Lecture 4: Optimal Control

Note Title 11-06-2008

$$\dot{x} = f(x,u,t) \qquad x(t_0), \qquad t_0 \le t \le t_f$$

fixed

$$x(t) \in \mathbb{R}^n \quad \text{and} \quad u(t) \in \mathbb{R}^m$$

$$\min J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(x,u,t)\,dt$$

Lagrange Multiplier Method

$$J = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} \Big[L(x,u,t) + \lambda^T(t)\{ f(x,u,t) - \dot{x} \} \Big] dt$$

Define a scalar function H (the Hamiltonian) as follows:

$$H[x,u,\lambda,t] = L(x,u,t) + \lambda^T(t) f(x,u,t)$$

Then: $$J = \phi(x(t_f), t_f) - \lambda^T(t_f) x(t_f) + \lambda^T(t_0) x(t_0) + \int_{t_0}^{t_f} \{ H(x,u,t) + \dot{\lambda}^T(t) x(t) \} dt$$

$$\delta J = \left[\frac{\partial \phi}{\partial x}(x(t_f), t_f) - \lambda^T(t_f) \right] \delta x(t_f) + \lambda^T(t_0)\, \cancel{\delta x(t_0)} + \int_{t_0}^{t_f} \left[\left\{ \frac{\partial H}{\partial x}(x,u,t) + \dot{\lambda}^T(t) \right\} \delta x(t) + \frac{\partial H}{\partial u}(x,u,t)\, \delta u(t) \right] dt$$

$x(t_0)$ fixed so $\delta x(t_0) = 0$

As in C.O.V. cases, δx & δu are dependent. Instead of computing this dependence, choose

$$\dot{\lambda}^T(t) = -\frac{\partial H}{\partial x}(x,u,t) = -\left[\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}\right]$$

$$\lambda^T(t_f) = \frac{\partial \phi}{\partial x}(x(t_f), t_f)$$

Then:

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial u}(x,u,t)\,\delta u(t)\right] dt$$

At extremum: $\delta J = 0 \quad \forall\, \delta u(t)$

$$\Rightarrow \frac{\partial H}{\partial u}(x,u,t) = 0$$

Summary of Necessary Conditions:

Euler - Lagrange Eqns

$$\dot{x} = f(x,u,t)$$

$$\dot{\lambda}^T = -\left[\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}\right]$$

$2n$ ODE's

$x(t_0) \leftarrow$ given

$\lambda^T(t_f) = \frac{\partial \phi}{\partial x}(x(t_f), t_f)$

Split bdd conditions.

$$\frac{\partial H}{\partial u} = 0$$

$\hookrightarrow m$ eqns

$$\begin{array}{ccccc} x; & u, & \lambda & & \\ \downarrow & \downarrow & \downarrow & & \\ n & m & n & = & 2n+m \end{array}$$

* H has a special property if H is not an explicit function of t.

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial u}\dot{u} + \dot{\lambda}^T f$$

$$= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial u}\dot{u} + \left[\frac{\partial H}{\partial x} + \dot{\lambda}^T\right] f$$

$[\because \dot{x} = f]$

$$= \frac{\partial H}{\partial t} + \frac{\partial H}{\partial u}\dot{u} \qquad \left[\text{since } \dot{\lambda}^T = -\frac{\partial H}{\partial x}\right]$$

If H is not an explicit function of t, then $\frac{\partial H}{\partial t} = 0$

& $\frac{\partial H}{\partial u} = 0$ by E-L eqns

So $\frac{dH}{dt} = 0 \Rightarrow H =$ constant on the optimal trajectory.

Example: Hamilton's Principle in Mechanics: The motion of a conservative system from t_0 to t_f is s.t.

$$I = \int_{t_0}^{t_f} L(u, q)\, dt \text{ has a stationary}$$

value. $L = \underbrace{T(u,q)}_{\text{Kinetic Energy}} - \underbrace{V(q)}_{\text{Potential Energy}}$

q = state of system (generalized position vector)

$u = \dot{q}$ = generalized velocity vector

State eqn: $\dot{q} = u$

$$H = L + \lambda^T u$$

E-L eqns: $\dot{\lambda}^T = -\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}$ — (1)

