Numerical Methods of Indirect Methods

Note Title 11-06-2008

Neighboring Extremal Algorithms

Case : Some state variables fixed at fixed t_f

$$\min \quad J = \phi[x(t_f)] + \int_{t_0}^{t_f} L(x,u,t)\,dt$$

where $\dot{x} = f(x,u,t)$ | $t_0, t_f \to$ fixed

$x(t_0) \leftarrow$ fixed

$$\psi[x(t_f)] = \begin{bmatrix} x_1(t_f) - x_1^f \\ \vdots \\ x_q(t_f) - x_q^f \end{bmatrix} = 0$$

E-L eqns : $\dot{\lambda}^T = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x}$

$$\lambda_j(t_f) = \left(\frac{\partial \phi}{\partial x_j}\right)_{t=t_f} \quad j = q+1, \cdots, n$$

$$0 = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u}$$

sp. $\begin{pmatrix} x_1(t_0) \\ \vdots \\ x_n(t_0) \end{pmatrix}$

unsp. $\begin{pmatrix} \lambda_1(t_0) \\ \vdots \\ \lambda_n(t_0) \end{pmatrix}$

$\left.\begin{matrix} x_1(t_f) \\ \vdots \\ x_q(t_f) \end{matrix}\right\}$ sp

$\left.\begin{matrix} x_{q+1}(t_f) \\ \vdots \\ x_n(t_f) \end{matrix}\right\}$ unsp

$\left.\begin{matrix} \lambda_1(t_f) \\ \vdots \\ \lambda_q(t_f) \end{matrix}\right\}$ unsp

sp $\overbrace{\begin{matrix} \lambda_{q+1}(t_f) \\ \vdots \\ \lambda_n(t_f) \end{matrix}}$

Step 1 : **Guess** $\lambda_1(t_0), \cdots, \lambda_n(t_0)$

step 2 : Integrate

$$\dot{x} = f(x,u,t)$$
$$\dot{\lambda}^T = -\frac{\partial H}{\partial x}$$

forward from t_0 to t_f using $\frac{\partial H}{\partial u} = 0$ to determine $u(t)$ at each step.

Step 3 : Calculate $[x_1(t_f) \cdots x_q(t_f)]$ and $[\lambda_{q+1}(t_f) \cdots \lambda_n(t_f)]$

Step 4 : Determine the $n \times n$ transition matrix $\frac{\partial \mu(t_f)}{\partial \lambda(t_0)}$, where

def^n
$$\delta\mu(t_f) = \begin{bmatrix} \delta x_1(t_f) \\ \vdots \\ \delta x_q(t_f) \\ \delta\lambda_{q+1}(t_f) \\ \vdots \\ \delta\lambda_n(t_f) \end{bmatrix} = \frac{\partial \mu(t_f)}{\partial \lambda(t_0)} \delta\lambda(t_0)$$

Q) How to compute this? → later

Step 5: Choose $\delta\mu(t_f) = -\varepsilon[\mu(t_f) - \mu^f]$
$0 \le \varepsilon \le 1$

Step 6: For chosen $\delta\mu(t_f)$, find

$$\delta\lambda(t_0) = \left[\frac{\partial \mu(t_f)}{\partial \lambda(t_0)}\right]^{-1} \delta\mu(t_f)$$

Step 7 : Using
$$\lambda(t_0)_{new} = \lambda(t_0)_{old} + \delta\lambda(t_0)$$

repeat steps 1 to 7 until $\mu(t_f)$ has the specified values.

Q> How to calculate $\dfrac{\partial \mu(t_f)}{\partial \lambda(t_0)}$?

Many methods are possible. The simplest one is :

→ Direct numerical integration:
Integrate n-times
$$\left.\begin{aligned} \dot{x} &= f(x,u,t) \\ \dot{\lambda}^T &= -\frac{\partial H}{\partial x} \end{aligned}\right) \quad \text{using } \frac{\partial H}{\partial u} = 0$$

→ For each int. change one component of $\lambda_i^\circ(t_0)$ by a small amount from the original guess in step 1.
→ Compute $\delta\mu(t_f)$ and divide by $\delta\lambda_i^\circ(t_0)$.

Functions of state variables specified at free tf

$$\min \quad J = \Phi[x(t_f), t_f] + \int_{t_0}^{t_f} L(x,u,t)\,dt$$

$$\dot{x} = f(x,u,t)$$
$$x(t_0) = x^0, \; t_0 \leftarrow \text{fixed} \qquad t_f \leftarrow \text{free}$$
$$\psi[x(t_f), t_f] = 0$$

$$\dot{\lambda}^T = -\left[\frac{\partial H}{\partial x}\right] \qquad 0 = \frac{\partial H}{\partial u}$$

$$\lambda^T(t_f) = \left.\frac{\partial \Phi}{\partial x}\right|_{t=t_f} \qquad \Phi = \phi + \nu^T \psi$$

$$\Omega = \left(\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} f + L\right)_{t=t_f} = 0$$

Step 1: Guess $x(t_f)$ (n values), ν (q values) and t_f

Step 2: Calculate 1> $\psi(x(t_f), t_f)$ — directly

2> $\lambda(t_f)$ from $\lambda^T(t_f) = \left.\frac{\partial \Phi}{\partial x}\right|_{t=t_f}$

