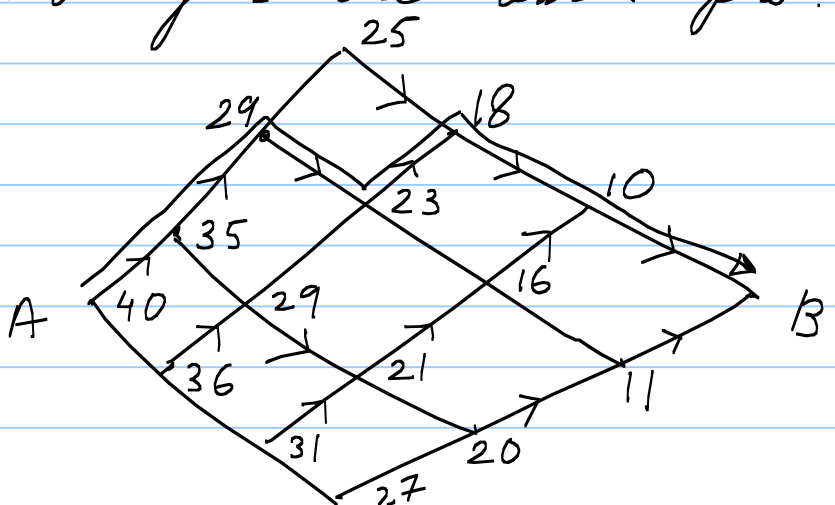
From x :

$6+10=16$

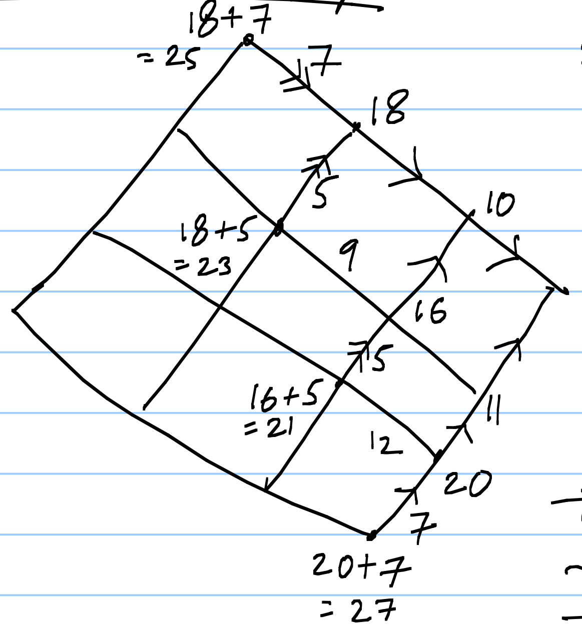
$7+11=18$

Hence any path going through x necessarily go through 6, 10

Repeat for all other pts:



Intermediate Step



Advantage: Had to check only 15 members

→ If n = no. of segments on side ($n=3$ here)

→ Exhaustive search $\sim \dfrac{\lfloor 2n}{(\lfloor n)^2} = \dfrac{\lfloor 6}{\lfloor 3 \lfloor 3} = 20$

→ DP computation → $(n+1)^2 - 1 = 15$

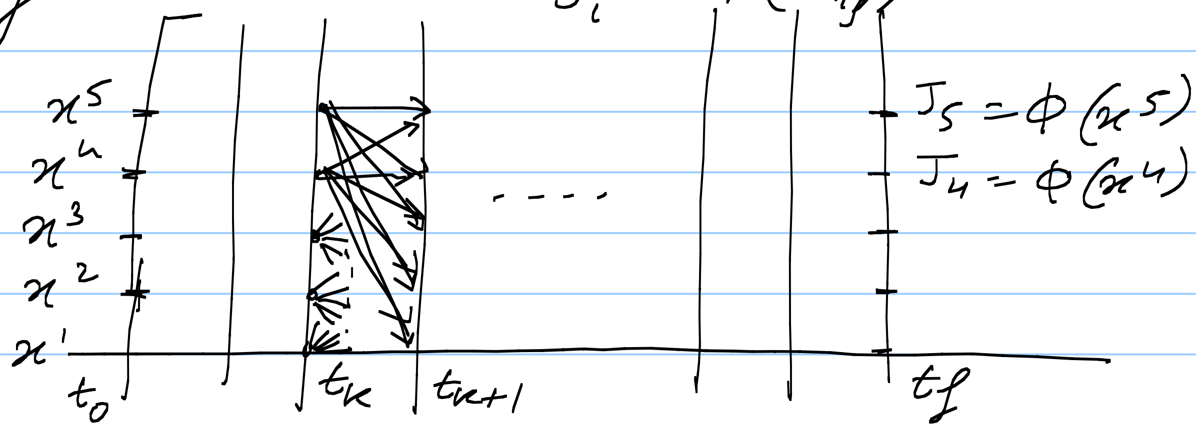
Exercise: Derive these formula.