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \quad \text{——} \quad (2)$$

From (2): $\dot{\lambda}^T = -\frac{d}{dt}\left(\frac{\partial L}{\partial u}\right) = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \quad (\because u = \dot{q})$

so $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0}$

→ Lagrange's Eqn of Motion for conservative system

We know if H is not an explicit function of time, H = constant.

$$H = L - \frac{\partial L}{\partial u} u = T - V - \frac{\partial T}{\partial u} u = \text{const}$$

T is a quadratic form in u, so

$$\frac{\partial T}{\partial u} u = 2T \qquad \boxed{\frac{\partial (u^T Q u)}{\partial u} u = u^T Q u}$$

$\Leftrightarrow H = T - V - 2T$

$-H = (T + V) = \text{constant}.$

kinetic + pot energy = constant

Some state variables specified at fixed terminal time.

If $x_i(t_f)$ is specified $\leftrightarrow \delta x_i(t_f) = 0$

Then not necessarily $\left[\frac{\partial \phi}{\partial x_i} - \lambda_i\right]_{t=t_f} = 0$

Hence $\lambda_i(t_f)$ is no longer known. But $x_i(t_f)$ is known

Similarly if $x_k(t_0)$ is NOT specified then not necessarily $\delta x_k(t_0) = 0$.

But we can choose $\lambda_k(t_0) = 0$

~~change~~ $x_k(t_0)$ not known but $\lambda_k(t_0) = 0$

Assume: $x_i(t_f)$ specified for $i = 1, \cdots, q$

Justification for $\frac{\partial H}{\partial u} = 0$

$\delta x_i(t_0) = 0$
$i = 1, \cdots, n$

Needed since $\delta u(t)$ is no longer arbitrary. It must produce $\delta x_i(t_f) = 0$ for $i = 1, \cdots, q$.

$\phi = \phi[x_{q+1}(t_f), \cdots, x_n(t_f)]$ since $x_1(t_f) \cdots x_q(t_f)$ are specified.

$$\delta J = \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial u} + (\lambda^J)^T \frac{\partial f}{\partial u}\right] \delta u(t)\, dt$$

$$(\dot{\lambda}^J)^T(t) = -\frac{\partial H}{\partial x} \quad ; \quad \lambda_j^J(t_f) = \begin{cases} 0 & j = 1, \cdots, q \\ \frac{\partial \phi}{\partial x_j}(x(t_f), t_f) & j = q+1, \cdots, n \end{cases}$$

We calculate $\delta x_i(t_f)$:

$J = x_i(t_f)$ i.e. $\phi = x_i(t_f)$

$i = q+1, \cdots, n$ $\qquad L = 0$

$$\delta J = \delta x_i(t_f) = \int_{t_0}^{t_f} \left[(\lambda^i)^T \frac{\partial f}{\partial u}\right] \delta u(t)\, dt \quad \text{---} (1)$$

$$(\dot{\lambda}^{(i)})^T(t) = -(\lambda^i)^T \frac{\partial f}{\partial x}$$

$$\lambda^{i^\circ}_j(t_f) = \begin{Bmatrix} 0 & j \neq i^\circ \\ 1 & j = i^\circ \end{Bmatrix} \quad j = 1, \cdots, n$$

We shall construct $\delta u(t)$ history that decreases J i.e. $\delta J < 0$ + satisfy $\delta x_i(t_f) = 0$ $i = 1, \cdots, q$

Add $\nu_{i^\circ} \, \delta x_i(t_f)$ to δJ.

$$\delta J + \sum_{i=1}^{q} \nu_{i^\circ} \underbrace{\delta x_i(t_f)}_{i=1,\cdots,q} = \int_{t_0}^{t_f} \left\{ \frac{\partial L}{\partial u} + \left[\lambda^J + \sum_{i=1}^{q} \nu_i \lambda^{i^\circ} \right]^T \frac{\partial f}{\partial u} \right\} \delta u \, dt$$

Now choose

$$\delta u = -k \left\{ \left(\frac{\partial f}{\partial u} \right)^T \left[\lambda^J + \sum \nu_{i^\circ} \lambda^{i^\circ} \right] + \left(\frac{\partial L}{\partial u} \right)^T \right\}$$

