Lecture 3: Calculus of Variations

Review key definitions: Functions, functional, Linearity, Norms of functions.

Functional : $J(x(t)) : x(t) \longmapsto R$

e.g. $J(x(t)) = \int_{t_0}^{t_f} x^2(t)\,dt$

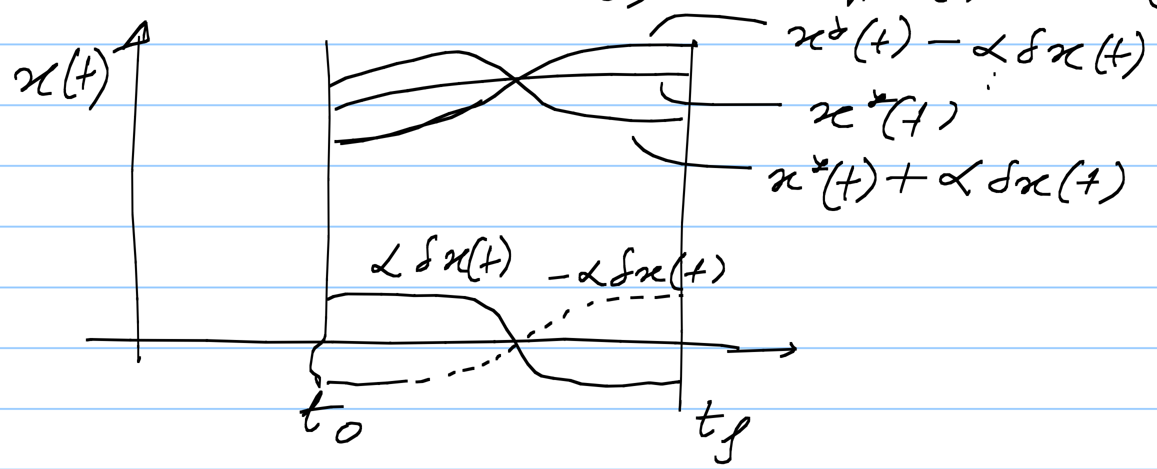
Norm of a function $\Leftrightarrow$ distance between two functions

$$d = \| x_1(t) - x_2(t) \|$$

Common norm: $\|x(t)\|_2 = \left[\int_{t_0}^{t_f} x^T(t)\, x(t)\, dt \right]^{1/2}$

Maximum/Minimum of a functional

$J(x(t))$ has a local min at $x^*(t)$ if $J(x(t)) \geq J(x^*(t))$ for all admissible $x(t)$ in $\|x(t) - x^*(t)\| \leq \varepsilon$



Increment of a functional

$$\Delta J = J(x(t) + \delta x(t)) - J(x(t))$$

A variation of the functional is a linear approximation of the increment:

$$\Delta J(x, \delta x) = \delta J(x, \delta x) + \underset{\swarrow \text{ scalar}}{g(x,\delta x)} \cdot \|\delta x\|$$

where δJ is linear in δx.

If $\lim_{\|\delta x\| \to 0} \{ \underbrace{g(x, \delta x)}_{\to \text{scalar}} \} = 0$ then J is said to be differentiable on x and δJ is the variation of J at x

E.g.1: $J(x) = \int_0^1 [x^2(t) + 2x(t)]dt$

where $x(t): [0,1] \to \mathbb{R}$ continuous.

Increment: $\Delta J(x, \delta x) = J(x + \delta x) - J(x)$

$$= \int_0^1 (x(t) + \delta x(t))^2 dt + 2\int_0^1 (x(t) + \delta x(t))dt - \int_0^1 [x^2(t) + 2x(t)]dt$$

$$= \int_0^1 \{ \underbrace{[2x(t) + 2]\delta x(t)}_{\text{linear in } \delta x(t)} dt + \int_0^1 \underbrace{[\delta x(t)]^2}_{g(x,\delta x)\|\delta x\|} \} dt$$

Let $\|\delta x(t)\| := \max_{0 \le t \le 1} |\delta x(t)|$

Then $\frac{\|\delta x(t)\|}{\|\delta x(t)\|} \int_0^1 [\delta x(t)]^2 dt = \|\delta x\| \cdot \underbrace{\int_0^1 \frac{[\delta x(t)]^2}{\|\delta x(t)\|} dt}_{g(x,\delta x)}$

Verify whether $\lim_{\|\delta x\| \to 0} g(x, \delta x) = 0$?

$$\int_0^1 \frac{[\delta x(t)]^2}{\|\delta x(t)\|} dt = \int_0^1 \frac{|\delta x(t)| \cdot |\delta x(t)|}{\|\delta x(t)\|} dt \leq \int_0^1 |\delta x(t)| dt$$

Now if $\|\delta x(t)\| \to 0$, $\delta x(t) \to 0$ for all $t \in [0,1]$

$$\Rightarrow \lim_{\|\delta x(t)\| \to 0} \left\{ \int_0^1 |\delta x(t)| dt \right\} = 0$$

Hence $\delta J(x, \delta x) = \int_0^1 \{ [2x(t) + 2] \delta x(t) \} dt$

Q. How else to get this expression?
A. Taylor series

<u>Fundamental Thm of Calculus of Variations</u>

<u>Thm:</u> If x^* is an extremal, the variation of J must vanish at x^* i.e $\delta J(x^*, \delta x) = 0$ for all admissible δx.

<u>Proof:</u> Assume x^* is an extremal but $\delta J(x^*, \delta x) \neq 0$ for some δx

$$\Delta J(x^*, \delta x) = \delta J(x^*, \delta x) + g(x^*, \delta x) \cdot \|\delta x\|$$

where $q(x^*, \delta x) \to 0$ as $\|\delta x\| \to 0$

Hence $\exists\, \varepsilon > 0$ s.t. $\Delta J(x^*; \delta x) \approx \delta J(x^*; \delta x)$
$\Rightarrow \text{sign}(\Delta J(x^*; \delta x)) = \text{sign}(\delta J(x^*; \delta x))$
$\forall\ \|\delta x\| < \varepsilon$.