Solving Optimal Control Problems with DP:

$$\min J = \Phi[x(t_f)] + \int_{t_0}^{t_f} L(x(t), u(t), t)\, dt$$

s.t. $\dot{x} = f(x, u, t)$, $x(t_0)$ = fixed
t_f = fixed

Step 1: → Develop grid over space/time
→ check $x^i(t_f)$ and evaluate final costs → $J_i^* = \Phi(x^i_{t_f})$



Step 2: Back up 1 step in time and consider all possible ways of completing the problem

→ Approximation of the Integral

Suppose 1) We are at x^i at time t_k
2) We apply control u_k^{ij} to move to x^j at time $t_{k+1} = t_k + \Delta t$

Cost $$\Delta J(x_k^i, x_{k+1}^j) = \int_{t_k}^{t_{k+1}} L(x(t), u(t), t)\,dt \Big|_{u(t)=u_k^{ij}}$$

$$\approx L(x_k^i, u_k^{ij}, t_k)\,\Delta t \quad \text{— (1)}$$

→ u_k^{ij} can be solved from st. eqn.

$$x_{k+1}^j \approx x_k^i + f(x_k^i, u_k^{ij}, t_k)\,\Delta t$$

↳ solve for u_k^{ij} (this can be a problem sometimes)

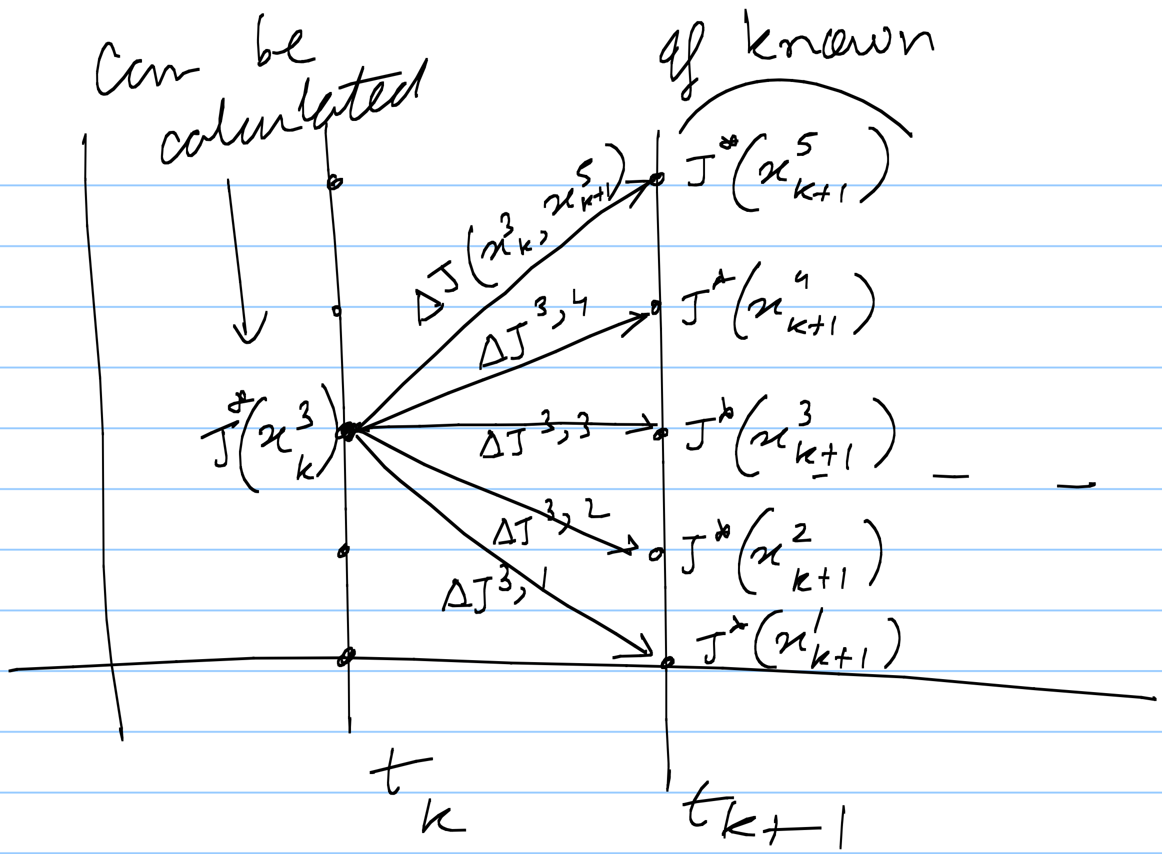
E.g. if $f(x,u,t) = f(x,t) + q(x,t)\,u$,