$k > 0$ scalar, Then

$$\delta J = -k \int_{t_0}^{t_f} \left\| \left(\frac{\partial f}{\partial u} \right)^T \cdots \left(\frac{\partial L}{\partial u} \right)^T \right\|^2 dt < 0$$

Now ν_i's can be solved to satisfy
$i = 1, \cdots, q$ $\quad \delta x_i(t_f) = 0 \quad i = 1, \cdots, q$
$\hookrightarrow q$ constraint

substituting δu in (1),

$$0 = \delta x_i(t_f) = \int_{t_0}^{t_f} (\lambda^i)^T \frac{\partial f}{\partial u} \left[\left(\frac{\partial f}{\partial u} \right)^T \lambda^J + \left(\frac{\partial L}{\partial u} \right)^T \right] dt$$
$$+ \nu_{j^\circ} \int_{t_0}^{t_f} (\lambda^{i^\circ})^T \frac{\partial f}{\partial u} \left(\frac{\partial f}{\partial u} \right)^T \lambda^{(j)} dt$$

Solving: $\quad \nu = Q^{-1} g \quad$ [Q is $q \times q$, g is a q vector]

$$Q_{ij} = \int_{t_0}^{t_f} (\lambda^i)^T f_u f_u^T \lambda^j \, dt \qquad i,j = 1, \dots, q$$

$$g_i = \int_{t_0}^{t_f} (\lambda^i)^T \frac{\partial f}{\partial u} \left[\left(\frac{\partial f}{\partial u}\right)^T \lambda^J + \left(\frac{\partial L}{\partial u}\right)^T \right] dt$$

With this $\delta u(t)$, $\delta J < 0$

This cannot be done only when:

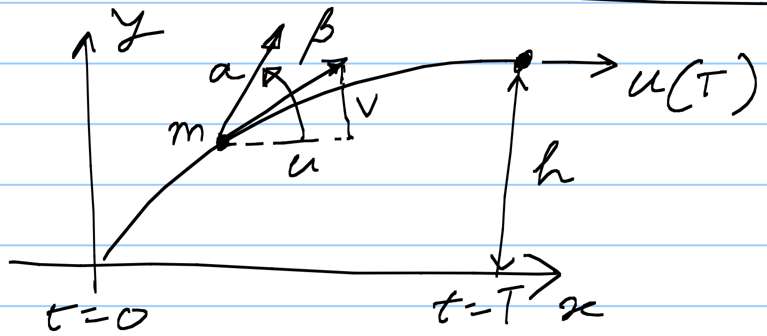
$$\frac{\partial L}{\partial u} + \left[\lambda^J + \sum \nu_i \lambda^i\right]^T \frac{\partial f}{\partial u} = 0 \qquad t_0 \le t \le t_f$$

Since all eqns involving λ's are linear the necessary conditions become:

$$\frac{\partial H}{\partial u} = 0 \qquad H = L + \lambda^T f \qquad \text{Exercise: Derive}$$

$$\dot{\lambda}^T = \frac{\partial H}{\partial x} \qquad \lambda_j(t_f) = \begin{cases} \nu_j & j = 1, \dots, q \\ \dfrac{\partial \phi}{\partial x_j}(x(t_f), t_f) \end{cases}$$

Example:



Planar motion, Force = ma, velocity: u, v

Thrust direction angle $= \beta(t)$ is the control variable.

Eqns of motion: $\dot{u} = a\cos\beta$
$\dot{v} = a\sin\beta$
$\dot{x} = u$
$\dot{y} = v$

$a \to$ constant acceleration, T - fixed.

Aim: maximize $u(T)$ s.t.

$u(0) = 0$	$u(T) \leftarrow$ to be maximized, free
$v(0) = 0$	$v(T) = 0$
$x(0) = 0$	$x(T)$ — free
$y(0) = 0$	$y(T) = h$ (fixed)

$$J = u(T) + \int_{t_0}^{t_f} 0\, dt$$

$$L = 0, \quad \phi = u(T)$$

$$H = \lambda_u a\cos\beta + \lambda_v a\sin\beta + \lambda_x u + \lambda_y v$$

$$\frac{\partial H}{\partial \beta} = -\lambda_u \sin\beta + \lambda_v \cos\beta = 0$$

Optimal control: $\tan\beta = \dfrac{\lambda_v}{\lambda_u}$ — (1)

