Optimal Control: Path Constraints

Note Title 11-06-2008

Integral Constraints:

$$\underset{\text{given}}{K} = \int_{t_0}^{t_f} N(x,u,t)\,dt$$

Define: $x_{n+1} = N(x,u,t)$

$$x_{n+1}(t_0) = 0 \quad \& \quad x_{n+1}(t_f) = K$$

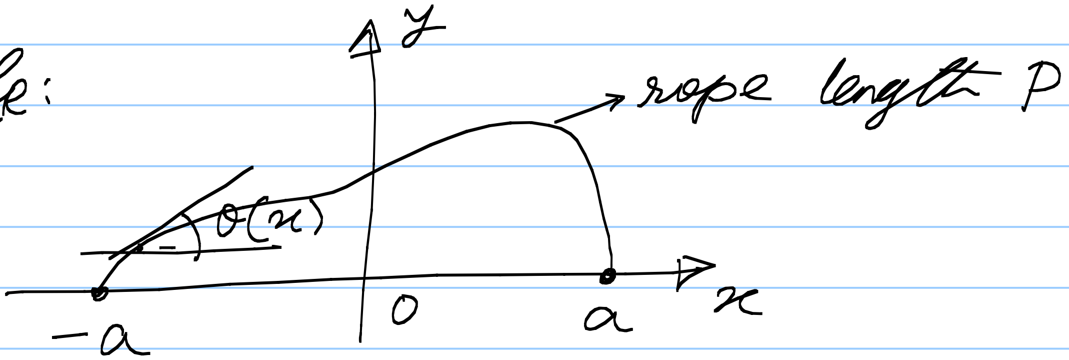
E-L eqns: $H = L + \lambda^T f + \mu N$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} - \mu \frac{\partial N}{\partial x}$$

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \mu \frac{\partial N}{\partial u}$$

$$\dot{\mu} = -\frac{\partial H}{\partial x_{n+1}} = 0 \Rightarrow \mu = \text{constant}$$

Example:



$$A = \int_{-a}^{a} y\,dx \longrightarrow \text{Find } \theta(x) \text{ to maximize } A.$$

$$P = \int_{-a}^{a} \sec\theta\,dx \qquad \frac{dy}{dx} = \tan\theta$$

Hamiltonian:

$$H = y + \lambda \tan\theta + \mu \sec\theta$$

E-L eqns are:

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$$\frac{d\lambda}{dx} = -\frac{\partial H}{\partial y} = -1 \Rightarrow \lambda = -x + \underset{\text{constant}}{c} \quad \text{—} (1)$$

$$0 = \frac{\partial H}{\partial \theta} = \mu \tan\theta \sec\theta + \lambda \sec^2\theta$$

$$\Rightarrow \sin\theta = -\frac{\lambda}{\mu} \quad \text{—} (2)$$

Eliminating λ from (1) & (2):

$$x = \mu \sin\theta + c$$

H is not an explicit function of x $\therefore$
H = constant

Then from (2) & (3):

$$H = y + \left(-\mu \sin\theta \tan\theta\right) + \mu \frac{1}{\cos\theta}$$

$$= y + \mu\left[\frac{1-\sin^2\theta}{\cos\theta}\right]$$

$$= y + \mu\cos\theta$$

$$\Leftrightarrow \quad y = -\mu\cos\theta + H \quad (H = \text{constant})$$

Now, $P = \int_A^B \sec\theta \frac{dx}{d\theta} d\theta$

But $\boxed{\frac{dx}{d\theta} = \mu\cos\theta}$ $= \mu \int_A^B d\theta$

$$= \mu(\theta_B - \theta_A) \quad \text{—} (4)$$

Unknowns: $c, H, \mu, \theta_A, \theta_B$

<u>Bdd conditions</u>: $x(\theta_A) = -a$, $x(\theta_B) = a$

$y(\theta_A) = 0$, $y(\theta_B) = 0$ & (4)

After solving; we get

$$x = -\frac{P}{2\alpha}\sin\theta$$

$$y = \frac{P}{2\alpha}(\cos\theta - \cos\alpha)$$

or $$x^2 + \left(y + \frac{P\cos\alpha}{2\alpha}\right)^2 = \frac{P^2}{4\alpha^2}$$

$\Leftrightarrow$ Circular arc.