then
$$u_k^{ij} = q(x_k^i, t_k)^{-1}\left[\frac{x_{k+1}^j - x_k^i}{\Delta t} - f(x_k^i, t_k)\right]$$

Replace in (1) to get $\Delta J(x_k^i, x_{k+1}^j)$

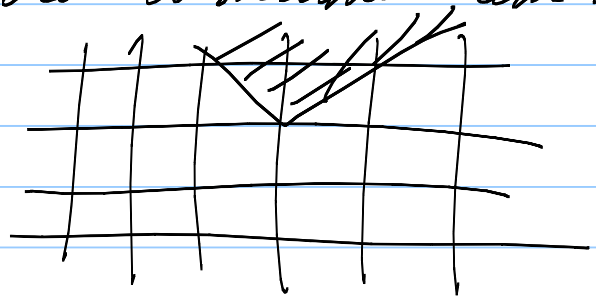
Then, if we know the optimal path from x_{k+1}^j, can establish optimal path (and cost) from x_k^i:

$$J^*(x_k^i) = \min_{x_{k+1}^j}\left[\Delta J(x_k^i, x_{k+1}^j) + J^*(x_{k+1}^j)\right]$$

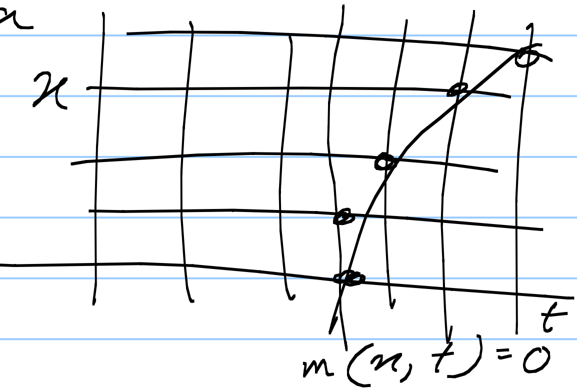


Repeat for each step backward in time until we reach $x(t_0)$, where only one of x is allowed. (since $x(t_0) = x_0$ is given.

Notes: 1) Control/state constraints can be handled



2) Free end time with additional end pt. constraints can be handled.
Let $m(x(t_f), t_f) = 0$
→ Start with a approximate $sol^{\underline{n}}$ of $m(x,t) = 0$ & work backwards.



3) scales badly for higher dimensional cases
4) Similar but different schemes for computation exist. (Section 3.6 in Kirk)

<u>Example</u>: (with a slightly different computational scheme)

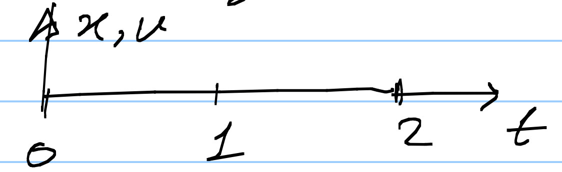
$$\min \quad J = x^2(2) + 2\int_0^2 u^2(t)\,dt$$

s.t. $\quad \dot{x} = u$

$0 \leq x \leq 1.5$

$-1 \leq u \leq 1$

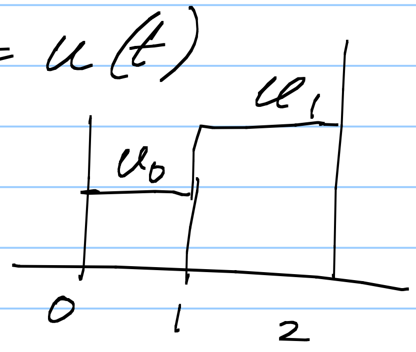
Discretize state on $t \in [0, 2]$, $\Delta t = 1$
$N = 2$.