(1) $\dot{\lambda}_u = -\lambda_x$	$\lambda_u(T) = 1$	From above
(2) $\dot{\lambda}_v = -\lambda_y$	$\lambda_v(T) = \nu_v$	$v(T) = 0$
(3) $\dot{\lambda}_x = 0$	$\lambda_x(T) = 0$	
(4) $\dot{\lambda}_y = 0$	$\lambda_y(T) = \nu_y$	$y(T) = h$

$\nu_v, \nu_y \to$ unknowns

$\lambda_u(t) = -c_1 t + c_2$ | $\lambda_x = c_1$
$\lambda_v(t) = -c_2 t + c_4$ | $\lambda_y = c_2$

$\dot{\lambda}_x = 0$ & $\lambda_x(T) = 0 \Rightarrow \lambda_x(t) = 0$
$\Rightarrow \left.\begin{array}{l}\dot{\lambda}_u = 0 \\ \lambda_u(T) = 1\end{array}\right\} \Rightarrow \lambda_u(t) = 1$

From (1): ~~total~~ law:

$$\tan\beta = \frac{\lambda_v(t)}{\lambda_u(t)} = -c_2 t + c_4$$
$$= \tan\beta_0 - ct$$

where $\tan\beta_0 = \nu_v + \nu_y T$, $c = \nu_y$

[Since, $\lambda_v(T) = \nu_v = -c_2 T + c_4$
$\hookrightarrow = \lambda_y = \nu_y$]

Still we need to calculate

$\tan\beta_0$ & $c \Leftrightarrow \nu_v$ & ν_y

This can be done from
$v(T) = 0$ & $y(T) = h$.
(skipped)

Functions of State Variables fixed at t_f (fixed)

$$\psi[x(t_f), t_f] = 0 \qquad q \text{ eqns}$$

$\left[\begin{array}{ll} q \le n-1 & \text{if } L = 0 \\ q \le n & \text{if } L \ne 0 \end{array}\right]$

$$J = \phi[x(t_f), t_f] + \nu^T \psi[x(t_f), t_f]$$

$$+\int_{t_0}^{t_f}\{L(x,u,t)+\lambda^T(f-\dot{x})\}dt$$

If we define $\Phi = \phi + \nu^T\psi$ then same eqns as earlier applies:

$\dot{x} = f(x,u,t)$ $\longrightarrow n$ ODE

$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T\frac{\partial f}{\partial x}$ $\longrightarrow n$ ODE

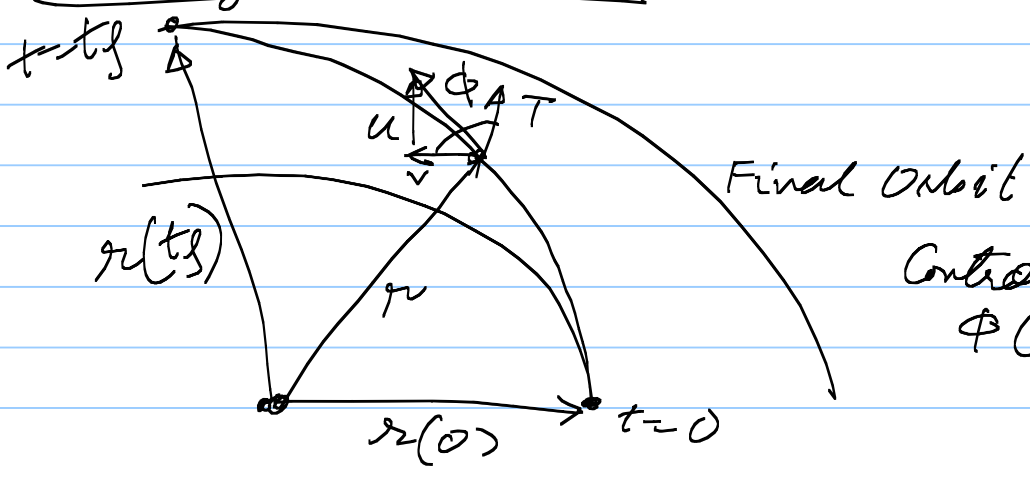
$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T\frac{\partial f}{\partial u} = 0$ $\longrightarrow m$ alg. eqns.