Select $\delta x = \alpha\, \delta x_1$ where $\alpha > 0$ & $\|\alpha\, \delta x_1\| < \varepsilon$. Let $\delta J(x^*; \alpha\, \delta x_1) < 0$

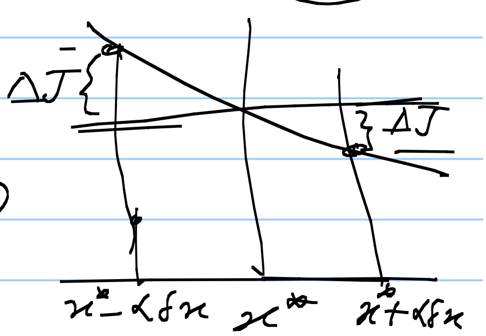
Since δJ is a linear functional of δx,
$$\delta J(x^*, \alpha\, \delta x_1) = \alpha\, \delta J(x^*, \delta x_1) < 0 \longrightarrow (1)$$
$$\Rightarrow \Delta J(x^*, \alpha\, \delta x_1) < 0 \longrightarrow (*)$$

Now consider $\delta x = -\alpha\, \delta x_1$. Clearly $\|-\alpha\, \delta x_1\| < \varepsilon$. Hence
$$\text{sign}(\Delta J(x^*, -\alpha\, \delta x_1)) = \text{sign}(\delta J(x^*, -\alpha\, \delta x_1))$$

But $\delta J(x^*, -\alpha\, \delta x_1) = -\alpha\, \delta J(x^*, \delta x_1)$. $\longrightarrow (2)$

From (1) & (2),
$$\delta J(x^*, -\alpha\, \delta x_1) > 0$$
$$\Rightarrow \Delta J(x^*, -\alpha\, \delta x_1) > 0 \longrightarrow (**)$$

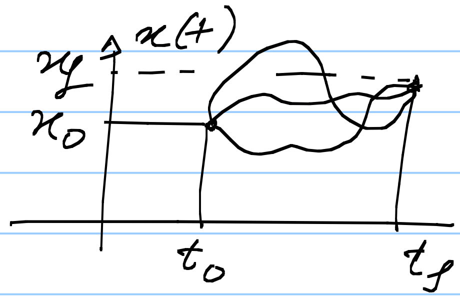
Contradiction in (*) & (**)



Scalar Variational Example

$$J(x(t)) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

Assume: 1) g has continuous first and second partial derivatives w.r.t $x(t)$, $\dot{x}(t)$, t.



2) t_0, t_f fixed
3) x_0, x_f fixed.

$$\Delta J(x, \delta x) = J(x+\delta x) - J(x)$$

$$= \int_{t_0}^{t_f} g(x(t) + \delta x(t), \dot{x}(t) + \delta \dot{x}(t), t)\,dt - \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\,dt$$

Note that $\Delta J(x, \delta x, \dot{x}, \delta \dot{x})$ is not written since $\delta \dot{x} = \frac{d}{dt}[\delta x]$, $\dot{x} = \frac{d}{dt} x$

Expanding in a Taylor series:

$$\Delta J = \int_{t_0}^{t_f} \Big\{ g(x(t), \dot{x}(t), t) + \left[\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t)\right]\delta x + \left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right]\delta \dot{x}(t) + \frac{1}{2}\left[\text{Quadratic \& higher term}\right] - g(x(t), \dot{x}(t), t)\Big\}\,dt$$

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \Big\{ \left[\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t)\right]\delta x(t) + \left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right]\delta \dot{x}(t)\Big\}\,dt$$

(1)

$$\delta x(t) = \int_{t_0}^{t_f} \delta \dot{x}(t)\,dt + \overset{=0}{\delta x(t_0)} \quad \Big| \; \begin{matrix}\delta x_f = 0 \\ \delta x_0 = 0\end{matrix}$$

To write (1), entirely in terms of $\delta x(t)$ integrate by parts: the $\delta \dot{x}(t)$ term

$$\delta J(x, \delta x) = \left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right]\delta x(t)\Big|_{t_0}^{t_f}$$
$$+ \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial x}(x(t), \dot{x}(t), t)\right] - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t)\right] \right\} \delta x(t)\,dt$$

Applying the fund. thm: $\delta J = 0$
(also $\delta x_f = \delta x_0 = 0$)

$$\delta J = \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x, \dot{x}, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x, \dot{x}, t)\right] \right\} \delta x(t)\,dt$$
$$= 0$$

Since this is true all $\delta x(t)$,

$$\boxed{\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) + \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] = 0 \quad \forall\, t \in [t_0, t_f]}$$

↳ Euler Eqn.

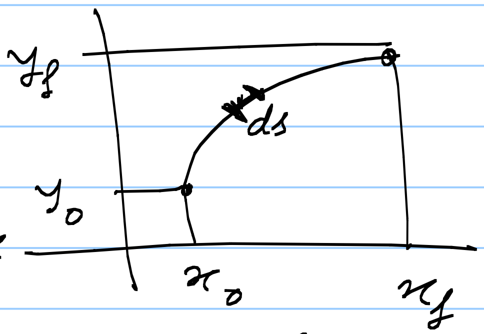
→ In general 2nd order non-linear diff. eqn.

→ Moreover the bdd condition are split (x_0, x_f known)

↳ Nonlinear TPBVP.