3> $\Omega[x(t_f), u(t_f), \nu, t_f]$

This $u(t_f)$ is calculated from $\frac{\partial H}{\partial u} = 0$ (using $\lambda(t_f)$)

Step 3:
$$\dot{x} = f(x, u, t)$$
$$\dot{\lambda}^T = -\frac{\partial H}{\partial x}$$
Integrate backwards from t_f to t_0 using

1> $u(t)$ from $\frac{\partial H}{\partial u} = 0$

2> $x(t_f) + \lambda(t_f)$ ← from steps 1 & 2

Step 4: Calculate $x(t_0)$

Step 5: Calculate $[(n+q+1) \times (n+q+1)]$ transition matrix

$$\frac{\partial [x(t_0), \psi, \Omega]}{\partial [x(t_f), \nu, t_f]} \quad \text{where}$$

$$\begin{bmatrix} \delta x(t_0) \\ d\psi \\ d\Omega \end{bmatrix} = \frac{\partial[x(t_0), \psi, \Omega]}{\partial[x(t_f), \nu, t_f]} \begin{bmatrix} \delta x(t_f) \\ d\nu \\ dt_f \end{bmatrix}$$

Step 6: Choose $\delta x(t_0)$, $d\psi$, $d\Omega$ so as to bring the next soln. closer the $x(t_0)$, $\psi = 0$, $\Omega = 0$.

$$\begin{bmatrix} \delta x(t_0) \\ d\psi \\ d\Omega \end{bmatrix} = -\varepsilon \begin{bmatrix} x(t_0) - x^0 \\ \psi[x(t_f), t_f] \\ \Omega[x(t_f), t_f] \end{bmatrix} \quad 0 < \varepsilon \le 1$$

Step 7: With chosen values of $\delta x(t_0)$, $d\psi$ & $d\Omega$, invert the transition matrix to calculate $\delta x(t_f)$, $d\nu$, dt_f

Step 8: Using $\begin{bmatrix} x(t_f) \\ \nu \\ t_f \end{bmatrix}_{new} = \begin{bmatrix} x(t_f) \\ \nu \\ t_f \end{bmatrix}_{old} + \begin{bmatrix} dx(t_f) \\ d\nu \\ dt_f \end{bmatrix}$

$$dx(t_f) = \delta x(t_f) + \dot{x}(t_f)\, dt_f$$

repeat steps 1–8 until $x(t_0) = x^0$
$\psi[x(t_f), t_f] = 0$ & $\Omega(\cdots) = 0$

First Order Gradient Algorithms:

Some state variable specified at fixed terminal time:

Step 1: Guess $u(t)$

Step 2: Integrate $\dot{x} = f(x,u,t)$ forward with $x(t_0)$ & $u(t)$ from step 1.
Record $x(t)$, $u(t)$, $\psi(x(t_f))$

Step 3: Backward integrate the following eqns:

$$\dot{p}^T = -\frac{\partial L}{\partial x} - p^T \frac{\partial f}{\partial x} \quad \Big| \quad p_i(t_f) = \begin{cases} 0 & i = 1, \dots, q \\ \left.\frac{\partial \phi}{\partial x_i}\right|_{t=t_f} & i = q+1, \dots, n \end{cases}$$

$$\dot{R}^T = -R^T \left[\frac{\partial f}{\partial x}\right] \qquad R_{ij}(t_f) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$i = 1, \dots, n$
$j = 1 \dots q$

Step 4: Simultaneously with Step 3, compute the following integrals:

$$I_{\psi\psi} = \int_{t_0}^{t_f} R^T \frac{\partial f}{\partial u} W^{-1} \left(\frac{\partial f}{\partial u}\right)^T R \, dt \quad (q \times q)$$

$$I_{J\psi} = I_{\psi J}^T = \int_{t_0}^{t_f} \left(p^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u}\right) W^{-1} \left(\frac{\partial f}{\partial u}\right)^T R \, dt$$

(q-row vector)

$$I_{JJ} = \int_{t_0}^{t_f} \left(p^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u}\right) W^{-1} \left[\left(\frac{\partial f}{\partial u}\right)^T p + \left(\frac{\partial L}{\partial u}\right)^T\right] dt$$

↳ scalar

where $W(t)$ is a $(m \times m)$ positive definite weighting matrix.

Step 5: Choose $\delta\psi$ to cause next sol^{n} to be closer to $\psi[x(t_f)]=0$

$\longrightarrow$ $\delta\psi = -\varepsilon\,\psi[x(t_f)]$, $0<\varepsilon\leq 1$

$\longrightarrow$ Then calculate ν from

$$\nu = -\left[I_{\psi\psi}\right]^{-1}\left(\delta\psi + I_{\psi J}\right)$$

Step 6: Repeat Steps 1 – 6, using improved $[u(t)]_{new} = [u(t)]_{old} + \delta u(t)$

where

$$\delta u(t) = -\left[W(t)\right]^{-1}\left[\frac{\partial L}{\partial u} + \left[p(t) + R(t)\nu\right]^T \frac{\partial f}{\partial u}\right]^T$$

Stop when $\psi[x(t_f)]=0$ and

$$I_{JJ} - I_{J\psi}\, I_{\psi\psi}^{-1}\, I_{\psi J} = 0$$