$x_k(t_0) = 0$ or $\lambda_k(t_0) = 0$, $k = 1, \cdots, n$
$\hookrightarrow n$ bdd cond

$\lambda^T(t_f) = \left[\frac{\partial \phi}{\partial x} + \nu^T\frac{\partial \psi}{\partial x}\right]\Big|_{t=t_f}$ $\longleftarrow n$ bdd cn

$\psi(x(t_f), t_f) = 0$ $\longleftarrow q$ side condition

Maximum Radius Orbit transfer in fixed time



Control: $\phi(t)$

r = radius from attracting center
u = radial velocity
v = tangential comp.
m = mass , $\dot{m}$ = constant (fuel consump rate)
ϕ = thrust direction angle
μ = grav. constant

Find $\phi(t)$ to max $r(t_f)$
s.t.

$$\text{S.E.} \begin{cases} \dot{r} = u \\ \dot{u} = \dfrac{v^2}{r} - \dfrac{\mu}{r^2} + \dfrac{T\sin\phi}{m_0 - |\dot{m}|t} \\ \dot{v} = -\dfrac{uv}{r} + \dfrac{T\cos\phi}{m_0 - |\dot{m}|t} \end{cases}$$

Initial cond.:
$r(0) = r_0$, $u(0) = 0$, $v(0) = \sqrt{\dfrac{\mu}{r_0}}$

Terminal Constraints:

$$\Psi_1 = u(t_f) = 0$$
$$\Psi_2 = v(t_f) - \sqrt{\frac{\mu}{r(t_f)}} = 0$$

$$H = \lambda_r u + \lambda_u\left(\frac{v^2}{r} - \frac{\mu}{r^2} + \frac{T\sin\phi}{m_0 - |\dot{m}|t}\right) + \lambda_v\left(-\frac{uv}{r} + \frac{T\cos\phi}{m_0 - |\dot{m}|t}\right)$$

$$\Phi = r(t_f) + \nu_1 u(t_f) + \nu_2\left[v(t_f) - \sqrt{\frac{\mu}{r(t_f)}}\right]$$

E-L Equations:
$$\dot{\lambda}_r = -\lambda_u\left(-\frac{v^2}{r^2} + \frac{2\mu}{r^3}\right) - \lambda_v\left(\frac{uv}{r^2}\right)$$

$$\dot{\lambda}_u = -\lambda_r + \lambda_v \frac{v}{r}$$

$$\dot{\lambda}_v = -\lambda_u \frac{2v}{r} + \lambda_v \frac{u}{r}$$

$$0 = \left(\lambda_u \cos\phi - \lambda_v \sin\phi\right) \frac{T}{m_0 - |\dot{m}| t}$$

$$\Rightarrow \tan\phi = \frac{\lambda_u}{\lambda_v}$$

<u>Terminal conditions.</u> $\lambda_r(t_f) = 1 + \dfrac{\nu_2 \sqrt{\mu}}{2[r(t_f)]^{3/2}}$

$$\lambda_u(t_f) = \nu_1$$
$$\lambda_v(t_f) = \nu_2$$

Some state variables specified at an unspecified terminal time (including min. time problem)

$$J = \phi[x(t_f), t_f] + \int_{t_0}^{t_f} \left[L(x,u,t) + \lambda^T f - \lambda^T \dot{x}\right] dt$$

$$\delta J = \left[\frac{\partial \phi}{\partial t} dt_f + \frac{\partial \phi}{\partial x} dx\right]_{t=t_f} + (L)_{t=t_f} dt_f \quad \leftarrow \text{Why?}$$