<u>Control Variable Equality Constraint</u>

$$C(u, t) = 0$$

$$u(t) \in R^m \qquad m \geq 2$$

* One method is to eliminate one variable in terms of the others.

* $$H = L + \lambda^T f + \mu C$$

Only change: $0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \mu \frac{\partial C}{\partial u}$

<u>Equality Constraints on functions of the control + state variable</u>

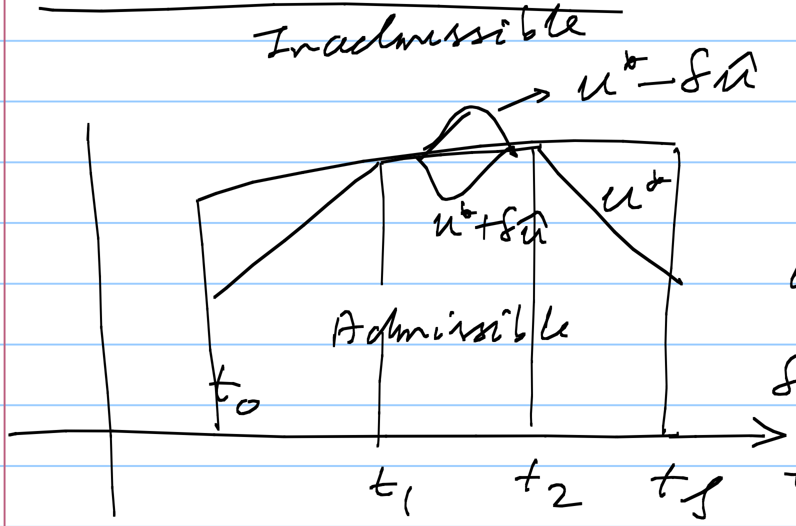
$$C(x, u, t) = 0$$

$\rightarrow$ $$H = L + \lambda^T f + \mu^T C$$

$$0 = \frac{\partial H}{\partial u} = \lambda^T \frac{\partial f}{\partial u} + \frac{\partial L}{\partial u} + \mu^T \frac{\partial C}{\partial u}$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} - \mu^T \frac{\partial c}{\partial x}$$

Inequality Constraints on the Control Variable:



For $\delta u = \delta\hat{u}$, $u^* + \delta\hat{u}$ is admissible but $u^* - \delta\hat{u}$ is NOT.

So $\delta J(u^*, \delta\hat{u}) \geq 0$ and not necessarily $= 0$.

So necessary condition:

$$\delta J(u^*, \delta u) \geq 0.$$

Now as usual:

$$\delta J = \int_{t_0}^{t_f} \frac{\partial H}{\partial u}\, \delta u(t)\, dt$$

$$= \int_{t_0}^{t_f} \left[H(x^*, u^* + \delta u, \lambda^*, t) - H(x^*, u^*, \lambda^*, t) \right] dt \geq 0 \quad (1)$$

for all admissible $\delta u(t)$.

Claim: (1) $\Rightarrow$ $H(x^*(t), u^*(t) + \delta u(t), \lambda^*(t), t)$

$$\geq H(x^*(t), u^*(t), \lambda^*(t), t)$$

for all admissible $\delta u(t)$ and all $t \in [t_0, t_f]$

Proof:

Let $\quad u(t) = \begin{cases} u^*(t) & t\in[t_1, t_2] \\ u^*(t)+\delta u(t) & t\in[t_1,t_2] \end{cases}$

δu is admissible & "small".