Approximate the continuous system:

$$\dot{x} \simeq \frac{x(t+\Delta t) - x(t)}{\Delta t} = u(t)$$

$$x_{k+1} = x_k + (\Delta t)\,u_k$$

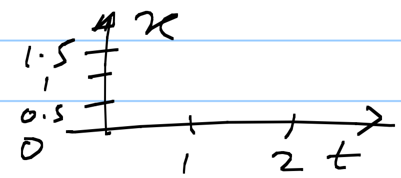


<u>Cost approximation</u>

$$J = x^2(2) + 2\sum_{k=0}^{1} u_k^2\,\Delta t$$

Quantize x_k between $0 \leq x_k \leq 1.5$

$$x_k \in \{0, 0.5, 1, 1.5\}$$



Quantize u_k (different from previous grap)

$|u_k| \leqslant 1$, $u(k) \in \{-1, -0.5, 0, 0.5, 1\}$

Start DP iteration : 1

x_2^j	$J_2^* = (x_2^j)^2$
0	0
0.5	0.25
1	1
1.5	2.25

$\Delta J_{12} = 2u_1^2$

Transition cost from $t=1$ to $t=2$

$u(1)$	$x_2^j = x_1^i + u(1)$			
x_1^i	0	0.5	1	1.5
0	0	0.5	1	1.5
0.5	-0.5	0	0.5	1
1	-1	-0.5	0	0.5
1.5	-1.5	-1	-0.5	0

ΔJ_{12}^{ij}	x_2^j			
x_1^i	0	0.5	1	1.5
0	0	0.5	2	∞
0.5	0.5	0	0.5	2
1	2	0.5	0	0.5
1.5	∞	2	0.5	0

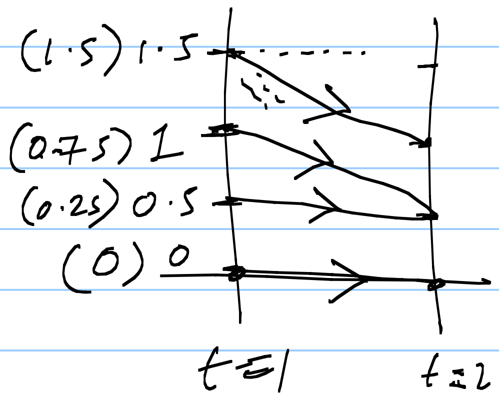
Q. What happens if $x_1^i \to x_2^j$ reqs u_k not in the quantized set?

Then $J_1 = \Delta_{12}^{ij} + J_2^*(x_2^j)$

J_1	x_2^j			
x_1^i	0	0.5	1	1.5
0	0	0.75	3	∞
0.5	0.5	0.25	1.5	4.25
1	2	0.75	1	2.75
1.5	∞	2.25	1.5	2.25

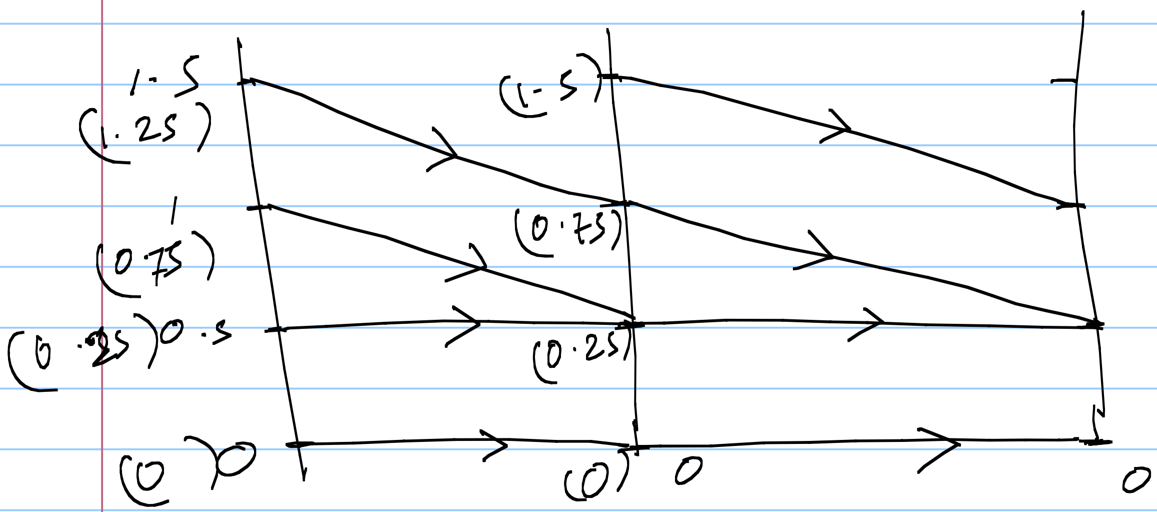
Best Actions

$x_1^i \to x_2^j$
$0 \to 0$
$0.5 \to 0.5$
$1 \to 0.5$
$1.5 \to 1$



Repeat process to find J_0. Best actions.