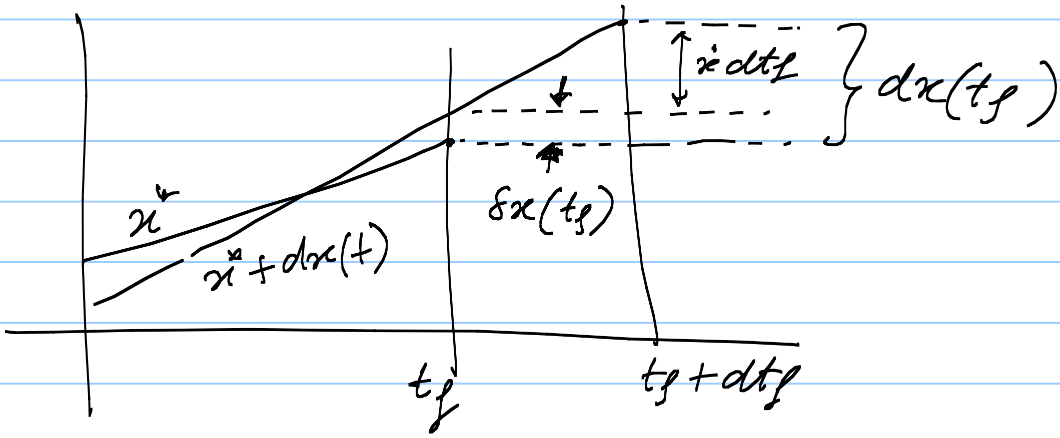
$$+ \int_{t_0}^{t_f} \Big[\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}\right)\delta x + \left(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u}\right)\delta u$$
$$+ \lambda^T \delta \dot{x}\Big] dt$$

$$\delta J = \left[\left(\frac{\partial \phi}{\partial t} + L\right) dt_f + \frac{\partial \phi}{\partial x} dx\right]_{t=t_f} - \left[\lambda^T \delta x\right]_{t=t_f}$$

$$+ \left[\lambda^T \delta x\right]_{t=t_0} + \int_{t_0}^{t_f} \Big[\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T\right)\delta x$$

$$+ \left(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u}\right) \delta u \Big] dt$$

$$dx(t_f) = \delta x(t_f) + \dot{x}(t_f)\, dt_f$$



$$\delta x(t_f) = dx(t_f) - \dot{x}(t_f)\, dt_f$$

$$\delta J = \left[\left(\frac{\partial \phi}{\partial t} + L + \lambda^T \dot{x}\right) dt_f + \left(\frac{\partial \phi}{\partial x} - \lambda^T\right) dx\right]_{t=t_f} + \lambda^T(t_0)\, \underbrace{\delta x(t_0)}_{\to 0}$$

since $x(t_0)$ fixed.

$$+ \int_{t_0}^{t_f} \Big[\left(\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x} + \dot{\lambda}^T\right) \delta x$$

$$+ \left(\frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u}\right) \delta u \Big] dt$$

As earlier: assume

$x_i(t_f)$ $\quad i = 1, \cdots, q$ are specified.

$$\phi = \phi\left[x_j(t_f), t_f\right] \quad j = q+1, \cdots, n$$

Choose $\lambda(t) \equiv \lambda^\phi(t)$ s.t.

$$(\dot{\lambda}^\phi)^T = -\left[\frac{\partial L}{\partial x} + (\lambda^\phi)^T \frac{\partial f}{\partial x}\right]$$

$$\lambda_j^\phi(t_f) = \begin{cases} 0 & j = 1, \cdots, q \\ \left.\frac{\partial \phi}{\partial x_j}\right|_{t=t_f} & j = q+1, \cdots, n \end{cases} \quad (*)$$

Then:

$$\delta J = \left[\frac{\partial \phi}{\partial t} + L + (\lambda^\phi)^T f\right]_{t=t_f} dt_f + \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial u} + (\lambda^\phi)^T \frac{\partial f}{\partial u}\right] \delta u \, dt \quad (1)$$

Now $dx_i(t_f) = 0$ for $i = 1, \cdots, q$. Use the same trick as before:

$$J \equiv x_i(t_f) \; ; \quad L = 0$$

Replace these values in (1) above and use the necessary conditions (*):

$$dJ = dx_i(t_f) = [f_i]_{t=t_f} dt_f + \int_{t_0}^{t_f} [\lambda^i(t)]^T \frac{\partial f}{\partial u} \delta u \, dt$$

where

$$(\dot{\lambda}^i)^T = (\lambda^i)^T \frac{\partial f}{\partial x}$$

$$\lambda_j^i(t_f) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$dJ + \sum \gamma_i \, dx_i(t_f)$

$$= \left[\frac{\partial \phi}{\partial t} + L + (\lambda^\phi)^T f + \sum \gamma_i f_i\right]_{t=t_f} dt_f + \int_{t_0}^{t_f} \left[\frac{\partial L}{\partial u} + \left(\lambda^\phi + \sum \gamma_i \lambda^i\right)^T \frac{\partial f}{\partial u}\right] \delta u \, dt$$