Example: Find the curve that gives the shortest distance between 2 pts. in a plane (x_0, y_0), (x_f, y_f)

$$J = \int_{x_0}^{x_f} ds$$



$$= \int_{x_0}^{x_f} \left[\sqrt{(dx)^2 + (dy)^2}\right] = \int_{x_0}^{x_f} \left[\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right] dx$$

$$\frac{dy}{dx} = \dot{y} \qquad J = \int_{x_0}^{x_f} \sqrt{1 + \dot{y}^2}\, dx =: \int_{x_0}^{x_f} g(\dot{y})\, dx$$

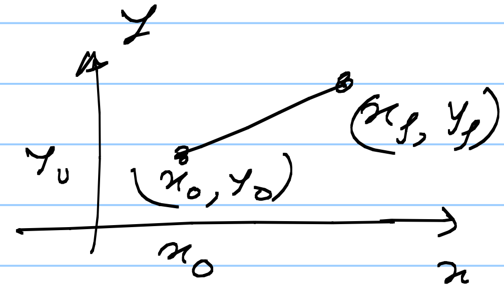
$$\frac{\partial g}{\partial y} = 0 \ , \quad \frac{d}{dx}\left[\frac{\partial g}{\partial \dot{y}}\right] = \frac{d}{d\dot{y}} \frac{d\dot{y}}{dx} \frac{\partial g}{\partial \dot{y}}$$

$$= \frac{d}{d\dot{y}}\left[\frac{\dot{y}}{(1 + \dot{y}^2)^{1/2}}\right] \ddot{y}$$

$$= \frac{\ddot{y}}{(1 + \dot{y}^2)^{3/2}} = 0$$

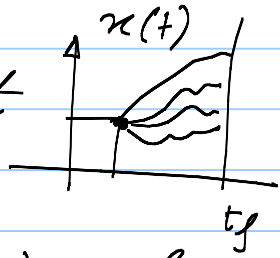
$$\Rightarrow \ddot{y} = 0$$

$$\Rightarrow y = c_1 x + c_2$$



Final Time Specified, $x(t_f)$ free

$$\min_{x(t)} \ J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$



t_0, $x(t_0)$, t_f $\leftarrow$ fixed, $x(t_f) \leftarrow$ free

Recall: the expression for $\delta J(x, \delta x)$:

$$\delta J(x, \delta x) = \left[\frac{\partial g}{\partial x}(x, \dot{x}, t)\right] \delta x(t) \Big|_{t_0}^{t_f}$$

$$+ \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial x}(x, \dot{x}, t)\right] - \frac{d}{dt}\left[\frac{\partial g}{\partial x}(x, \dot{x}, t)\right] \right\} \delta x(t)\, dt$$

$\delta x(t_0) = 0$, But $\delta x(t_f)$ is arbitrary

But that means, $\delta x(t_f) = 0$ is also a possibility

$$\Rightarrow \left[\frac{\partial g}{\partial x}(x, \dot{x}, t)\right] - \frac{d}{dt}\left[\frac{\partial g}{\partial x}(x, \dot{x}, t)\right] = 0$$

$\forall t \in [t_0, t_f]$, ↪ Euler's Eqn

In addition: $\frac{\partial g}{\partial x}(x(t_f), \dot{x}(t_f), t_f)\, \delta x(t_f) = 0$

Since $\delta x(t_f)$ is arbitrary:

$$\boxed{\frac{\partial g}{\partial x}(x(t_f), \dot{x}(t_f), t_f) = 0}$$ → natural boundary condition for Euler's eqn.

→ Still TPBVP. $\left(x(t_0) = x_0 + \right)$

Example: $J(x) = \int_0^2 \left[\dot{x}^2(t) + 2x(t)\dot{x}(t) + 4x^2(t)\right] dt$

$x(0) = 1$, $x(2)$ ← free

Euler eqn: $-\ddot{x}^*(t) + 4x^*(t) = 0$

Sol^n → $x^*(t) = c_1 e^{-2t} + c_2 e^{2t}$ — (1)

Bdd conditions: $x(0) = 1$

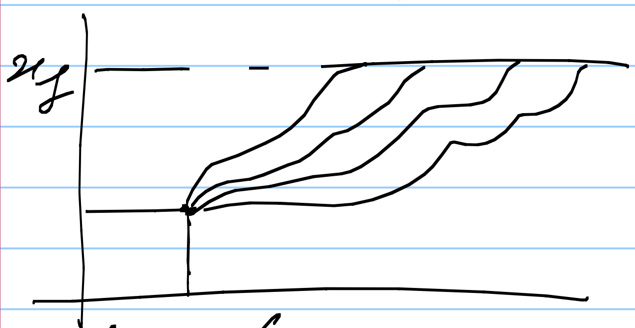
and $\frac{\partial g}{\partial \dot{x}}(x^*(2), \dot{x}^*(2)) = 0$

i.e. $\dot{x}^*(2) + x^*(2) = 0$ — (2)

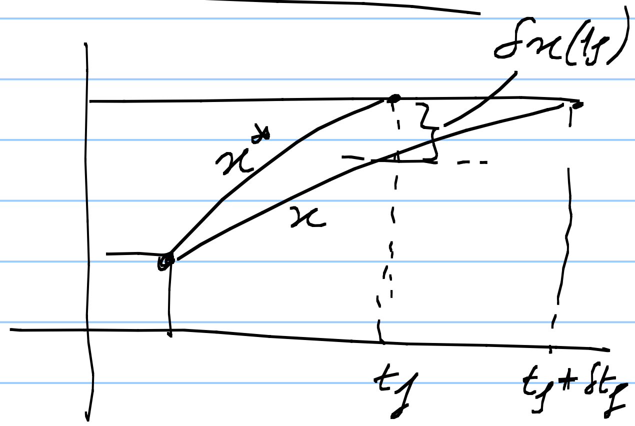
From (1), $\dot{x}^*(t) = -2c_1 e^{-2t} + 2c_2 e^{2t}$

Replacing in ②: $-c_1 e^{-4} + 3c_2 e^{4} = 0$
$x(0) = 1$ gives $c_1 + c_2 = 1$
Solve for c_1, c_2.