Let $\quad H(x^*, u, \lambda^*, t) < H(x^*, u^*, \lambda^*, t)$

$$\Rightarrow \int_{t_0}^{t_f} \left[H(x^*, u, \lambda^*, t) - H(x^*, u^*, \lambda^*, t)\right] dt < 0$$

Since $[t_1, t_2]$ can be anywhere in $[t_0, t_f]$

$$\boxed{\begin{gathered} H(x^*, u^*, \lambda^*, t) \leq H(x^*, u, \lambda^*, t) \\ \forall\, t\in[t_0, t_f] \end{gathered}} \quad -(1)$$

Pontryagin's Minimum Principle

For all the problems considered, if input inequality constraints are present, replace $\frac{\partial H}{\partial u}=0$ with (1).

$\rightarrow$ All the other necessary conditions remain valid.

Note: 1> $u^*(t)$ is the control that makes H globally min. at each t.

2> Eqn (1) is necessary but not sufficient.

3> The above derivation was done for fixed t_f & no terminal constraints. However (1) holds for free t_f & terminal constraints also:

Example: $\dot{x}_1 = x_2$ | $x(t_0) = x_0$

$\dot{x}_2 = -x_2 + u$

$$-1 \le u(t) \le 1 \quad \forall t \in [t_0, t_f]$$

$$\min \quad J = \int_{t_0}^{t_f} \frac{1}{2}\left[x_1^2(t) + u^2(t)\right] dt$$

$$H = \frac{1}{2}x_1^2 + \frac{1}{2}u^2 + \lambda_1 x_1 + \lambda_2(-x_2 + u)$$

$$= (\qquad\qquad) + \frac{1}{2}u^2 + \lambda_2 u$$

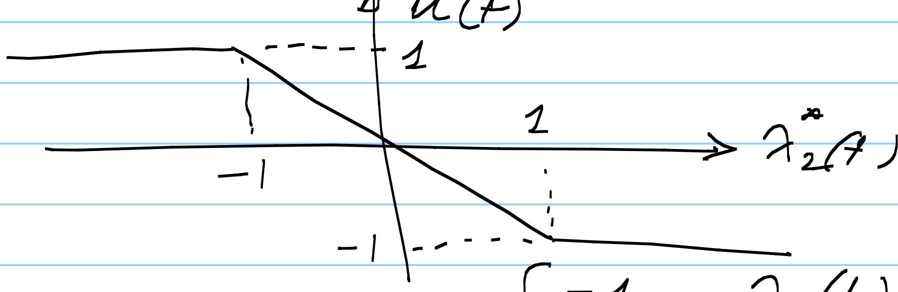
For parts where $|u(t)| < 1$,

$$u^*(t) = -\lambda_2(t)$$

This will happen when $|\lambda_2(t)| < 1$

If $|\lambda_2(t)| > 1$, then

$$u^*(t) = \begin{cases} -1 & \text{for } \lambda_2(t) > 1 \\ 1 & \text{for } \lambda_2(t) < -1 \end{cases}$$



Hence $u^*(t) = \begin{cases} -1 & \lambda_2(t) > 1 \\ -\lambda_2(t) & -1 \le \lambda_2(t) \le 1 \\ +1 & \lambda_2(t) < -1 \end{cases}$