Choose: $dt_f = -k_1 \left\{ \frac{\partial \phi}{\partial t} + L + [\lambda^\phi]^T f + \sum \nu_i f_i \right\}\Big|_{t=t_f}$

$$\delta u = -k_2 \left[\left(\frac{\partial L}{\partial u}\right)^T + \left[\lambda^\phi + \sum \nu_i \lambda^i\right]^T \frac{\partial f}{\partial u} \right]$$

Then:

$$dJ = -k_1 \|\ ----\ \|^2 - k_2 \int_{t_0}^{t_f} \|\ ----\ \|^2 dt \leq 0$$

Thus $dJ = 0 \Leftrightarrow\ = 0$ and $= 0$

ν_i's can be solved from $dx_i(t_f) = 0$ under controllability assumption.

The resulting necessary conditions are:

$$\left[\frac{\partial \phi}{\partial t} + H\right]\Big|_{t=t_f} = 0 \qquad H = L + \lambda^T f$$

$$\frac{\partial H}{\partial u} = 0 \qquad t_0 < t < t_f$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\left[\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}\right]$$

$$\lambda_j(t_f) = \begin{cases} \nu_j & j = 1, \cdots, q \\ \frac{\partial \phi}{\partial x_j}\Big|_{t=t_f} & j = q+1, \cdots, n \end{cases}$$

Min. Time Problem: $\phi = 0$, $L = 1$

E-L eqns: $\dot{x} = f(x, u, t)$ $\quad x(t_0)$ given

$$\dot{\lambda}^T = -\lambda^T \frac{\partial f}{\partial x}$$ $x_j(t_f)$ $j = 1, \cdots, q$ fixed

$\lambda_j(t_f) = 0$ $j = q+1, \cdots, n$ (n terminal condition)

$$\lambda^T \frac{\partial f}{\partial u} = 0 \longrightarrow m\text{ - algebraic eqns}$$

$$[\lambda^T f]_{t=t_f} = -1$$ Trn $\left(\frac{\partial \phi}{\partial t} + H\right)_{t=t_f} = 0$

$[0 + 1 + \lambda^T f]_{t=t_f} = 0$

$H = 1 + \lambda^T f$ = constant over entire trajectory

So $H(t) = H(t_f) = 1 + \lambda^T(t_f)\, f(t_f) = 0$
$= 0$

Example: Ship travelling through strong currents:

Currents: $u = u(x,y)$
velocity component in x direction

$v = v(x,y)$
velocity comp in y direction

Mag. of ship velocity w.r.t water $= V$.

$$\dot{x} = V\cos\theta + u(x,y)$$
$$\dot{y} = V\sin\theta + v(x,y)$$

$\theta \to$ heading angle (control)

Problem: choose $\theta(t)$ to go from fixed A to B in min possible time

$$H = \lambda_x (V\cos\theta + u) + \lambda_y (V\sin\theta + v) + 1 \quad \text{—— (1)}$$

$$\downarrow L$$

$$\dot{\lambda}_x = -\frac{\partial H}{\partial x} = -\lambda_x \frac{\partial u}{\partial x} - \lambda_y \frac{\partial v}{\partial x}$$

$$\dot{\lambda}_y = -\frac{\partial H}{\partial y} = -\lambda_x \frac{\partial v}{\partial y} - \lambda_y \frac{\partial v}{\partial y} \quad \text{—— (5)}$$

$$0 = \frac{\partial H}{\partial \theta} = V(-\lambda_x \sin\theta + \lambda_y \cos\theta) \quad \text{—— (2)}$$

$$\Rightarrow \tan\theta = \frac{\lambda_y}{\lambda_x}$$

H is not an explicit function of time $\Rightarrow$ H = constant $= 0$.