Final time free, $x(t_f)$ specified



$t_f \leftarrow$ free
$t_0, x(t_0), x(t_f)$ fixed



Let $x^*(t)$ be the extremal curve terminating at x_f, t_f.
$x(t) \longrightarrow$ terminates at $(x_f, t_f + \delta t_f)$

$$\Delta J = \int_{t_0}^{t_f+\delta t_f} g(x(t), \dot{x}(t), t)\,dt - \int_{t_0}^{t_f} g(x^*(t), \dot{x}^*(t), t)\,dt$$

$$= \int_{t_0}^{t_f} \left[g(x(t), \dot{x}(t), t) - g(x^*(t), \dot{x}^*(t), t) \right] dt$$

$$+ \int_{t_f}^{t_f+\delta t_f} g(x(t), \dot{x}(t), t)\,dt$$

$$= \int_{t_0}^{t_f} \Big[g\big(x^*(t) + \delta x(t), \dot{x}(t) + \delta \dot{x}(t), t\big)$$
$$- g(x^*(t), \dot{x}^*(t), t) \Big] dt$$

$$+ \int_{t_f}^{t_f+\delta t_f} g(x(t), \dot{x}(t), t)\,dt$$

Expanding in Taylor series: about x^*, $\dot{x}^*$

$$= \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t)\right] \delta x(t) + \left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] \delta \dot{x}(t) \right\} dt$$

$$+ \int_{t_f}^{t_f + \delta t_f} g(x, \dot{x}, t)\, dt \quad + H.O.T.$$

Now, $\displaystyle\int_{t_f}^{t_f+\delta t_f} g(x, \dot{x}, t)\, dt = g(x(t_f), \dot{x}(t_f), t_f)\, \delta t_f + H.O.T(\delta t_f)$

Integrating by parts & using + $\delta x(t_0) = 0$

$$\Delta J = \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right] \delta x(t_f)$$

$$+ g(x(t_f), \dot{x}(t_f), t_f)\, \delta t_f + \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] \right\} \delta x(t)\, dt$$

Next we expand $g(x(t_f), \dot{x}(t_f), t_f)$ in term of $g(x^*(t_f), \dot{x}^*(t_f), t_f)$

$$g(x(t_f), \dot{x}(t_f), t_f) = g(x^*(t_f), \dot{x}^*(t_f), t_f)$$

$$+ \left[\frac{\partial g}{\partial x}(x^*(t_f), \dot{x}^*(t_f), t_f)\right] \delta x(t_f)$$

$$+ \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right] \delta \dot{x}(t_f) + HOT$$

Substituting back in ΔJ

$$\Delta J = \left[\frac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]\delta x(t_f)$$

$$+ \left[g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]\delta t_f$$

$$+ \int_{t_0}^{t_f}\left\{\frac{\partial g}{\partial x}\left(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}\left(x^*, \dot{x}^*, t\right)\right]\right\}\delta x(t)\,dt$$

$$+ \text{HOT}$$

* $\delta x(t_f)$ depends on δt_f — a linear approximation of this dependence:

$$\delta x(t_f) + \dot{x}^*(t_f)\,\delta t_f = 0 \quad \text{(approx upto 1st order)}$$

$$\delta x(t_f) = -\dot{x}^*(t_f)\,\delta t_f$$

$$\delta J(x^*, \delta x) = 0 = \left\{\left[-\frac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]\dot{x}^*(t_f)\right.$$

$$\left. + g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right\}\delta t_f$$

$$+ \int_{t_0}^{t_f}\left\{\frac{\partial g}{\partial x}\left(x^*, \dot{x}^*, t\right) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}\left(x^*, \dot{x}^*, t\right)\right]\right\}\delta x(t)\,dt$$

Since both δt_f & $\delta x(t)$ are arbitrary

$$\begin{cases} \dfrac{\partial g}{\partial x}\left(x^*(t), \dot{x}^*(t), t\right) - \dfrac{d}{dt}\left[\dfrac{\partial g}{\partial \dot{x}^*}\left(x^*(t), \dot{x}^*(t), t\right)\right] = 0 \\[2ex] g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) - \left[\dfrac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]\dot{x}^*(t_f) = 0 \end{cases}$$

<u>Example</u>: Find the extremal for the functional

$$J(x) = \int_1^{t_f} \left[2x(t) + \frac{1}{2}\dot{x}^2(t)\right] dt$$

$x(1) = 4$, $x(t_f) = 4$ and $t_f > 1$ free.

Euler's Eqn: $2 - \ddot{x}^*(t) = 0$

Sol$\underline{n}$: $x^*(t) = t^2 + c_1 t + c_2$

Also since t_f is free,

$$\left[2x^*(t_f) + \frac{1}{2}\dot{x}^{*2}(t_f)\right] - \dot{x}^{*2}(t_f) = 0$$

$$\Leftrightarrow \quad 2x^*(t_f) - \frac{1}{2}\left[\dot{x}^*(t_f)\right]^2 = 0 \quad \text{must hold}$$

Given:

$$x^*(1) = 4 = 1 + c_1 + c_2 \Leftrightarrow c_1 + c_2 = 3 \quad (1)$$

$$x^*(t_f) = 4 = t_f^2 + c_1 t_f + c_2 \quad (2)$$

$$2x^*(t_f) - \frac{1}{2}\left[\dot{x}^*(t_f)\right]^2 = 2\left[t_f^2 + c_1 t_f + c_2\right] - \frac{1}{2}\left[2t_f + c_1\right]^2 = 0$$

$$\Rightarrow \quad 2t_f^2 + 2c_1 t_f + 2c_2 - \frac{1}{2}\left[4t_f^2 + 4t_f c_1 + c_1^2\right] = 0$$