$x_0^i \to x_1^j$	J_0
$0 \to 0$	0
$0.5 \to 0.5$	0.25
$1 \to 0.5$	0.75
$1.5 \to 1$	1.25



Discrete LQR : Analytical Results with DP

$$x_{k+1} = A x_k + B u_k$$

Find $u^*(x_k, k)$ that minimizes

$$J = \frac{1}{2} x_N^T H x_N + \frac{1}{2} \sum_{k=0}^{N-1} \left[x_k^T Q x_k + u_k^T R u_k \right]$$

$H, Q \geq 0$ {real, sym, $n \times n$}

$R > 0$ { " " , $m \times m$}

Clearly $J_N^*[x_N] = \frac{1}{2} x_N^T H x_N =: \frac{1}{2} x_N^T P_N x_N$

Q. How to find $J_{N-1}^*(x_{N-1})$?

$$J_{N-1}^*[x_{N-1}] = \min_{u_{N-1}} \frac{1}{2} \left\{ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} \right\} + J_N^*(x_N)$$

$$= \min_{u_{N-1}} \frac{1}{2} \left\{ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + x_N^T P_N x_N \right\}$$

[Using $x_N = A x_{N-1} + B u_{N-1}$]

$$= \min_{u_{N-1}} \frac{1}{2} \left\{ x_{N-1}^T Q x_{N-1} + u_{N-1}^T R u_{N-1} + [A x_{N-1} + B u_{N-1}]^T P_N [A x_{N-1} + B u_{N-1}] \right\}$$

$$\frac{\partial J_{N-1}^*(x_{N-1})}{\partial u_{N-1}} = u_{N-1}^T R + [A x_{N-1} + B u_{N-1}]^T P_N B$$

Take transpose & set equal to zero:

$$[R + B^T P_N B] u_{N-1} + B^T P_N A x_{N-1} = 0$$

$$\Rightarrow u_{N-1}^* = -[\underbrace{R + B^T P_N B}_{\text{Pos. def} \Rightarrow \text{inv exists}}]^{-1} B^T P_N A x_{N-1}$$

$$= -F_{N-1} x_{N-1} \quad \{\text{feedback sol}^{\underline{n}}\}$$

Exercise: check whether this is a min

i.e. check $\dfrac{\partial^2 J^*_{N-1}[x_{N-1}]}{\partial u^2_{N-1}}$?

Replacing u^* in $J^*_{N-1}(x_{N-1})$:

$$J^*_{N-1}(x_{N-1}) = \frac{1}{2} x^T_{N-1} \{ Q + F^T_{N-1} R F_{N-1} + [A - BF_{N-1}]^T P_N [A - BF_{N-1}] \} x_{N-1}$$

$$= \frac{1}{2} x^T_{N-1} P_{N-1} x_{N-1}$$

Now using induction; at some time k, if the control is of the form

$$u^*_k = -F_k x_k \; ; \; F_k = [R + B^T P_{k+1} B]^{-1} B^T P_{k+1} A$$

and $J^*_k[x_k] = \frac{1}{2} x_k^T P_k x_k$

Going backwards to $k-1$,

$$J^*_{k-1}(x_{k-1}) = \min_{u_{k-1}} \left\{ \frac{1}{2} x^T_{k-1} Q x_{k-1} + u^T_{k-1} R u_k + J^*_k [x_k] \right\}$$

Then:

$$\frac{\partial J^*_{k-1}[x_{k-1}]}{\partial u_{k-1}} = u^T_{k-1} R + [A x_{k-1} + B u_{k-1}]^T P_k B$$

so $u^*_{k-1} = -[R + B^T P_k B]^{-1} B^T P_k A x_{k-1}$

$$= -F_{k-1} x_{k-1}$$

Then:

$$J^*_{k-1}[x_{k-1}] = \frac{1}{2} x^T_{k-1} P_{k-1} x_{k-1}$$

and
$$P_{k-1} = Q + F^T_{k-1} R F_{k-1} + [A - BF_{k-1}]^T P_k [A - BF_{k-1}]$$

So always true.