From (1) & (2),

$$\lambda_x = \frac{-\cos\theta}{V + u\cos\theta + v\sin\theta} \quad \text{—— (3)}$$

$$\lambda_y = \frac{-\sin\theta}{V + u\cos\theta + v\sin\theta} \quad \text{—— (4)}$$

Using (3) & (4) in (5):

$$\begin{cases} \dot{\theta} = \sin^2\theta \dfrac{\partial v}{\partial x} + \sin\theta\cos\theta\left(\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial y}\right) - \cos^2\theta \dfrac{\partial u}{\partial y} \quad \text{—— (6)} \\ \dot{x} = V\cos\theta + u \\ \dot{y} = V\sin\theta + u \end{cases}$$

Note: If u & v are constants, the (6) implies that $\theta(t)$ = constant. (Min time paths are st. lines)

Functions of state variables specified at free t_f (including min-time prob)

$$J = \phi[x(t_f), t_f] + \int_{t_0}^{t_f} L[x(t), u(t), t]\,dt$$

$$\Psi[x(t_f), t_f] = 0 \quad \leftarrow q \text{ eqns.}$$

$$\dot{x} = f[x(t), u(t), t] \longrightarrow t_0 \text{ fixed}$$

$$J = [\phi + \nu^T \Psi]_{t=t_f} + \int_{t_0}^{t_f} \{L + \lambda^T (f - \dot{x})\}\,dt$$

$$H = L + \lambda^T f \quad ; \quad \Phi = \phi + \nu^T \Psi$$

$$dJ = \left[\left(\frac{\partial \Phi}{\partial t} + L\right) dt + \frac{\partial \Phi}{\partial x} dx\right]_{t=t_f}$$

$$+ \int_{t_0}^{t_f} \left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u + \lambda^T \delta \dot{x}\right] dt$$

Integrating by parts & using $\delta x = dx - \dot{x}\,dt$

$$dJ = \left[\frac{\partial \Phi}{\partial t} + L + \lambda^T \dot{x}\right]_{t=t_f} dt_f + \left[\left(\frac{\partial \Phi}{\partial x} - \lambda^T\right) dx\right]_{t=t_f}$$

$$+ \left[\lambda^T \delta x\right]_{t=t_0} + \int_{t_0}^{t_f} \left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right] dt$$

Hence necessary conditions are:

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\lambda^T \frac{\partial f}{\partial x} - \frac{\partial L}{\partial x}$$

$$\lambda^T(t_f) = \left[\frac{\partial \Phi}{\partial x}\right]_{t=t_f} = \left[\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right]_{t=t_f}$$

$$\left(\frac{\partial \Phi}{\partial t} + L + \lambda^T \dot{x}\right)_{t=t_f} = \left(\frac{d\Phi}{dt} + L\right)_{t=t_f} = 0$$

Exercise:

$$\frac{d\Phi}{dt} = \frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x}\dot{x}$$

As a result of this choice of $\lambda(t)$

$$dJ = \int_{t_0}^{t_f} \frac{\partial H}{\partial u}\,\delta u\,dt + \lambda^T(t_0)\,\cancel{dx(t_0)}$$

0 for $x(t_0)$ fixed

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} = 0 \qquad t_0 \le t \le t_f$$

Minimum Time Problem:

$$\phi[x(t_f), t_f] = 0, \quad L = 1$$

$$\left(\nu^T \frac{d\psi}{dt} + 1\right) = 0$$

Summary of necessary conditions

$$\dot{x} = f(x, u, t) \longrightarrow n \text{ ODE}$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\left[\frac{\partial L}{\partial x} + \lambda^T \frac{\partial f}{\partial x}\right] \longrightarrow n \text{ ODE}$$

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} \longrightarrow m \text{ optimality conditions}$$

$x_k(t_0)$ fixed or $\lambda_k(t_0) = 0 \rightarrow n$ bdd cond

$$\lambda^T(t_f) = \left[\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right]_{t=t_f} \rightarrow n \text{ bdd cond.}$$

$$\left[\frac{\partial \phi}{\partial t} + \nu^T \frac{\partial \psi}{\partial t} + \left(\frac{\partial \phi}{\partial x} + \nu^T \frac{\partial \psi}{\partial x}\right) f + L\right]_{t=t_f} = 0$$

$\rightarrow 1$ bdd condition

$$\psi[x(t_f), t_f] = 0$$

$\rightarrow q$ bdd cond

Unknowns: x, λ, u, ν, t_f

$$n + n + m + q + 1$$

$$= 2n + m + q + 1$$