$$\Leftrightarrow \quad 2c_2 - \frac{c_1^2}{2} = 0 \quad --- (3)$$

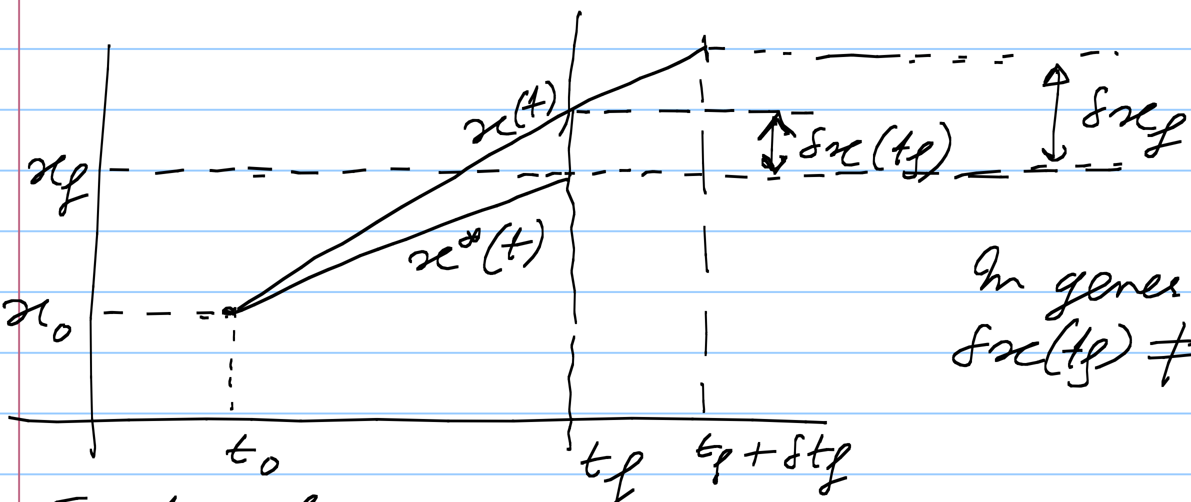
Solving (1), (2), (3) for t_f, c_1, c_2 give

$x^*(t) = t^2 - 6t + 9$ and $t_f = 5$.

Both t_f and $x(t_f)$ free

$$J(x) = \int_{t_0}^{t_f} g(x(t), \dot{x}(t), t)\, dt$$

t_0, $x(t_0) = x_0$ specified | t_f, $x(t_f)$ $\rightarrow$ free



In general
$\delta x(t_f) \neq \delta x_f$

First few steps are same as before:

$$\Delta J = \left[\frac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right] \delta x(t_f)$$
$$+ \left[g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right] \delta t_f$$
$$+ \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}\left(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}\left(x^*, \dot{x}^*, t\right)\right]\right\} \delta x(t)\, dt$$
$$+ \text{HOT} \qquad (2)$$

We must relate $\delta x(t_f)$, δt_f & δx_f

From figure:

$$\delta x_f \doteq \delta x(t_f) + \dot{x}^*(t_f)\, \delta t_f$$
or
$$\delta x(t_f) = \delta x_f - \dot{x}^*(t_f)\, \delta t_f \qquad (1)$$

Using (1) in (2),

$$\delta J = 0 = \frac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\Big] \delta x_f$$
$$+ \left[g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) - \left\{\frac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right\} \dot{x}^*(t_f)\right] \delta t_f$$
$$+ \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}\left(x^*(t), \dot{x}^*(t), t\right) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}\left(x^*, \dot{x}^*, t\right)\right]\right\} \delta x(t)\, dt$$

Many assumptions are possible:

<u>Assumption 1</u>: t_f & $x(t_f)$ are unrelated

i.e. δx_f & δt_f are independent.

$$\Rightarrow \begin{cases} \dfrac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) = 0 \\ \text{and } g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) \\ \qquad - \dfrac{\partial g}{\partial \dot{x}}\left[\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]\dot{x}^*(t_f) = 0 \end{cases}$$

$$\Rightarrow g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) = 0$$

<u>Assumption 2</u>: t_f & $x(t_f)$ are related

Assume: $x(t_f) = \theta(t_f)$

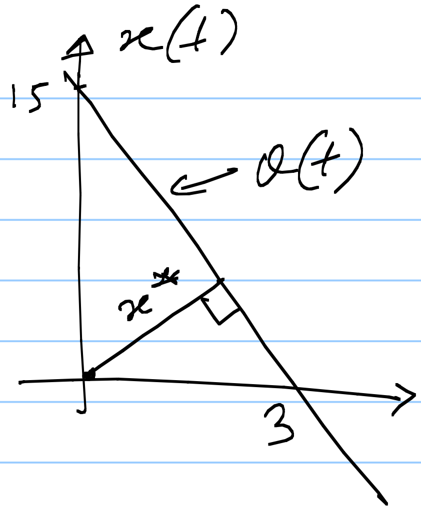
Then $\delta x_f = \dfrac{d\theta}{dt}(t_f)\,\delta t_f$. Replacing:

$$\left[\frac{\partial g}{\partial \dot{x}}\left(x^*(t_f), \dot{x}^*(t_f), t_f\right)\right]\left[\frac{d\theta}{dt}(t_f) - \dot{x}^*(t_f)\right] + g\left(x^*(t_f), \dot{x}^*(t_f), t_f\right) = 0$$

↳ Transversality Condition.

<u>Example</u>: Find the extremal curve for $J(x) = \displaystyle\int_{t_0}^{t_f} \sqrt{1+\dot{x}^2(t)}\, dt$

$t_0 = 0$, $x(0) = 0$, t_f & $x(t_f)$ ← free but $x(t_f)$ is required to lie on the line $\theta(t) = -5t + 15$



Euler's eqn:

$$\frac{d}{dt}\left[\frac{\dot{x}^*(t)}{\sqrt{1+\dot{x}^{*2}(t)}}\right]=0$$

$$\Leftrightarrow \quad \ddot{x}^*(t)=0$$

$$x^*(t)=c_1 t+c_2$$

$x^*(0)=0 \Rightarrow c_2=0$

To calculate c_1 use the transversality condition:

$$\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f),\dot{x}^*(t_f),t_f)\right]\left[\frac{d\theta}{dt}(t_f)-\dot{x}^*(t_f)\right] + g(x^*(t_f),\dot{x}^*(t_f),t_f)=0$$

$$\frac{\dot{x}^*(t_f)}{\sqrt{1+\dot{x}^{*2}(t_f)}}\cdot\left[-5-\dot{x}^*(t_f)\right]+\sqrt{\left[1+\dot{x}^{*2}(t_f)\right]}=0$$

Simplifying, $-5\dot{x}^*(t_f)+1=0$

i.e. $-5c_1+1=0 \Rightarrow c_1=\frac{1}{5}$

To find t_f: use $x(t_f)=\theta(t_f)$

ie $\frac{1}{5}t_f=-5t_f+15$

or $t_f=2.88$

Vector Functions:

$$J(x_1,x_2,\dots,x_n)=\int_{t_0}^{t_f} g(x_1,\dots,x_n,\dot{x}_1,\dots,\dot{x}_n,t)\,dt$$