<u>Note:</u> 1) The optimal control is state feedback (but time varying)

2) P_k, F_k can be computed offline.

<u>Infinite time LQR</u> (will be proved later)

Assume (A, B) stabilizable.
As $N \to \infty$, $P_k \to P_{ss} > 0$
↳ constant + bdd

and satisfies:

$$P_{ss} = Q + A^T \left\{ P_{ss} - P_{ss} B [R + B^T P_{ss} B]^{-1} B^T P_{ss} \right\} A$$

$$F_{ss} = [R + B^T P_{ss} B]^{-1} B^T P_{ss} A$$

→ Discrete form of Algebraic Riccati Eqn.

→ Easy to solve numerically.

<u>DP in continuous time: HJB</u>

Consider the cost over $[t, t_f]$ where $t \leq t_f$

$$J\big(x(t), t, \underset{t \le \tau \le t_f}{u(\tau)}\big)$$

$$= \phi\big(x(t_f), t_f\big) + \int_t^{t_f} L\big(x(\tau), u(\tau), \tau\big)\, d\tau$$

$$J^*\big(x(t), t\big) = \min_{\substack{u(\tau) \\ t \le \tau \le t_f}} J\big(x(t), u(\tau), t\big)$$

Split $[t, t_f]$ into $[t, t+\Delta t]$ and $[t+\Delta t, t_f]$

$$J^*\big(x(t), t\big) = \min_{\substack{u(\tau) \\ t \le \tau \le t_f}} \Big\{ \phi\big(x(t_f), t_f\big) + \int_t^{t+\Delta t} L(x, u, \tau)\, d\tau + \underline{\int_{t+\Delta t}^{t_f} L(x, u, \tau)\, d\tau} \Big\}$$

By the principle of optimality:

$$J^*\big(x(t), t\big) = \min_{\substack{u(\tau) \\ t \le \tau \le t+\Delta t}} \Big\{ \int_t^{t+\Delta t} L\, d\tau + J^*(x+\Delta t, t+\Delta t) \Big\} \quad (1)$$

Assume $J^*\big(x(t+\Delta t), t+\Delta t\big)$ has bdd 2nd derivatives in both arguments:

$$J^*\big(x(t+\Delta t), t+\Delta t\big) \approx J^*\big(x(t), t\big) + \left[\frac{\partial J^*}{\partial t}\big(x(t), t\big)\right]\Delta t + \left[\frac{\partial J^*}{\partial x}\big(x(t), t\big)\right]\big\{x(t+\Delta t) - x(t)\big\} \quad (2)$$

For small Δt, (2) can be written as:

$$J^*(x(t+\Delta t), t+\Delta t) \approx J^*(x(t), t)$$
$$+ \frac{\partial J^*}{\partial t}(x(t), t)\Delta t + \frac{\partial J}{\partial x}(x(t), t) f(x(t), u(t), t)\Delta t$$

Substituting in (1):

$$J^*(x(t), t) = \min_{u(t) \in \mathcal{U}} \{ L(x(t), u(t), t)\Delta t$$
$$+ J^*(x(t), t) + \frac{\partial J^*}{\partial t}(x(t), t)\Delta t$$
$$+ \frac{\partial J}{\partial x}(x(t), t) f(x(t), u(t), t)\Delta t \}$$

$$\Leftrightarrow 0 = \frac{\partial J^*}{\partial t}(x(t), t) + \min_{u(t) \in \mathcal{U}} \{ L(x(t), u(t), t)$$
$$+ \frac{\partial J^*}{\partial x}(x(t), t) f(x(t), u(t), t) \}$$

This is a non-linear PDE: to be solved backward in time starting from $t = t_f$ & bdd. condition

$$J^*(x(t_f), t_f) = \Phi(x(t_f), t_f)$$

For simplicity define the Hamiltonian $\mathcal{H}$ as

$$\mathcal{H}\left(x(t), u(t), \frac{\partial J^*}{\partial x}, t\right)$$
$$= L(x(t), u(t), t) + \frac{\partial J^*}{\partial x}(x(t), t) f(x(t), u(t), t)$$

Then HJB is

$$-\frac{\partial J^*}{\partial t}(x(t),t) = \min_{u(t)\in \mathcal{U}} \mathcal{H}\left(x(t), u(t), \frac{\partial J^*}{\partial x}(x,t), t\right)$$

→ Powerful necessary + sufficient condition for optimality
→ have to solve analytically
→ numerical methods are available.