Example : $J = \frac{a^2}{2} \|x(T)\|^2 + \frac{1}{2} \int_0^T \|u\|^2 dt$

$$\dot{x} = g(t) u \quad (g(t) : \text{known})$$

$$|u(t)| \leq 1$$

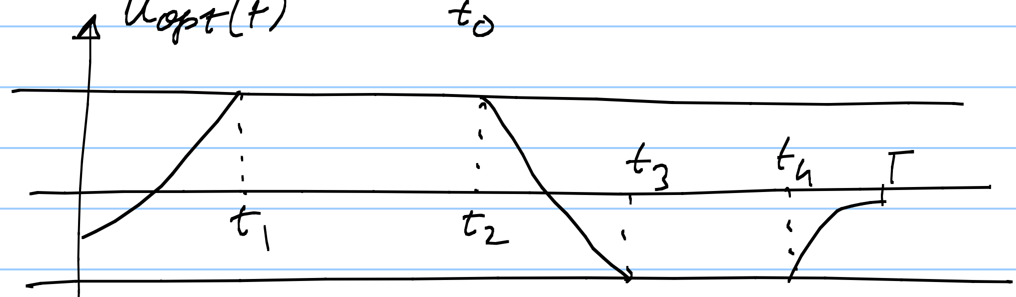
$$H = \frac{1}{2}\|u\|^2 + \lambda^T g u$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = 0 \qquad \lambda(t) = \lambda(T) = a^2 x(T)$$

Using PMP: $u_{opt} = -\,\mathrm{sat}[a^2 g(t) x(T)]$

$$\mathrm{sat}(\alpha) = \begin{cases} \alpha & |\alpha| < 1 \\ 1 & |\alpha| \geq 1 \end{cases}$$

$$x(T) = x_0 - \int_{t_0}^T g(t)\, \mathrm{sat}[a^2 g(t) x(T)]\, dt$$

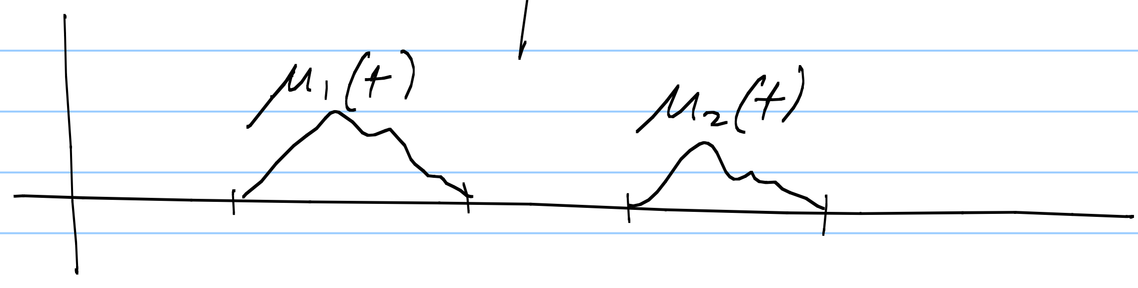


Alternate way : $H = \frac{1}{2}\|u\|^2 + \lambda^T g u + \mu_1(u-1) + \mu_2(-u-1)$

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u + \lambda^T g + \mu_1 - \mu_2 = 0$$

$\mu_1 > 0$ if $u = 1$ | $\mu_2 > 0$ if $u = -1$

$= 0$ otherwise | $= 0$ otherwise



$$\mu_1(t) = \begin{cases} -(1 + a^2 g(t) x(T)) & t_1 \leq t \leq t_2 \\ 0 & \text{other time} \end{cases}$$

$$\mu_2(t) = \begin{cases} -1 + a^2 g(t) x(T) & t_3 \leq t \leq t_4 \\ 0 & \text{other time} \end{cases}$$

$1 + a^2 g(t) x(T) = 0$ at $t = t_1$ & t_2

$-1 + a^2 g(t) x(T) = 0$ at $t = t_3$ & t_4

Inequality Constraints on Functions of Control + state variables

$$c(x, u, t) \leq 0$$

$$H = L + \lambda^T f + \mu c \quad \text{where } \mu = \begin{cases} > 0 & c = 0 \\ = 0 & c < 0 \end{cases}$$

and E-L eqns become

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = \begin{cases} \frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} - \mu \frac{\partial c}{\partial x} & c = 0 \\ -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} & c < 0 \end{cases}$$

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \mu \frac{\partial c}{\partial u} = 0 \quad \text{—— (1)}$$

For $c < 0$, $\mu = 0$ in (1) determine $u(t)$

For $c = 0$, $\mu > 0$.