$t_0,\ t_f,\ x(t_0),\ x(t_f) \leftarrow$ fixed.

$$J(x) = \int_{t_0}^{t_f} g(x, \dot{x}, t)\,dt$$ with $x(t)$ as a vector

$$\Delta J = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial x}(x,\dot{x},t)\right] \delta x(t) + \left[\frac{\partial g}{\partial \dot{x}}(x,\dot{x},t)\right] \delta \dot{x}(t) \right\} dt$$

$$\left[\frac{\partial g}{\partial x_1} \cdots \frac{\partial g}{\partial x_n}\right]\left[\quad\right] \qquad \left[\quad\right]\left[\quad\right]$$

After integration by parts:

$$\delta J(x, \delta x) = \int_{t_0}^{t_f} \left\{ \frac{\partial g}{\partial x}(x(t), \dot{x}(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x(t), \dot{x}(t), t)\right] \right\} \delta x(t)\,dt$$

$$\left\{ \left[\quad\right] - \left[\quad\right] \right\} \begin{bmatrix} \delta x_1 \\ \vdots \\ \delta x_n \end{bmatrix}$$

Since each of $\delta x_1, \ldots, \delta x_n$ is arbitrary every coefficient of δx_i are necessarily zero. i.e.

$$\frac{\partial g}{\partial x}(x^*(t), \dot{x}^*(t), t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*(t), \dot{x}^*(t), t)\right] = 0$$

i.e.

$$n \text{ eqns } \begin{cases} \dfrac{\partial g}{\partial x_1}(x^*, \dot{x}^*, t) - \dfrac{d}{dt}\left[\dfrac{\partial g}{\partial \dot{x}_1}(x^*, \dot{x}^*, t)\right] = 0 \\ \vdots \\ \dfrac{\partial g}{\partial x_n}(x^*, \dot{x}^*, t) - \dfrac{d}{dt}\left[\dfrac{\partial g}{\partial \dot{x}_1}(x^*, \dot{x}^*, t)\right] = 0 \end{cases}$$

For the problem with free end pt:

$$J(x(t)) = \int_{t_0}^{t_f} g(x, \dot{x}, t)\,dt$$

$t_0, x(t_0) \leftarrow$ fixed | $t_f, x_f \leftarrow$ free possibly with some constraints

Eulers eqn

$$\frac{\partial g}{\partial x}(x^*, \dot{x}^*, t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{x}}(x^*, \dot{x}^*, t)\right] = 0$$

$\boxed{n \text{ eqns}}$

Boundary condition

$$\left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\delta x_f + \left\{ g(x^*(t_f), \dot{x}^*(t_f), t_f) - \left[\frac{\partial g}{\partial \dot{x}}(x^*(t_f), \dot{x}^*(t_f), t_f)\right]\dot{x}^*(t_f)\right\}\delta t_f = 0$$

scalar

scalar

Exercise: 1) Read Table on pg 151

2) What about split end pt conditions? E.g. t_f fixed, $x_i(t_f)$, $i = 1, \cdots, r$ $\leftarrow$ fixed and $x_i(t_f)$, $i = r+1, \cdots, n$ are free.

Constrained Minimization

$$J(w(t)) = \int_{t_0}^{t_f} g(w(t), \dot{w}(t), t)\,dt$$

Assume $\omega(t_0), \omega(t_f), t_0, t_f$ are specified such that

$$\left\{ \begin{array}{l} f_1(\omega(t),t)=0 \\ \vdots \\ f_n(\omega(t),t)=0 \end{array} \right. \qquad \omega(t) = \begin{bmatrix} \omega_1(t) \\ \vdots \\ \omega_n(t) \\ \vdots \\ \omega_{n+m}(t) \end{bmatrix} \begin{array}{l} \text{dep.} \\ \\ \text{ind.} \end{array} \quad (n+m)\times 1$$

Only m - components are independent.

One could solve for $\omega_1, \cdots, \omega_n$ in terms of $\omega_{n+1}, \cdots, \omega_{n+m}$ and then use the earlier eqns.
→ Impossible in most cases.

Lagrange Multipliers

Define:

$$J_a(\omega(t), p(t)) = \int_{t_0}^{t_f} \left\{ g(\omega(t), \dot{\omega}(t), t) + p^T(t)\left[f(\omega(t), t)\right] \right\} dt$$

$$p^T(t) = \left[p_1(t) \quad p_2(t) \quad \cdots \quad p_n(t) \right]$$

Constraints $f(\omega, t) = 0$ must be satisfied for all $t \in [t_0, t_f]$. Hence $p(t)$ are functions of time.

$$\delta J_a(\omega, p, \delta\omega, \delta p) = \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial \omega}(\omega, \dot{\omega}, t) + p^T(t) \left\{ \underbrace{\frac{\partial f}{\partial \omega}(\omega, t)}_{(n \times n+m)} \right\} \right] \delta\omega(t) \right.$$

$$+\left[\frac{\partial g}{\partial \dot{\omega}}(\omega,\dot{\omega},t)\right]\delta\dot{\omega}(t)+\left[f^T(\omega,t)\right]\delta p(t)\Big\}dt$$

After integrating by parts:

$$\delta J_a = \int_{t_0}^{t_f}\Big\{\Big[\frac{\partial g}{\partial \omega}(\omega,\dot{\omega},t)+p^T(t)\frac{\partial f}{\partial \omega}(\omega,t)$$
$$-\frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\omega}}(\omega,\dot{\omega},t)\right]\Big]\delta\omega(t)$$
$$+\left[f^T(\omega(t),t)\right]\delta p(t)\Big\}dt$$

Now, $\delta J_a(\omega^*, p, \delta\omega, \delta p) = 0$

Also $f(\omega^*, t) = 0 \quad t\in[t_0, t_f]$

<u>Problem</u>: $\delta\omega(t)$'s are not independent

$p(t)$ can be chosen to suit our needs.
(since in any case $f(\omega^*, t) = 0$)

We choose $p(t) = p^*(t_f)$ s.t. the coefficients of n components of $[\delta\omega_1(t), \dots, \delta\omega_n(t)]$ are zero identically over $[t_0, t_f]$
The remaining m-components $[\delta\omega_{n+1}, \dots, \delta\omega_{n+m}]$ can then be chosen independently.