Example: $\dot{x} = x + u$

$$J = \frac{1}{4}x^2(T) + \int_0^{T} \frac{1}{4}u^2\,dt$$

T → specified

$$\mathcal{H}\left(x(t), u(t), \frac{\partial J^*}{\partial x}, t\right)$$
$$= \frac{1}{4}u^2(t) + \frac{\partial J^*}{\partial x}\left[x(t)+u(t)\right]$$

Since $u(t)$ is unconstrained,

$$\frac{\partial H}{\partial u} = \frac{1}{2}u(t) + \frac{\partial J^*}{\partial x}(x(t),t) = 0$$

or $$u^*(t) = -2\frac{\partial J^*}{\partial x}(x(t),t)$$

Substituting back into HJB,

$$0 = \frac{\partial J^*}{\partial t} + \frac{1}{4}\left[-2\frac{\partial J^*}{\partial x}\right]^2 + \left[\frac{\partial J^*}{\partial x}\right]\left[x(t)\right] - 2\left[\frac{\partial J^*}{\partial x}\right]^2$$

$$\Rightarrow \boxed{0 = \frac{\partial J^*}{\partial t} - \left[\frac{\partial J^*}{\partial x}\right]^2 + \left[\frac{\partial J^*}{\partial x}\right] x(t)} \quad \text{— } (\mathcal{D})$$

Bdd. condition: $J^*(x(T), T) = \frac{1}{4} x^2(T)$

Q. How to solve $(\mathcal{D})$?

A. A possible way is to guess the sol^n !!

Assume: $J^*(x(t), t) = \underbrace{\frac{1}{2} K(t)}_{} x^2(t)$

K → unknown scalar function of t.

$$\frac{\partial J^*}{\partial x} = K(t) x(t) \quad \text{—— } (1)$$

$$\Rightarrow u^*(t) = -2K(t)\, x(t)$$

Assume $K(T) = \frac{1}{2} \Rightarrow \underbrace{J^*(x(T), T) = \frac{1}{4} x^2(T)}_{\text{matches with req.}}$

Next:

$$\frac{\partial J^*}{\partial t} = \frac{1}{2} \dot{K}(t) x^2(t) \quad \text{—— } (2)$$

Using (1) & (2) in $(\mathcal{D})$,

$$0 = \frac{1}{2} \dot{K}(t)\, x^2(t) - K^2(t)\, x^2(t) + K(t)\, x^2(t)$$

This eqn. must be satisfied $\forall\, x(t) \Rightarrow$

$$\frac{1}{2} \dot{K}(t) - K^2(t) + K(t) = 0$$

Solving for $K(t) = \dfrac{e^{(T-t)}}{e^{(T-t)} + e^{-(T-t)}}$

$$u^* = -2\left[\frac{e^{T-t}}{e^{T-t}+e^{-(T-t)}}\right]x(t)$$

Continuous Linear Regulator

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

$$J = \frac{1}{2}x^T(t_f)Hx(t_f) + \int_{t_0}^{t_f}\frac{1}{2}\left[x^T(t)Qx(t) + u^T(t)R(t)u(t)\right]dt$$

$H(t), Q(t) \geq 0$

$R(t) > 0$ $\qquad t_0, t_f \leftarrow$ specified.

No constraints on $u(t), x(t)$

To use HJB: form the Hamiltonian:

$$\mathcal{H}\left(x(t), u(t), \frac{\partial J^*}{\partial x}, t\right) = \frac{1}{2}x^T(t)Q(t)x(t) + \frac{1}{2}u^T(t)R(t)u(t) + \frac{\partial J^*}{\partial x}(x(t),t)\left[A(t)x(t) + B(t)u(t)\right]$$

To find $u(t)$ which minimizes $\mathcal{H}$:

$$\frac{\partial \mathcal{H}}{\partial u} = 0 \Rightarrow \frac{\partial \mathcal{H}}{\partial u} = u(t)^T R(t) + \frac{\partial J^*}{\partial x}(x(t),t)B(t) = 0$$

or $\quad u^*(t) = -R(t)^{-1}B^T(t)\dfrac{\partial J^*}{\partial x}(x(t),t)^T$

[One can check $\dfrac{\partial^2 \mathcal{H}}{\partial u^2} = R(t) > 0$]

Substituting back in $\mathcal{H}$:

$$\mathcal{H}\left(x(t), u^*(t), \frac{\partial J^*}{\partial x}, t\right) = \frac{1}{2} x^T Q x +$$

$$\frac{1}{2} \frac{\partial J^*}{\partial x} B R^{-1} B^T \left[\frac{\partial J^*}{\partial x}\right]^T + \frac{\partial J^*}{\partial x}\left[A x - B R^{-1} B^T \left[\frac{\partial J^*}{\partial x}\right]^T\right]$$

$$= \frac{1}{2} x^T Q x + \frac{\partial J^*}{\partial x} A x - \frac{1}{2} \frac{\partial J^*}{\partial x} B R^{-1} B^T \left[\frac{\partial J^*}{\partial x}\right]^T$$

Then the HJB is:

$$0 = \frac{\partial J^*}{\partial t} + \frac{1}{2} x^T Q x - \frac{1}{2} \frac{\partial J^*}{\partial x} B R^{-1} B^T \left[\frac{\partial J^*}{\partial x}\right]^T$$
$$+ \frac{\partial J^*}{\partial x} A x$$

<u>To solve</u>: recall $J^*(x(t_f), t_f) = \frac{1}{2} x^T(t_f) H x(t_f)$

<u>A good guess</u>: $J^*(x(t), t) = \frac{1}{2} x^T(t) K(t) x(t)$

where $K(t) = K^T(t) > 0$

Substituting:

$$0 = \frac{1}{2} x^T \dot{K} x + \frac{1}{2} x^T Q x - \frac{1}{2} x^T K B R^{-1} B^T K x$$
$$+ \frac{1}{2} x^T K A x \quad \left(\begin{matrix}\text{using} \\ K = K^T\end{matrix}\right)$$

Now $KA = \frac{1}{2}\underbrace{\left[KA + (KA)^T\right]}_{\text{Symmetric part}} + \frac{1}{2}\underbrace{\left[KA - (KA)^T\right]}_{\text{Asymmetric Part}}$

$$x^T\left[KA - (KA)^T\right] x = \underbrace{x^T (KA) x}_{\text{scalar}} - \underbrace{x^T (KA)^T x}_{\text{scalar}} = 0$$

Thus:

$$0 = \frac{1}{2} x^T \dot{K} x + \frac{1}{2} x^T Q x - \frac{1}{2} x^T K B R^{-1} B^T K x + \frac{1}{2} x^T K A x + \frac{1}{2} x^T A^T K x$$

This eqn must hold for all $x(t)$: so

$$\boxed{\begin{aligned} 0 = \dot{K}(t) + Q(t) - K(t) B(t) R^{-1}(t) B^T(t) K(t) \\ + K(t) A(t) + A^T(t) K(t) \\ \text{with bdd. condition } \quad K(t_f) = H \end{aligned}}$$

↳ Differential Riccati Eqn

1> Solve for $K(t)$ numerically
2> Since $K(t)$ is symmetric only $n(n+1)/2$ differential eqns need to be solved.

3> $$u^*(t) = -\underbrace{R^{-1}(t) B^T(t) K(t)} x(t)$$

$$u^*(t) = -F(t)\, x(t)$$

linear state feedback

For A, B, C constant and $t_f \to \infty$

Assume: (A, B) stabilizable, the DRE settles down to a steady state value, which is the solution of:

$$K_{ss} A + A^T K_{ss} + Q - K_{ss} B R^{-1} B^T K_{ss} = 0$$

↳ Algebraic Riccati Eq.

$$u(t) = -R^{-1}B^{T}K_{ss}\, x(t) = -F_{ss}\, x(t)$$

<u>Scalar LQR Example</u> : $\dot{x} = ax + bu$

$Q > 0,\ R > 0$

$$J = \int_0^\infty \left[Q x^2(t) + R u^2(t) \right] dt$$

The ARE: $2aK_{ss} + Q - \dfrac{K_{ss}^2 b^2}{R} = 0$

$$K_{ss} = \frac{a \pm \sqrt{a^2 + \frac{b^2 Q}{R}}}{b^2/R}$$

only + is valid.

Then $F_{ss} = -R^{-1}B^{T}K_{ss} = \dfrac{a + \sqrt{a^2 + \frac{b^2Q}{R}}}{b}$

The c.l. eqns: $\dot{x} = (a - bK_{ss})x$

$$= \left(a - a - \sqrt{a^2 + \frac{b^2Q}{R}}\right)x$$

$$= -\sqrt{a^2 + \frac{b^2Q}{R}}\; x \Leftarrow \text{stable}$$

Q. How does the $\frac{Q}{R}$ ratio affect the c.l.?