Hence, now for $p(t) = p^*(t)$ <u>all</u> the coefficients of the components of $\delta\omega(t)$ must be zero.

Hence:

$$\frac{\partial g}{\partial \omega}(\omega^*(t), \dot{\omega}^*(t), t) + \left[\frac{\partial f}{\partial \omega}(\omega^*(t), t)\right]^T p^*(t) - \frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\omega}}(\omega^*(t), \dot{\omega}^*(t), t)\right] = 0$$

Define: $g_a(\omega, \dot{\omega}, t) = g(\omega, \dot{\omega}, t) + p^T(t)[f(\omega, t)]$

Then:

$$\frac{\partial g_a}{\partial \omega}(\omega^*, \dot{\omega}^*, p^*, t) - \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\omega}}(\omega^*, \dot{\omega}^*, p^*, t)\right] = 0$$

$\hookrightarrow$ $(n+m)$ - 2nd order ODE's

Additionally, $f(\omega^*(t), t) = 0$

$\hookrightarrow$ n - algebraic eqns

$\omega^*(t)$ and $p^*(t)$

$\hookrightarrow$ $(n+m)$ variable $\hookrightarrow$ n variables.

<u>Differential Eqn Constraints</u>:

$$J(\omega) = \int_{t_0}^{t_f} g(\omega(t), \dot{\omega}(t), t)\, dt$$

$$\omega(t) = [\omega_1(t) \cdots \omega_n(t)\ \omega_{n+1}(t) \cdots \omega_{n+m}(t)]^T$$

$$\left\{\begin{array}{l} f_1(\omega(t), \dot{\omega}(t), t) = 0 \\ \vdots \\ f_n(\omega(t), \dot{\omega}(t), t) = 0 \end{array}\right.$$

$\leftarrow$ Because of these only m of the $n+m$ $\omega_i(t)$ are ind.

Q. What about initial conditions?

$$J_a(\omega, p) = \int_{t_0}^{t_f} \left[g(\omega, \dot{\omega}, t) + p^T(t) f(\omega, \dot{\omega}, t)\right] dt$$

$\delta J_a(\omega, \delta\omega, p, \delta p)$

$$= \int_{t_0}^{t_f} \left\{ \left[\frac{\partial g}{\partial \omega}(\omega, \dot{\omega}, t) + p^T(t) \frac{\partial f}{\partial \omega}(\omega, \dot{\omega}, t)\right] \delta\omega(t)\right.$$

$$+\left[\frac{\partial g}{\partial \dot{\omega}}(\omega,\dot{\omega},t)+p^T(t)\frac{\partial f}{\partial \dot{\omega}}(\omega,\dot{\omega},t)\right]\delta\dot{\omega}(t)$$

$$+\left[f^T(\omega,\dot{\omega},t)\right]\delta p(t)\Big\}\,dt$$

After integration by parts:

$$\delta J_a=\int_{t_0}^{t_f}\Big\{\Big[\frac{\partial g}{\partial \omega}(\omega,\dot{\omega},t)+p^T(t)\frac{\partial f}{\partial \omega}(\omega,\dot{\omega},t)$$

$$-\frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\omega}}(\omega,\dot{\omega},t)+p^T(t)\frac{\partial f}{\partial \dot{\omega}}(\omega,\dot{\omega},t)\right]\Big]\delta\omega(t)$$

$$+\left[f^T(\omega,\dot{\omega},t)\right]\delta p(t)\Big\}\,dt$$

On the extremal:

$\delta J_a(\omega^*,p)=0$ along with $f(\omega^*,\dot{\omega}^*,t)=0$

Using similar logic as above we select n components of $p(t)=p^*(t)$ n - coefficients of $\delta\omega(t)$ are identically zero.

Remaining $\delta\omega(t)$'s are ind. So all coeff's are zero. Hence:

$$\frac{\partial g}{\partial \omega}(\omega^*,\dot{\omega}^*,t)+p^{*T}(t)\frac{\partial f}{\partial \omega}(\omega^*,\dot{\omega}^*,t)$$

$$-\frac{d}{dt}\left[\frac{\partial g}{\partial \dot{\omega}}(\omega^*,\dot{\omega}^*,t)+p^{*T}(t)\frac{\partial f}{\partial \dot{\omega}}(\omega^*,\dot{\omega}^*,t)\right]=0$$

If $g_a=g+p^Tf$, then

$$\frac{\partial g_a}{\partial \omega}(\omega^*,\dot{\omega}^*,p^*,t)-\frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\omega}}(\omega^*,\dot{\omega}^*,p^*,t)\right]=0$$

Isoperimetric Constraints:

$$J(\omega) = \int_{t_0}^{t_f} g(\omega, \dot{\omega}, t)\, dt \quad s.t.$$

$$\int_{t_0}^{t_f} e_1(\omega, \dot{\omega}, t)\, dt = c_1$$

$$\vdots$$

$$\int_{t_0}^{t_f} e_r(\omega, \dot{\omega}, t)\, dt = c_r$$

Define, $z_i(t) := \int_{t_0}^{t} e_i(\omega(t), \dot{\omega}(t), t)\, dt \quad i = 1, \ldots, r$

with $z_i(t_0) = 0$ & $z_i(t_f) = c_i$

Then,

$$\dot{z}_i(t) = e_i(\omega, \dot{\omega}, t) \quad i = 1, 2, \ldots, r$$

or $\dot{z}(t) = e(\omega, \dot{\omega}, t) \leftarrow r$ diff eqns (same as last case)

Define:

$$g_a(\omega(t), \dot{\omega}(t), p(t), \dot{z}(t), t)$$

$$= g(\omega, \dot{\omega}, t) + p^T(t)\left[e(\omega, \dot{\omega}, t) - \dot{z}\right]$$

Then: $\frac{\partial g_a}{\partial \omega}(\omega^*, \dot{\omega}^*, p^*, \dot{z}^*, t) - \frac{d}{dt}\left[\frac{\partial g_a}{\partial \dot{\omega}}(\ldots)\right] = 0$

<u>AND</u> $\underbrace{\frac{\partial g_a}{\partial z}(\omega^*, \dot{\omega}^*, p^*, \dot{z}^*, t)}_{=0} - \frac{d}{dt}\underbrace{\left[\frac{\partial g_a}{\partial \dot{z}}(\cdot\ \cdot)\right]}_{\dot{p}^*(t)} = 0$

$$\dot{p}^*(t) = 0$$

Hence $\dot{p}^*(t) = 0$ is a necessary condⁿ

i.e. $p^*(t) = \text{constant}$

in addition $\dot{z}^*(t) = e(\omega^*, \dot{\omega}^*, t)$

Examples: $\dot{x}_1 = x_2 - x_1$
$\dot{x}_2 = -2x_1 - 3x_2 + u(t)$

Find $u(t)$ to minimize

$$J(x,u) = \int_{t_0}^{t_f} \frac{1}{2}\left[x_1^2 + x_2^2 + u^2\right] dt \qquad \left.\begin{matrix} x_1(t_0) \\ x_2(t_0) \\ x_1(t_f) \\ x_2(t_f) \end{matrix}\right\} \text{specified}$$

$x_1 = \omega_1, \; x_2 = \omega_2, \; u = \omega_3$

$$J(\omega) = \int_{t_0}^{t_f} \frac{1}{2}\left[\omega_1^2 + \omega_2^2 + \omega_3^2\right] dt$$

s.t. $\dot{\omega}_1 - (\omega_2 - \omega_1) = 0$ $\quad f_1 = 0$
$\dot{\omega}_2 + 2\omega_1 + 3\omega_2 - \omega_3 = 0$ $\quad f_2 = 0$

$$g_a(\omega, \dot{\omega}, p) = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 + \frac{1}{2}\omega_3^2$$
$$+ p_1(t)\left[\omega_2 - \omega_1 - \dot{\omega}_1\right] + p_2(t)\left[-2\omega_1 - 3\omega_2 + \omega_3 - \dot{\omega}_2\right]$$

From the necessary condns derived above:

$$\dot{p}_1^*(t) = -\omega_1^*(t) + p_1^*(t) + 2p_2^*(t)$$
$$\dot{p}_2^*(t) = -\omega_2^*(t) - p_1^*(t) + 3p_2^*(t)$$

Algebraic eqn: $\omega_3^*(t) + p_2^*(t) = 0$

Additionally: $\dot{\omega}_1^* = \omega_2^* - \omega_1^*$
$\dot{\omega}_2^* = -2\omega_1^* - 3\omega_2^* + \omega_3^*$

Replacing $\omega_3^* = -p_2^*$,

$$\begin{bmatrix} \dot{\omega}_1^* \\ \dot{\omega}_2^* \\ \dot{p}_1^* \\ \dot{p}_2^* \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -2 & -3 & 0 & -1 \\ -1 & 0 & 1 & 2 \\ 0 & -2 & -1 & 3 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ p_1 \\ p_2 \end{bmatrix} \qquad \left.\begin{matrix} \omega_1(t_0), \; t_0, t_f \\ \omega_2(t_0) \\ \omega_1(t_f) \\ \omega_2(t_f) \end{matrix}\right\} \text{known}$$

Example:

$$\dot{x}_1 = -x_1 + x_2 + u$$
$$\dot{x}_2 = -2x_1 - 3x_2 + u$$

$$\text{Min } J(x,u) = \int_{t_0}^{t_f} \frac{1}{2}\left[x_1^2 + x_2^2\right] dt$$

$$\text{s.t. } \int_{t_0}^{t_f} u^2(t)\,dt = c \quad \text{[Total energy expended]}$$

$$\omega_1 = x_1, \quad \omega_2 = x_2, \quad \omega_3 = u$$

$$J(\omega) = \int_{t_0}^{t_f} \frac{1}{2}\left[\omega_1^2 + \omega_2^2\right] dt,$$

$$\text{s.t. } \dot{\omega}_1 = -\omega_1 + \omega_2 + \omega_3 \qquad \Big| \quad \int_{t_0}^{t_f} \omega_3^2\, dt = c$$
$$\dot{\omega}_2 = -2\omega_1 - 3\omega_2 + \omega_3$$

$$g_a = \frac{1}{2}\omega_1^2 + \frac{1}{2}\omega_2^2 + p_1\left[-\omega_1 + \omega_2 + \omega_3 - \dot{\omega}_1\right]$$
$$+ p_2\left[-2\omega_1 - 3\omega_2 + \omega_3 - \dot{\omega}_2\right] + p_3\left[\omega_3^2 - \dot{z}\right]$$

Using the necessary conditions:

Diff. eq.

$$\begin{cases} \dot{p}_1^* = p_1^* + 2p_2^* - \omega_1^* \\ \dot{p}_2^* = -p_1^* + 3p_2^* - \omega_2^* \\ \dot{p}_3^* = 0 \end{cases}$$

$$\begin{cases} \dot{\omega}_1^* = -\omega_1^* + \omega_2^* + \omega_3^* \\ \dot{\omega}_2 = -2\omega_1^* - 3\omega_2^* + \omega_3^* \end{cases}$$

$$\dot{z}^* = \omega_3^{*2}$$

Algebraic

$$p_1^* + p_2^* + 2\omega_3^* p_3^* = 0$$

$$z^*(t_0) = 0$$
$$z^*(t_f) = c$$

$$\left.\begin{matrix} \omega_1(t_0), & \omega_1(t_f) \\ \omega_2(t_0), & \omega_2(t_f) \end{matrix}\right\} \text{known}$$